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On deviations from rational functions of entire functions of finite lower order

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Abstract. Let f be a transcendental entire function of finite lower order, and let q_ν be rational functions. For $0 < \gamma < \infty$ let

$$B(\gamma) := \begin{cases} \dfrac{\pi\gamma}{\sin \pi\gamma} & \text{if } \gamma \le 0.5, \\ \pi\gamma & \text{if } \gamma > 0.5. \end{cases}$$

We estimate the upper and lower logarithmic density of the set

$$\Big\{ r : \sum_{1\le\nu\le k} \log^+ \max_{|z|=r} |f(z) - q_\nu(z)|^{-1} < B(\gamma)T(r,f) \Big\}.$$

The theory of value distribution of meromorphic functions was introduced in the 1920's in the papers of the Finnish mathematician Rolph Nevanlinna. The fundamental role in this theory is played by two functions. The first of them,

$$m(r,a,f) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta & \text{for } a = \infty, \\ \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \log^+ \left| \dfrac{1}{f(re^{i\theta}) - a} \right| d\theta & \text{for } a \ne \infty, \end{cases}$$

measures the mean proximity of $f(z)$ to the value a (here $\log^+ x = \max(\log x, 0)$). The second one,

$$N(r,a,f) = \int_0^r [n(t,a,f) - n(0,a,f)] \frac{dt}{t} + n(0,a,f)\log r,$$

counts the a-points of $f(z)$ (here $n(t,a,f)$ is the number of a-points of $f(z)$ in the disc $\{z : |z| \le t\}$, counted together with their multiplicity).

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The *first fundamental theorem* of Nevanlinna states that the sum of those two functions is to some degree independent of the choice of $a \in \overline{\mathbb{C}}$, that is, for a fixed f and $r \to \infty$, choosing a different value a changes the sum $m(r,a,f) + N(r,a,f)$ by a bounded term.

THEOREM A ([13]). *For a meromorphic function $f(z)$ and for a point $a \in \overline{\mathbb{C}}$ the following equality holds*:

$$m(r,a,f) + N(r,a,f) = T(r,f) + O(1) \quad (r \to \infty). \tag{1}$$

The function $T(r,f) := m(r,\infty,f) + N(r,\infty,f)$ is called *Nevanlinna's characteristic function* of the meromorphic function $f(z)$.

The *second fundamental theorem* of Nevanlinna shows that for most values a the main role in the invariant sum (1) is played by the counting function $N(r,a,f)$.

THEOREM B ([13]). *Let $\{a_k\}_{k=1}^{q} \subset \overline{\mathbb{C}}$ be a finite set. Then*

$$\sum_{k=1}^{q} m(r,a_k,f) \le 2T(r,f) + O(\log(rT(r,f)))$$

for $r \to \infty$, except possibly for r in a set of finite linear measure.

The quantity

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)}$$

is called *Nevanlinna's defect* of the meromorphic function $f(z)$ at the point $a \in \overline{\mathbb{C}}$. We refer to the set $D(f) = \{a \in \overline{\mathbb{C}} : \delta(a,f) > 0\}$ as the set of *defective values* of $f(z)$. The first fundamental theorem of Nevanlinna implies that $0 \le \delta(a,f) \le 1$ for all $a \in \overline{\mathbb{C}}$. The second fundamental theorem, on the other hand, means that the set $D(f)$ is at most countable and $\sum_{a\in\overline{\mathbb{C}}} \delta(a,f) \le 2$.

In 1986 the following extension of the second fundamental theorem of Nevanlinna was shown by G. Frank and G. Weissenborn.

THEOREM C ([5]). *Let $f(z)$ be a transcendental meromorphic function. Then for any distinct rational functions $q_1(z), \dots, q_k(z)$ we have*

$$m(r,f) + \sum_{\nu=1}^{k} m\left(r, \frac{1}{f - q_\nu}\right) \le (2 + o(1))T(r,f)$$

for $r \to \infty$, except possibly for r in a set of finite linear measure.

Also in 1986 N. Steinmetz proved a more general result.

THEOREM D ([16]). *Let $f(z)$ be a nonconstant meromorphic function and let $\{a_\nu\}_{\nu=1}^{k}$ be a set of pairwise distinct meromorphic functions such* *that for* $1 \le \nu \le k$ *we have* $T(r, a_\nu) = o(T(r, f))$ *as* $r \to \infty$. *Then*

$$m(r,f) + \sum_{\nu=1}^{k} m\left(r, \frac{1}{f - a_\nu}\right) \le (2 + o(1))T(r,f)$$

for $r \to \infty$, *except possibly for* r *in a set of finite linear measure.*

In 1969 Petrenko raised a question: how will Nevanlinna's theory change if we measure the proximity of a meromorphic function $f(z)$ to a value a applying a different metric? In order to find the answer he introduced the function of deviation:

$$\mathcal{L}(r,a,f) = \begin{cases} \max\limits_{|z|=r} \log^+ |f(z)| & \text{for } a = \infty, \\ \max\limits_{|z|=r} \log^+ \left| \dfrac{1}{f(z) - a} \right| & \text{for } a \ne \infty. \end{cases}$$

The quantity

$$\beta(a,f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r,a,f)}{T(r,f)}$$

is called the *magnitude of deviation* and $\Omega(f) = \{a \in \overline{\mathbb{C}} : \beta(a,f) > 0\}$ the set of positive deviations of $f(z)$.

It is easy to notice that for all $a \in \overline{\mathbb{C}}$ we have $\delta(a,f) \le \beta(a,f)$. Therefore $D(f) \subset \Omega(f)$. In the case of meromorphic functions of finite *lower order*

$$\lambda := \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

the properties of $\beta(a,f)$ strongly resemble the properties of $\delta(a,f)$. Petrenko himself obtained a sharp upper estimate for $\beta(a,f)$ and also an estimate for $\sum_{a\in\overline{\mathbb{C}}} \beta(a,f)$.

THEOREM E ([15]). *If* $f(z)$ *is a meromorphic function of finite lower order* λ, *then for all* $a \in \overline{\mathbb{C}}$ *we have*

$$\beta(a,f) \le B(\lambda) := \begin{cases} \dfrac{\pi\lambda}{\sin \pi\lambda} & \text{if } \lambda \le 0.5, & (2) \\ \pi\lambda & \text{if } \lambda > 0.5, & (3) \end{cases}$$

$$\sum_{a\in\overline{\mathbb{C}}} \beta(a,f) \le 816\pi(\lambda+1)^2.$$

The value $B(\lambda)$ is called *Paley's constant.* In 1932 Paley [14] conjectured that (3) holds for any entire function $f(z)$ and $a = \infty$. This was proved by Govorov [8] in 1969. It should be mentioned that (3) follows from a result of Gol'dberg and Ostrovskiĭ [7].

In 1990 Marchenko and Shcherba obtained a sharp estimate of the sum of deviations, which is an analogue of the estimate of the sum of defects.

Theorem F ([12]). *If $f(z)$ is a meromorphic function of finite lower order λ, then*

$$\sum_{a\in\overline{\mathbb{C}}}\beta(a,f)\le 2B(\lambda).$$

Let us now recall the definitions of the lower and upper logarithmic density of a set. Let $E\subset(0,\infty)$ be a measurable set. The quantities

$$\overline{\operatorname{logdens}}\,E=\limsup_{R\to\infty}\frac{1}{\log R}\int\limits_{E\cap[1,R]}\frac{dt}{t},$$

$$\underline{\operatorname{logdens}}\,E=\liminf_{R\to\infty}\frac{1}{\log R}\int\limits_{E\cap[1,R]}\frac{dt}{t}$$

are called respectively the *upper* and *lower logarithmic density* of the set E.

In 1998 Marchenko gave the following analogue of the second fundamental theorem of Nevanlinna.

Theorem G ([11]). *Let $f(z)$ be a meromorphic function of finite lower order λ and of order ϱ. Let $\{a_\nu\}_{\nu=1}^k$ be a finite set of distinct complex numbers. For $0<\gamma<\infty$ put*

$$E_1(\gamma)=\Big\{r:\sum_{\nu=1}^{k}\mathcal{L}(r,a_\nu,f)<2B(\gamma)T(r,f)\Big\}.$$

Then

$$\overline{\operatorname{logdens}}\,E_1(\gamma)\ge 1-\lambda/\gamma\quad\textit{and}\quad\underline{\operatorname{logdens}}\,E_1(\gamma)\ge 1-\varrho/\gamma.$$

Moreover, in [11] it is shown that for an entire function $f(z)$ the estimates from Theorem G also hold for the set

$$E_2(\gamma)=\Big\{r:\sum_{\nu=1}^{k}\mathcal{L}(r,a_\nu,f)<B(\gamma)T(r,f)\Big\}.$$

Let now $\{q_\nu\}_{\nu=1}^k$ be a finite set of distinct rational functions and let f be a transcendental entire function of finite lower order. We put

$$\beta(q_\nu,f)=\liminf_{r\to\infty}\frac{\mathcal{L}(r,q_\nu,f)}{T(r,f)},$$

where $\mathcal{L}(r,q_\nu,f)=\mathcal{L}\big(r,\infty,\frac{1}{f-q_\nu}\big)$. Our main result is the following theorem.

Theorem 1. *Let $f(z)$ be a transcendental entire function of finite lower order λ and of order ϱ, and let $0<\gamma<\infty$. Let also $\{q_\nu(z)\}_{\nu=1}^k$ be distinct rational functions. Put*

$$E(\gamma)=\Big\{r:\sum_{\nu=1}^{k}\mathcal{L}(r,q_\nu,f)<B(\gamma)T(r,f)\Big\}.$$

Then

$$\overline{\text{logdens}}\, E(\gamma) \geq 1 - \lambda/\gamma \quad \text{and} \quad \underline{\text{logdens}}\, E(\gamma) \geq 1 - \varrho/\gamma.$$

Write $\mathfrak{M}$ for the set of all rational functions.

COROLLARY. *Let $f(z)$ be a transcendental entire function of finite lower order λ. The set $\{q \in \mathfrak{M} : \beta(q, f) > 0\}$ is at most countable. Moreover, for any distinct rational functions $\{q_\nu(z)\}$ we have*

$$\sum_\nu \beta(q_\nu, f) \leq B(\lambda).$$

1. Auxiliary results. In order to prove Theorem 1 we need a version of the lemma on the logarithmic derivative, which follows from Lemma 4 of [11].

LEMMA 1. *Let $f(z)$ be a meromorphic function. Then, except possibly for r in a set of finite linear measure, for $k = 1, 2, \ldots$ we have*

$$\log^+ M\left(r, \frac{f^{(k)}}{f}\right) = O(\log(rT(r, f))) \quad (r \to \infty)$$

where $M(r, f) = \max_{|z|=r} |f(z)|$ and $f^{(k)}$ is the kth derivative of f.

We first prove Theorem 1 for linear functions, then for all polynomials, and finally we handle the general case.

So assume that $f(z)$ is a transcendental entire function of finite lower order λ and that for $1 \leq \nu \leq k$,

$$p_\nu(z) := a_\nu z + b_\nu$$

are polynomials of degree $\deg(p_\nu) \leq 1$ such that for $\nu \neq \eta$ we have $a_\nu \neq a_\eta$ or $b_\nu \neq b_\eta$. Choose $S_0 > 0$ such that if $|z| \geq S_0$, then $p_\nu(z) \neq p_\eta(z)$ for all $1 \leq \nu, \eta \leq k$, $\nu \neq \eta$. For $\nu \neq \eta$ we put

$$c_{\nu,\eta} = \min_{|z| \geq S_0} |p_\nu(z) - p_\eta(z)| > 0, \qquad \min_{1 \leq \nu,\eta \leq k} c_{\nu,\eta} = c > 0.$$

Let $\{R_n\}$ be a sequence of positive numbers such that

$$\lambda = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r} = \lim_{n\to\infty} \frac{\log T(3R_n, f)}{\log R_n}.$$

For $n \geq n_0$ we consider the set

$$G_n = \{z : S_0 < |z| < R_n,\ |f^{(2)}(z)| < 1/R_n^{\lambda+4}\},$$

where n_0 is chosen in such a way that for $n \geq n_0$ we have

$$T(3R_n, f) < R_n^{\lambda+1} \quad \text{and} \quad 20\pi^2/R_n < c/4. \tag{4}$$

Now for $1 \le \nu \le k$ we write $G_{n,\nu}$ for the set of those connected components of G_n which contain a point z_1 such that

$$|f(z_1) - a_\nu z_1 - b_\nu| < c/4 \tag{5}$$

and a point z_2 such that

$$|f^{(1)}(z_2) - a_\nu| < 1/R_n^{\lambda+4}. \tag{6}$$

We are going to show that for $\nu \neq \eta$ the sets $G_{n,\nu}$ and $G_{n,\eta}$ are disjoint. We apply the method introduced by Weitsman [17]. Let $l(t)$ be the length of the level line $|f^{(2)}(z)| = t$ in the disc $|z| \le R_n$. The rule of line and square (see [9]) gives

$$\int_0^\infty \frac{l^2(t)}{tp(t)}\, dt \le 2\pi^2 R_n^2,$$

where

$$p(t) = \frac{1}{2\pi} \int_0^{2\pi} n\left(R_n, \frac{1}{f^{(2)} - te^{i\varphi}}\right) d\varphi.$$

It follows easily that

$$\begin{aligned} n\left(R_n, \frac{1}{f^{(2)} - te^{i\varphi}}\right) &= n\left(R_n, e^{i\varphi}, \frac{f^{(2)}}{t}\right) \le \frac{1}{\log 2}\, N\left(2R_n, e^{i\varphi}, \frac{f^{(2)}}{t}\right) \\ &\le \frac{1}{\log 2}\, T\left(2R_n, \frac{f^{(2)}}{t}\right) + O(1) \\ &\le \frac{1}{\log 2}\left[T(2R_n, f^{(2)}) + \log^+ \frac{1}{t}\right] + O(1). \end{aligned}$$

Applying the lemma on the logarithmic derivative to the entire functions $f^{(1)}$ and f we obtain

$$\begin{aligned} T(2R_n, f^{(2)}) &= m(2R_n, f^{(2)}) \le m(2R_n, f^{(2)}/f^{(1)}) + m(2R_n, f^{(1)}) \\ &\le m(2R_n, f^{(2)}/f^{(1)}) + m(2R_n, f^{(1)}/f) + m(2R_n, f) \\ &\le 5 \log T(3R_n, f^{(1)}) + 5 \log T(3R_n, f) + T(2R_n, f) \\ &\le 3T(3R_n, f). \end{aligned}$$

Therefore

$$\begin{aligned} p(t) &= \frac{1}{2\pi} \int_0^{2\pi} n\left(R_n, \frac{1}{f^{(2)} - te^{i\varphi}}\right) d\varphi \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{1}{\log 2}\left(T(2R_n, f^{(2)}) + \log^+ \frac{1}{t}\right) + O(1)\right] d\varphi \\ &\le \frac{1}{\log 2}\left(T(2R_n, f^{(2)}) + \log^+ \frac{1}{t}\right) + O(1) \le \frac{1}{\log 2}\left(3T(3R_n, f) + \log^+ \frac{1}{t}\right). \end{aligned}$$

Let now $\gamma_1 = 1/R_n^{\lambda+4}$ and $\gamma_2 = 2\gamma_1$. Then

$$2\pi^2 R_n^2 \geq \int_0^\infty \frac{l^2(t)}{tp(t)}\,dt \geq \int_{\gamma_1}^{\gamma_2} \frac{l^2(t)}{tp(t)}\,dt > \frac{\log 2}{3T(3R_n, f) + \log^+ \frac{1}{\gamma_1}} \int_{\gamma_1}^{\gamma_2} \frac{l^2(t)}{t}\,dt.$$

This way we get

$$\begin{aligned}\int_{\gamma_1}^{\gamma_2} \frac{l^2(t)}{t}\,dt &< \frac{2\pi^2 R_n^2}{\log 2}\Big(3T(3R_n, f) + \log^+ \frac{1}{\gamma_1}\Big)\\ &= \frac{2\pi^2 R_n^2}{\log 2}\Big(3T(3R_n, f) + (\lambda+4)\log R_n\Big).\end{aligned}$$

Moreover

$$\int_{\gamma_1}^{\gamma_2} \frac{l^2(t)}{t}\,dt = \int_{-\log\gamma_2}^{-\log\gamma_1} l^2(e^{-t})\,dt = \int_{(\lambda+4)\log R_n - \log 2}^{(\lambda+4)\log R_n} l^2(e^{-t})\,dt.$$

Thus there exists $\alpha \in [(\lambda+4)\log R_n - \log 2, (\lambda+4)\log R_n]$ such that $l(e^{-\alpha}) < 4\pi R_n \sqrt{T(3R_n, f)}$. We put

$$G' = \{z : S_0 < |z| < R_n,\ \log|f^{(2)}(z)| < -\alpha\}.$$

It is easy to see that $G_{n,\nu} \subset G_n \subset G'$. Let $z \in G_{n,\nu}$. Then there exists a connected component G of $G_{n,\nu}$ such that $z \in G \subset G_{n,\nu} \subset G_n \subset G'$. We connect points z and z_2 by a line $\Gamma \subset G'$ whose length is not greater than the length of the boundary of G'. Thus we have

$$|f^{(1)}(z) - f^{(1)}(z_2)| = \Big|\int_\Gamma f^{(2)}(z)\,dz\Big| \leq \int_\Gamma |f^{(2)}(z)|\,|dz| < 4\pi R_n \sqrt{T(3R_n, f)}\,\frac{1}{R_n^{\lambda+4}}.$$

It follows that for $z \in G_{n,\nu}$,

$$\begin{aligned}|f^{(1)}(z) - a_\nu| &\leq |f^{(1)}(z) - f^{(1)}(z_2)| + |f^{(1)}(z_2) - a_\nu|\\ &< (4\pi R_n \sqrt{T(3R_n, f)} + 1)\frac{1}{R_n^{\lambda+4}}.\end{aligned} \tag{7}$$

Next, we connect z and z_1 by a line $\Gamma_1 \subset G'$ whose length is not greater than the length of the boundary of G'. This way we get

$$\begin{aligned}|f(z) - f(z_1) - a_\nu(z - z_1)| &= \Big|\int_{\Gamma_1} (f^{(1)}(\xi) - a_\nu)\,d\xi\Big|\\ &\leq \int_{\Gamma_1} |f^{(1)}(\xi) - a_\nu|\,|d\xi|\\ &< (4\pi R_n \sqrt{T(3R_n, f)} + 1)\frac{4\pi\sqrt{T(3R_n, f)}}{R_n^{\lambda+3}}\\ &< 20\pi^2\,\frac{T(3R_n, f)}{R_n^{\lambda+2}}.\end{aligned} \tag{8}$$

Now, taking into consideration the inequalities (4)–(8), we can estimate $|f(z) - p_\nu(z)|$ for $z \in G_{n,\nu}$. For $n \geq n_0$ we get

$$|f(z) - p_\nu(z)| \leq |f(z) - f(z_1) - a_\nu(z - z_1)| + |f(z_1) - p_\nu(z_1)| < c/2.$$

This shows that $G_{n,\nu} \cap G_{n,\eta} = \emptyset$ for $\nu \neq \eta$.

Now for $1 \leq \nu \leq k$ and $n \geq n_0$ we consider the functions

$$u_{n,\nu}(z) := \begin{cases} \log \dfrac{1}{|f^{(2)}(z)|}, & z \in G_{n,\nu}, \\ (\lambda + 4) \log R_n, & z \notin G_{n,\nu}. \end{cases}$$

Each $u_{n,\nu}(z)$ is a δ-subharmonic function in $S_0 < |z| < R_n$. Let us recall the definition and basic properties of Baernstein's function T^*. For a complex number $z = re^{i\theta}$ we put [1]:

$$m^*(z, u_{n,\nu}) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int\limits_E u_{n,\nu}(re^{i\varphi})\, d\varphi,$$
$$T^*(z, u_{n,\nu}) = m^*(z, u_{n,\nu}) + N(r, u_{n,\nu}),$$

where $r \in (S_0, R_n)$, $\theta \in [0, \pi]$, $|E|$ is Lebesgue's measure of the set E and $N(r, u_{n,\nu})$ counts the zeros of $f^{(2)}(z)$ in $G_{n,\nu} \cap \{z : |z| < r\}$. Write $\widetilde{u}_{n,\nu}(z)$ for the circular symmetrization of $u_{n,\nu}(z)$. It is easy to notice that

$$m^*(z, u_{n,\nu}) = \frac{1}{\pi} \int\limits_0^\theta \widetilde{u}_{n,\nu}(re^{i\varphi})\, d\varphi.$$

From Baernstein's theorem (see [1]) the function $T^*(z, u_{n,\nu})$ is subharmonic on

$$D = \{re^{i\theta} : S_0 < r < R_n,\ n > n_0,\ 0 < \theta < \pi\},$$

continuous on $D \cup (-R_n, S_0) \cup (S_0, R_n)$ and logarithmically convex in $r \in (S_0, R_n)$ for each fixed $\theta \in [0, \pi]$. What is more, for $r \in (S_0, R_n)$,

$$T^*(r, u_{n,\nu}) = N(r, u_{n,\nu}), \qquad T^*(re^{i\pi}, u_{n,\nu}) = T(r, u_{n,\nu}),$$
$$\frac{\partial}{\partial\theta} T^*(re^{i\theta}, u_{n,\nu}) = \frac{\widetilde{u}_{n,\nu}(re^{i\theta})}{\pi} \qquad \text{for } 0 < \theta < \pi,$$

where $T(r, u_{n,\nu})$ is the Nevanlinna characteristic of $u_{n,\nu}(z)$.

For $\alpha(r)$ a real-valued function of a real variable r we consider the operator

$$L\alpha(r) = \liminf_{h \to 0} \frac{\alpha(re^h) + \alpha(re^{-h}) - 2\alpha(r)}{h^2}.$$

If $\alpha(r)$ is twice differentiable in r, then

$$L\alpha(r) = r\,\frac{d}{dr}\, r\,\frac{d}{dr}\alpha(r).$$

As $T^*(re^{i\theta}, u_{n,\nu})$ is a convex function of $\log r$, for $S_0 < r < R_n$ and $\theta \in [0, \pi]$ we have

$$LT^*(re^{i\theta}, u_{n,\nu}) \geq 0.$$

LEMMA 2. *For almost all $\theta \in [0, \pi]$ and almost all $r \in (S_0, R_n)$,*

$$LT^*(re^{i\theta}, u_{n,\nu}) \geq -\frac{1}{\pi} \frac{\partial \widetilde{u}_{n,\nu}(re^{i\theta})}{\partial \theta}.$$

The proof of this lemma can be conducted along the same lines as the proof of Lemma 1 in [11].

We now put [12]

$$T_0^*(z, f) := \sum_{\nu=1}^{k} T^*(z, u_{n,\nu}).$$

It follows from the definition of the operator L and from the logarithmic convexity of each $T^*(z, u_{n,\nu})$ that

$$LT_0^*(z, f) \geq \sum_{\nu=1}^{k} LT^*(z, u_{n,\nu}) \geq 0.$$

Moreover, Lemma 2 implies that

$$LT_0^*(z, f) \geq -\frac{1}{\pi} \sum_{\nu=1}^{k} \frac{\partial \widetilde{u}_{n,\nu}(re^{i\theta})}{\partial \theta}.$$

For $\tau > 0$ we choose numbers α and ψ such that

$$0 < \alpha \leq \min\left(\pi, \frac{\pi}{2\tau}\right), \qquad -\frac{\pi}{2\tau} \leq \psi \leq \frac{\pi}{2\tau} - \alpha.$$

We set

$$\begin{aligned} h_n(r, \tau) := \frac{1}{\pi} \sum_{\nu=1}^{k} \widetilde{u}_{n,\nu}(r) \cos \tau\psi - \frac{1}{\pi} \sum_{\nu=1}^{k} \widetilde{u}_{n,\nu}(re^{i\alpha}) \cos \tau(\alpha + \psi) \\ - \tau \sin \tau(\alpha + \psi) T_0^*(re^{i\alpha}, f) + \tau \sin \tau\psi \sum_{\nu=1}^{k} N(r, u_{n,\nu}). \end{aligned}$$

LEMMA 3. *Let $A = \{r : S_0 < r < R_n,\ h_n(r, \tau) > 0\}$. Then*

$$\tau \int_A \frac{dt}{t} \leq \log T(2R_n, f) + \log\log R_n + O(1) \qquad (n \to \infty).$$

Proof. We put [4, 6]

$$\sigma(r) = \int_0^\alpha T_0^*(re^{i\theta}, f) \cos \tau(\theta + \psi)\, d\theta.$$

As $T_0^*(r,\theta,u_{n\nu})$ is a convex function of $\log r$, applying Fatou's lemma we obtain

$$
\begin{aligned}
(9)\qquad L\sigma(r) &= L\int_0^{\alpha} T_0^*(re^{i\theta},f)\cos\tau(\theta+\psi)\,d\theta \\
&\ge \int_0^{\alpha} LT_0^*(re^{i\theta},f)\cos\tau(\theta+\psi)\,d\theta \ge 0.
\end{aligned}
$$

It follows that $\sigma(r)$ is a convex function of $\log r$, so $r\sigma'_-(r)$ is increasing on (S_0,R_n). Therefore for almost all $r\in(S_0,R_n)$,

$$L\sigma(r) = r\,\frac{d}{dr}\,r\sigma'_-(r),$$

where $\sigma'_-(r)$ is the left derivative of $\sigma(r)$ at r. Lemma 2 and inequality (9) imply that for almost all $r\in(S_0,R_n)$,

$$L\sigma(r) = r\,\frac{d}{dr}\,r\sigma'_-(r) \ge -\frac{1}{\pi}\int_0^{\alpha}\sum_{\nu=1}^{k}\frac{\partial\widetilde{u}_{n,\nu}(re^{i\theta})}{\partial\theta}\cos\tau(\theta+\psi)\,d\theta.$$

If for $r\in(S_0,R_n)$ there are neither zeros nor poles of $f(z)$ on the circle $|z|=r$, the function $u_{n,\nu}(re^{i\theta})$ satisfies the Lipschitz condition in θ. Therefore $\widetilde{u}_{n,\nu}(re^{i\theta})$ also satisfies the Lipschitz condition on $[0,\pi]$ (see [9]). This means that $\widetilde{u}_{n,\nu}(re^{i\theta})$ is absolutely continuous on $[0,\pi]$. Integrating twice by parts we obtain

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^{\alpha}\sum_{\nu=1}^{k}\frac{\partial\widetilde{u}_{n,\nu}(re^{i\theta})}{\partial\theta}\cos\tau(\theta+\psi)\,d\theta \\
&\quad= \frac{1}{\pi}\sum_{\nu=1}^{k}\widetilde{u}_{n,\nu}(re^{i\alpha})\cos\tau(\alpha+\psi) - \frac{1}{\pi}\sum_{\nu=1}^{k}\widetilde{u}_{n,\nu}(r)\cos\tau\psi \\
&\qquad + \tau T_0^*(re^{i\alpha},f)\sin\tau(\alpha+\psi) - \tau\sum_{\nu=1}^{k}N(r,u_{n,\nu})\sin\tau\psi - \tau^2\sigma(r) \\
&\quad= -h_n(r,\tau) - \tau^2\sigma(r).
\end{aligned}
$$

From this, and from the monotonicity of $\widetilde{u}_{n,\nu}(re^{i\theta})$, we have

$$(10)\qquad h_n(r,\tau) + \tau^2\sigma(r) \ge 0 \quad \text{for } r\in(S_0,R_n).$$

This way for almost all $r\in(S_0,R_n)$ we obtain the inequality

$$r\,\frac{d}{dr}\,r\sigma'_-(r) \ge h_n(r,\tau) + \tau^2\sigma(r).$$

We divide this inequality by $r^{\tau+1}$ and integrate by parts over $[r,R_n]$ (see [10]) to obtain

$$\int_r^{R_n} \frac{h_n(t,\tau)}{t^{\tau+1}}\,dt \le \int_r^{R_n} \frac{1}{t^\tau}\,\frac{d}{dt}\,t\sigma'_-(t)\,dt + \tau^2 \int_r^{R_n} \frac{1}{t^{\tau+1}}\,\sigma(t)\,dt$$
$$\le \left(\frac{t\sigma'_-(t)}{t^\tau} + \tau\,\frac{\sigma(t)}{t^\tau}\right)\Bigg|_r^{R_n}, \qquad S_0 \le r \le R_n.$$

We now apply the method of P. Barry [2, 3]. Set

$$\Phi(r) = -\int_r^{R_n} \frac{h_n(t,\tau)}{t^{\tau+1}}\,dt, \qquad S_0 \le r \le R_n.$$

From the above considerations we have

$$\Phi(r) \ge -\frac{\sigma'_-(R_n)}{R_n^{\tau-1}} - \tau\,\frac{\sigma(R_n)}{R_n^\tau} + \frac{\sigma'_-(r)}{r^{\tau-1}} + \tau\,\frac{\sigma(r)}{r^\tau}. \tag{11}$$

We now put

$$\psi(r) = r^\tau\left[\Phi(r) + \frac{\sigma'_-(R_n)}{R_n^{\tau-1}} + \tau\frac{\sigma(R_n)}{R_n^\tau}\right].$$

Thus from (11) we obtain

$$\psi(r) \ge r\sigma'_-(r) + \tau\sigma(r), \qquad S_0 \le r < R_n.$$

By the above and (11) we get

$$r\psi'(r) = \tau\psi(r) + h_n(r,\tau) \ge \tau r\sigma'_-(r) + \tau^2\sigma(r) + h_n(r,\tau)$$
$$\ge \tau r\sigma'_-(r) \ge 0.$$

The function $T_0^*(re^{i\theta}, f)$ is increasing for $r \in (S_0, R_n)$ and hence $\sigma(r)$ is increasing on (S_0, R_n). Therefore $r\sigma'_-(r) \ge 0$ for all $r \in (S_0, R_n)$. Moreover, $\sigma(r) > 0$ for all $r \in (S_0, R_n)$. This way we have

$$\psi(r) \ge r\sigma'_-(r) + \tau\sigma(r) > 0.$$

If $r \in A$ then $r\psi'(r) > \tau\psi(r) > 0$. Therefore $\psi'(r)/\psi(r) > \tau/r$. Consequently,

$$\tau \int_{A\cap[1,R_n]} \frac{dr}{r} \le \int_{A\cap[1,R_n]} \frac{\psi'(r)}{\psi(r)}\,dr \le \int_{S_0}^{R_n} \frac{\psi'(r)}{\psi(r)}\,dr = \log\frac{\psi(R_n)}{\psi(S_0)}. \tag{12}$$

But $\psi(R_n) = R_n\sigma'_-(R_n) + \tau\sigma(R_n)$. The definition of $\sigma(r)$ implies that

$$\sigma(r) = \int_0^\alpha T_0^*(re^{i\theta}, f)\cos\tau(\theta+\psi)\,d\theta \le \int_0^\alpha T_0^*(re^{i\theta}, f)\,d\theta.$$

What is more,

$$\begin{aligned} T_0^*(z,f) = T_0^*(re^{i\theta},f) = \sum_{\nu=1}^{k} T^*(z,u_{n,\nu}) &= \sum_{\nu=1}^{k}(m^*(z,u_{n,\nu}) + N(r,u_{n,\nu})) \\ &\le T(r,f^{(2)}) + k(\lambda+4)\log R_n \\ &\le 2T(r,f) + k(\lambda+4)\log R_n. \end{aligned}$$

Therefore we have

$$\sigma(r) \le 2\pi T(r,f) + \pi k(\lambda+4)\log R_n.$$

From the monotonicity of $r\sigma'_-(r)$ we get

$$r\sigma'_-(r) \le \int_r^{2r} \sigma'_-(t)\,dt \le \sigma(2r) \le 2\pi T(2r,f) + \pi k(\lambda+4)\log R_n.$$

This way from (12) we obtain

$$\begin{aligned} \tau \int_{A\cap[1,R_n]} \frac{dr}{r} \le \log\frac{\psi(R_n)}{\psi(S_0)} &\le \log\psi(R_n) + O(1) \\ &= \log[R_n\sigma'_-(R_n) + \tau\sigma(R_n)] + O(1) \\ &\le \log T(2R_n,f) + \log\log R_n + O(1), \qquad n\to\infty, \end{aligned}$$

and the proof of Lemma 3 is complete.

2. Proof of Theorem 1. We show the estimate for the upper logarithmic density of $E(\gamma)$. The proof for the lower logarithmic density is similar, with R_n replaced by any positive number R, and the lower order λ replaced by the order ϱ.

We start with the following sum:

$$\begin{aligned} \sum_{\nu=1}^{k} \mathcal{L}(r,p_\nu,f) &= \sum_{\nu=1}^{k} \log^+ \max_{|z|=r} \frac{1}{|f(z)-p_\nu(z)|} \\ &= \sum_{\nu=1}^{k} \log^+ \frac{1}{|f(re^{i\theta_\nu})-p_\nu(re^{i\theta_\nu})|}, \end{aligned}$$

where $r\in(S_0,R_n)$ and p_ν are polynomials with $\deg(p_\nu)\le 1$ for $1\le\nu\le k$.

If $|f(re^{i\theta_\nu})-p_\nu(re^{i\theta_\nu})| \ge c/4$ then

$$\log^+ \frac{1}{|f(re^{i\theta_\nu})-p_\nu(re^{i\theta_\nu})|} \le \log^+\frac{4}{c}.$$

Let now $|f(re^{i\theta_\nu})-p_\nu(re^{i\theta_\nu})| < c/4$. Then we have

$$\log^+ \frac{1}{|(f-p_\nu)(re^{i\theta_\nu})|} \le \log^+\left|\frac{(f-p_\nu)^{(2)}(re^{i\theta_\nu})}{(f-p_\nu)(re^{i\theta_\nu})}\right| + \log^+\left|\frac{1}{f^{(2)}(re^{i\theta_\nu})}\right|,$$

or

$$\log^+ \frac{1}{|(f-p_\nu)(re^{i\theta_\nu})|} \le \log^+ \left|\frac{(f-p_\nu)^{(1)}(re^{i\theta_\nu})}{(f-p_\nu)(re^{i\theta_\nu})}\right| + \log^+ \left|\frac{1}{(f-p_\nu)^{(1)}(re^{i\theta_\nu})}\right|.$$

If $|f^{(2)}(re^{i\theta_\nu})| \ge 1/R_n^{\lambda+4}$, we obtain

$$\log^+ \frac{1}{|(f-p_\nu)(re^{i\theta_\nu})|} \le \log^+ \left|\frac{f^{(2)}(re^{i\theta_\nu})}{(f-p_\nu)(re^{i\theta_\nu})}\right| + (\lambda+4)\log R_n.$$

If, on the other hand, $|f^{(2)}(re^{i\theta_\nu})| < 1/R_n^{\lambda+4}$, we have either (for $|(f-p_\nu)^{(1)}(re^{i\theta_\nu})| \ge 1/R_n^{\lambda+4}$)

$$\log^+ \frac{1}{|(f-p_\nu)(re^{i\theta_\nu})|} \le \log^+ \left|\frac{(f-p_\nu)^{(1)}(re^{i\theta_\nu})}{(f-p_\nu)(re^{i\theta_\nu})}\right| + (\lambda+4)\log R_n,$$

or (if $|(f-p_\nu)^{(1)}(re^{i\theta_\nu})| < 1/R_n^{\lambda+4}$)

$$\log^+ \left|\frac{1}{f^{(2)}(re^{i\theta_\nu})}\right| \le \widetilde{u}_{n,\nu}(r).$$

It follows from the inequalities above that in general for $r \in (S_0, R_n)$ and $1 \le \nu \le k$ we have

$$\begin{aligned}\log^+ \frac{1}{|(f-p_\nu)(re^{i\theta_\nu})|} \le \widetilde{u}_{n,\nu}(r) &+ \log^+ M\left(r, \frac{(f-p_\nu)^{(1)}}{f-p_\nu}\right)\\ &+ \log^+ M\left(r, \frac{(f-p_\nu)^{(2)}}{f-p_\nu}\right) + (\lambda+4)\log R_n + \log^+ \frac{4}{c}.\end{aligned}$$

This way we obtain

$$\begin{aligned}\sum_{\nu=1}^{k} \mathcal{L}(r, p_\nu, f) \le \sum_{\nu=1}^{k} \widetilde{u}_{n,\nu}(r) &+ \sum_{\nu=1}^{k} \log^+ M\left(r, \frac{(f-p_\nu)^{(1)}}{f-p_\nu}\right)\\ &+ \sum_{\nu=1}^{k} \log^+ M\left(r, \frac{(f-p_\nu)^{(2)}}{f-p_\nu}\right) + k(\lambda+4)\log R_n + k\log^+ \frac{4}{c}.\end{aligned} \tag{13}$$

Now, in the case $\gamma \le \lambda$ the assertion of the theorem is obvious, so let $\gamma > \lambda$. We choose $\lambda < \tau < \gamma$ and set $\alpha = \min(\pi, \pi/2\tau)$ and $\psi = \pi/2\tau - \alpha$. Then we have

$$\begin{aligned}h_n(r,\tau) &= \frac{\sin\tau\alpha}{\pi}\left[\sum_{\nu=1}^{k} \widetilde{u}_{n,\nu}(r) - \frac{\pi\tau}{\sin\tau\alpha}\left(T_0^*(re^{i\alpha}) - \cos\tau\alpha \sum_{\nu=1}^{k} N(r, u_{n,\nu})\right)\right]\\ &\ge \frac{\sin\tau\alpha}{\pi}\left[\sum_{\nu=1}^{k} \widetilde{u}_{n,\nu}(r) - \frac{\pi\tau}{\sin\tau\alpha} T_0^*(re^{i\alpha})\right]\\ &\ge \frac{\sin\tau\alpha}{\pi}\left[\sum_{\nu=1}^{k} \widetilde{u}_{n,\nu}(r) - B(\tau)T(r, f^{(2)}) - B(\tau)\pi k(\lambda+4)\log R_n\right].\end{aligned}$$

We now consider the set

$$A_1=\Big\{r\in(S_0,R_n):\sum_{\nu=1}^{k}\widetilde{u}_{n,\nu}(r)-B(\tau)T(r,f^{(2)})-B(\tau)\pi k(\lambda+4)\log R_n>0\Big\}.$$

If $r\in A_1$, then $h_n(r,\tau)>0$ and by Lemma 3,

$$\tau\int\limits_{A_1\cap[1,R_n]}\frac{dt}{t}\le\log T(2R_n,f)+\log\log R_n+O(1).$$

If $r\notin A_1$, then

$$\sum_{\nu=1}^{k}\widetilde{u}_{n,\nu}(r)\le B(\tau)T(r,f^{(2)})+B(\tau)\pi k(\lambda+4)\log R_n.$$

Thus for $r\in[S_0,R_n]\setminus A_1$ from (13) we get

$$\sum_{\nu=1}^{k}\mathcal{L}(r,p_\nu,f)\le B(\tau)T(r,f^{(2)})+(B(\tau)\pi+1)k(\lambda+4)\log R_n$$
$$+\sum_{\nu=1}^{k}\log^+M\bigg(r,\frac{(f-p_\nu)^{(1)}}{f-p_\nu}\bigg)+\sum_{\nu=1}^{k}\log^+M\bigg(r,\frac{(f-p_\nu)^{(2)}}{f-p_\nu}\bigg)+k\log^+\frac{4}{c}.$$

Applying Lemma 1 we find that, except possibly for r in a set of finite linear measure, for $r\in[S_0,R_n]\setminus A_1$,

$$\sum_{\nu=1}^{k}\mathcal{L}(r,p_\nu,f)\le B(\tau)T(r,f)+o(T(r,f))+O(\log R_n).$$

Choose $\varepsilon(R)\to 0$ such that $T(R^{\varepsilon(R)},f)/\log R\to\infty$ as $R\to\infty$, and put $S_n=R_n^{\varepsilon(R_n)}$, where $\{R_n\}$ is the sequence from (4). Let also $r\in[S_n,R_n]$. Then from the definition of S_n we get

$$T(r,f)\ge T(S_n,f)=\log R_n\,\frac{T(R_n^{\varepsilon(R_n)},f)}{\log R_n}$$

for $r\in[S_n,R_n]$, which implies that $\log R_n=o(T(r,f))$ $(n\to\infty)$.

Therefore for $r\in[S_n,R_n]\setminus A_1$, except possibly for r in a set of finite linear measure, we have

$$\sum_{\nu=1}^{k}\mathcal{L}(r,p_\nu,f)\le(B(\tau)+o(1))T(r,f)<B(\gamma)T(r,f)\qquad(n\to\infty).$$

This, together with (13), leads to the estimate

$$\tau\int\limits_{E(\gamma)\cap[1,R_n]}\frac{dt}{t}\ge\tau\int\limits_{E(\gamma)\cap[S_n,R_n]}\frac{dt}{t}\ge\tau\int\limits_{[S_n,R_n]\setminus A_1}\frac{dt}{t}+O(1)$$
$$\ge\tau(1-\varepsilon(R_n))\log R_n-\log T(3R_n,f)-\log\log R_n+O(1)$$

as $n \to \infty$. We divide this inequality by $\log R_n$:

$$\frac{1}{\log R_n} \int\limits_{E(\gamma)\cap[S_0,R_n]} \frac{dt}{t} \geq (1-\varepsilon(R_n)) - \frac{\log T(3R_n,f)}{\tau \log R_n} - \frac{\log\log R_n + O(1)}{\tau \log R_n}.$$

From the definition of $\{R_n\}$ we obtain, for $\tau < \gamma$,

$$\overline{\operatorname{logdens}}\, E(\gamma) \geq 1 - \lambda/\tau.$$

Letting $\tau \to \gamma$ we get

$$\overline{\operatorname{logdens}}\, E(\gamma) \geq 1 - \lambda/\gamma.$$

Thus we obtain the statement for polynomials with $\deg(p_\nu) \leq 1$.

This result can be extended to polynomials of higher degrees the following way. Let $\{p_\nu\}_{\nu=1}^k$ be a set of distinct polynomials such that $\deg(p_\nu) \leq d$ for $1 \leq \nu \leq k$, and $d \geq 1$. We define the numbers S_0, c and the sequences $\{R_n\}$, $\{S_n\}$ as before. For $n \geq n_0$ we consider the set

$$G_n = \left\{z : S_0 < |z| < R_n,\ |f^{(d+1)}(z)| < R_n^{-[\frac{d+1}{2}(\lambda+1)+d+2]}\right\},$$

where n_0 is chosen in such a way that for $n \geq n_0$ we have

$$T(3R_n, f) < R_n^{\lambda+1} \quad \text{and} \quad \frac{[(2d+2)^{d+1} + (2d+2)^d]\pi^{d+1}}{R_n} < \frac{c}{4}.$$

Now for $1 \leq \nu \leq k$ we write $G_{n,\nu}$ for the set of those connected components of G_n which contain a point z_1 such that

$$|f(z_1) - p_\nu(z_1)| < c/4$$

and points $z_2, \ldots, z_{d+1}$ such that for $j = 2, \ldots, d+1$,

$$|f^{(j-1)}(z_j) - p_\nu^{(j-1)}(z_j)| < R_n^{-[\frac{d+1}{2}(\lambda+1)+d+2]}.$$

The fact that the sets $G_{n,\nu}$ and $G_{n,\eta}$ are disjoint for $\nu \neq \eta$ can be shown in a similar way to the case of polynomials of degree $\deg(p_\nu) \leq 1$. Thus for $1 \leq \nu \leq k$ and $n \geq n_0$ we may consider the functions

$$u_{n,\nu}(z) := \begin{cases} \log \dfrac{1}{|f^{(d+1)}(z)|}, & z \in G_{n,\nu}, \\ \left[\dfrac{d+1}{2}(\lambda+1) + d + 2\right] \log R_n, & z \notin G_{n,\nu}. \end{cases} \tag{14}$$

Lemmas 2 and 3 also hold for the functions $u_{n,\nu}(z)$ defined in (14). The rest of the proof of Theorem 1 can be done analogously to the case when $\deg(p_\nu) \leq 1$.

Let now $\{q_\nu(z)\}_{\nu=1}^k$ be a set of distinct rational functions and let $f(z)$ be a transcendental entire function of finite lower order. We choose a polynomial $p(z)$ such that for $1 \leq \nu \leq k$ each $p_\nu(z) := p(z) \cdot q_\nu(z)$ is a polynomial. We also set $F(z) := p(z) \cdot f(z)$.

As in the previous case, for $r \in (S_0, R_n)$ we consider the sum

$$\sum_{\nu=1}^{k} \mathcal{L}(r, q_\nu, f) = \sum_{\nu=1}^{k} \log^+ \max_{|z|=r} \frac{1}{|f(z) - q_\nu(z)|}.$$

We have

$$\begin{aligned}\sum_{\nu=1}^{k} \log^+ \max_{|z|=r} \frac{1}{|f(z) - q_\nu(z)|} &= \sum_{\nu=1}^{k} \log^+ \max_{|z|=r} \frac{p(z)}{|F(z) - p_\nu(z)|} \\ &\leq \sum_{\nu=1}^{k} \log^+ \max_{|z|=r} \frac{1}{|F(z) - p_\nu(z)|} \\ &\quad + k \log^+ \max_{|z|=r} |p(z)|.\end{aligned}$$

We obtain

$$\sum_{\nu=1}^{k} \mathcal{L}(r, q_\nu, f) \leq \sum_{\nu=1}^{k} \mathcal{L}(r, p_\nu, F) + o(T(r, f)) \quad (r \to \infty).$$

As we have already obtained the statement for polynomials, we may apply all the previous estimates to the transcendental entire function $F(z)$ and distinct polynomials $p_\nu(z)$ and thus complete the proof of Theorem 1.

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