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Cegrell classes on compact Kähler manifolds

by Sławomir Dinew (Kraków)

Abstract. We study Cegrell classes on compact Kähler manifolds. Our results generalize some theorems of Guedj and Zeriahi (from the setting of surfaces to arbitrary manifolds) and answer some open questions posed by them.

1. Introduction. Since the cornerstone results of Bedford and Taylor ([BT1] and [BT2]) pluripotential theory in domains of $\mathbb{C}^n$ has become a subject of very intensive studies.

Recently, in [K1], [GZ1] and [GZ2] pluripotential theory in the setting of compact Kähler manifolds has been developed. Such a theory has interesting applications in complex dynamics, differential and algebraic geometry and also in problems in "flat" theory (by flat we mean pluripotential theory in hyperconvex domains in $\mathbb{C}^n$). We refer to [GZ1], where some interactions between plurisubharmonic functions in $\mathbb{C}^n$ with logarithmic growth and the $\mathrm{PSH}(\mathbb{P}^n, \omega_{\mathrm{FS}})$ functions on the complex projective space $\mathbb{P}^n$ equipped with the Fubini–Study metric ω_{FS} are shown. In [GZ2] the authors defined the Monge–Ampère operator and proved various results concerning it. They claimed that their results still hold in arbitrary dimension, but they restricted themselves to the surface case, since the definition is much simpler in that case.

Here in Section 2 we give the general definition of the Monge–Ampère operator on a compact Kähler manifold and specify its domain of definition. Next we introduce Cegrell classes of $\mathrm{PSH}(X, \omega)$ functions and prove several properties generalizing some results in [GZ2]. Some proofs from this section rely heavily on their flat analogues (Propositions 2.1 and 2.9, Theorem 2.4). We shall also often refer to [GZ2] when an "n-dimensional" result follows directly from the surface case, and shall focus only on those points where analogies are less clear.

The results from this section will be used to prove our main theorem.

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Theorem 1.1 (Main result). *Every* $\mathrm{PSH}(X,\omega)$ *function* ϕ *with bounded* p*-energy is a limit of a decreasing sequence of functions* ϕ_j *which belong to* $L^\infty(X)\cap \mathrm{PSH}(X,\omega)$ *and whose* p*-energies tend to the* p*-energy of* ϕ *(Section* 3*).*

Note that an analogous result is true in flat theory (see [Ce1]), but the proof relies on several rather nontrivial results (i.e. Cegrell decomposition, existence results for the Dirichlet problem and a "global" comparison principle). That proof cannot be repeated in the Kähler manifold setting, mainly because there is no analogue of the global comparison principle (since all Monge–Ampère measures are probability measures in the Kähler case). Our proof, however, can be applied in both situations, so that as a byproduct we obtain a different proof of this result in the flat case (more technical but not requiring heavy machinery).

In Section 4 we generalize the (local) comparison principle from [K1] to Cegrell classes.

We refer to [GZ1] and [GZ2] for all notions used in this paper. More background in pluripotential theory can be found in [Kli], [K2].

2. Definitions. Let X be a compact n-dimensional Kähler manifold equipped with a fundamental Kähler form ω given in local coordinates by

$$\omega=\frac{i}{2}\sum_{k,j=1}^{n} g_{k\overline{j}}dz^k\wedge d\overline{z}^j.$$

We assume that the metric is normalized so that

$$\int_X \omega^n=1.$$

Recall that

$$\mathrm{PSH}(X,\omega):=\{\phi\in L^1(X,\omega): dd^c\phi\geq -\omega,\ \phi\in \mathcal{C}^{\uparrow}(X)\}$$

where as usual $d=\partial+\overline{\partial}$, $d^c=\frac{i}{2\pi}(\overline{\partial}-\partial)$ and $\mathcal{C}^{\uparrow}(X)$ denotes the space of upper semicontinuous functions. We call the functions that belong to $\mathrm{PSH}(X,\omega)$ *ω-plurisubharmonic* (ω-psh for short).

Throughout the paper we shall assume that all the functions ϕ we consider satisfy the extra condition

$$\sup_X \phi\leq -1. \tag{2.1}$$

This condition is not restrictive, since if we add a constant to a function in a certain Cegrell class, the new function also belongs to that class. Nevertheless, (2.1) will often be very helpful for our purposes.

We would like to define the Monge–Ampère operator

$$(\omega_u)^n:=(\omega+dd^c u)^n$$

acting on ω-psh functions. It is well known that this cannot be done for all ω-psh functions (for counterexamples in dimension 2 see [GZ2]). Therefore one should restrict oneself to a smaller class of ω-psh functions. We denote the maximal class of ω-psh functions for which the Monge–Ampère operator is well defined by $\mathcal{E}(X,\omega)$ or just by $\mathcal{E}$ for simplicity. In the flat theory we have a complete description of this class due to Błocki (see [Bl]). Using his ideas one can also describe the class $\mathcal{E}$ on Kähler manifolds.

Let us first recall some constructions for bounded ω-psh functions:

Proposition 2.1. *Let u be a bounded ω-psh function. Then one can define the (positive) currents*

$$\omega_u^k := \underbrace{(\omega + dd^c u) \wedge \cdots \wedge (\omega + dd^c u)}_{k\,times}, \qquad k = 1, \ldots, n.$$

Moreover ω_u^n is a probability measure for every bounded ω-psh function u.

Proof. It is enough to define the currents locally, i.e. in coordinate charts where we have a continuous potential for ω (a function v such that $dd^c v = \omega$). But then $u+v$ is simply a plurisubharmonic function. Hence one can use the classical results from [BT2] to define our currents. Note that the definition is coherent in an intersection of two charts.

The last assertion of the proposition follows from the fact that we can decompose

$$\omega_u^n = \omega^n + \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} (dd^c u)^k \wedge \omega^{n-k}$$

where the latter term happens to be a closed current. ■

For more details we refer to [GZ1] and [GZ2] (the latter in the case $n = 2$).

A natural question is under what kind of convergence this operator is continuous. To study continuity results one can define the capacity cap_ω by setting

$$\mathrm{cap}_\omega(A) := \sup\Big\{ \int_A \omega_u^n : u \in \mathrm{PSH}(X,\omega),\, 0 \le u \le 1\Big\}$$

where A is an arbitrary Borel subset of X (for more details see [K1]).

Recall that a sequence u_j *converges to u with respect to capacity* if

$$\forall t > 0 \qquad \lim_{j\to\infty} \mathrm{cap}_\omega(\{|u_j - u| > t\}) = 0.$$

Proposition 2.2. *The Monge–Ampère operator defined above is continuous on decreasing sequences in* $\mathrm{PSH}(X,\omega) \cap L^\infty(X)$*. It is also continuous with respect to convergence in capacity* cap_ω.

Proof. See [K2]. ■

Having in mind these results one can ask whether it is possible to define the Monge–Ampère operator also for unbounded functions. We would of course like to keep its basic properties, i.e. continuity on decreasing sequences. So we make the following definition:

Definition 2.3. Let $u \in \mathrm{PSH}(X, \omega)$. If for every sequence of ω-psh functions $u_j \in \mathrm{PSH}(U, \omega) \cap L^\infty(U)$ decreasing to u on some open subset U of X the associated sequence $\omega_{u_j}^n$ is weakly convergent (on U) and the limit measure $M(u)$ is independent of the sequence, we define $\omega_u^n := M(u)$ on the set U where the convergence holds.

Here $\mathrm{PSH}(U, \omega)$ denotes the set of germs of ω-psh functions defined on U. The definition is coherent on intersections. The class of functions u as in Definition 2.3 is the maximal class of ω-psh functions for which one can define the Monge–Ampère operator, which we have denoted by $\mathcal{E}$.

Remark. Of course one can use a sequence that converges to u everywhere on X. We choose the local definition not only in order to use connections with flat theory, but also because global approximation e.g. with smooth ω-psh functions is a very delicate matter and often requires restrictions on the form ω and the underlying manifold. In the local context we can approximate easily, using for example convolutions with smooth kernel.

Of course Definition 2.3 is of very small practical use (we have to check all convergent sequences). The following result makes this definition more manageable:

Theorem 2.4. *Let $u \in \mathrm{PSH}(X, \omega)$. The following conditions are equivalent*:

1. $u \in \mathcal{E}$.
2. *For every $x \in X$ there exists a neighbourhood U_x such that for every sequence $u_j \in \mathrm{PSH}(U_x, \omega) \cap L^\infty$ with $u_j \searrow u$ the sequences*
$$(-u_j)^{n-p-2} du_j \wedge d^c u_j \wedge (\omega_{u_j})^p \wedge \omega^{n-p-1}, \quad p \in \{0, \ldots, n-2\},$$
are weakly bounded.
3. *For every $x \in X$ there exists a neighbourhood U_x such that there exists a sequence $u_j \in \mathrm{PSH}(U_x, \omega) \cap L^\infty$ with $u_j \searrow u$ such that the sequences*
$$(-u_j)^{n-p-2} du_j \wedge d^c u_j \wedge (\omega_{u_j})^p \wedge \omega^{n-p-1}, \quad p \in \{0, \ldots, n-2\},$$
are weakly bounded.

Proof. Since the result is local, one can use the argument from Proposition 1.1 once more. If v is a local potential then $u_j + v \searrow u + v$. Now the result follows from Błocki's theorem in the flat case (see [Bl]). ∎

Using analogous arguments one can prove that most local results from the flat theory remain true in the Kähler manifold setting. In particular we have the following corollary:

COROLLARY 2.5. *If* $u \in \mathcal{E}$ *and* $w \in \mathrm{PSH}(X, \omega)$ *with* $u \le w$, *then* $w \in \mathcal{E}$.

Now following [Ce1] and [GZ1] we are ready to introduce the so called Cegrell classes:

DEFINITION 2.6. Let $\mathcal{E}^p$ denote the class of $u \in \mathrm{PSH}(X, \omega)$ such that there exist $u_j \in \mathrm{PSH}(X, \omega) \cap L^\infty(X, \omega)$ with $u_j \searrow u$ such that

$$\sup_j \int_X (-u_j)^p \omega_{u_j}^n < \infty$$

(in this paper for simplicity we assume $p \ge 1$, although the definition makes sense for every $p > 0$).

REMARK. In the flat setting one also considers classes with the additional property

$$\sup_j \int (dd^c u)^n < \infty.$$

This is obviously satisfied in our setting since all integrals $\int_X (\omega_{u_j})^n$ are equal to 1.

Of course bounded functions belong to $\mathcal{E}^p$, but $\mathcal{E}^p$ contains many unbounded functions. These, however, cannot be very singular, which will follow from the results below (most of them are generalizations of the analogous results in [GZ2]).

PROPOSITION 2.7. *Let* $u \in \mathcal{E}^p$. *Then* $u + c \in \mathcal{E}^p$ *for any constant* c.

REMARK. When we do not a priori assume that all functions considered are negative, the condition in the definition of Cegrell classes has to be modified slightly (as in [GZ2]), namely instead of $(-u_j)^p$ we integrate $|u_j|^p$.

Proof of Proposition 2.7. This rather simple observation justifies our initial assumption (2.1). Indeed, if u_j is a sequence from the definition of $\mathcal{E}^p$ (for a function u), then $u_j + c$ is such a sequence for $u + c$. One just has to use the Minkowski inequality, which is justified, since ω_{u_j+c} are all positive measures. ■

PROPOSITION 2.8. *If* $u_1, \dots, u_n \in \mathcal{E}$ *then* $\omega_{u_1} \wedge \cdots \wedge \omega_{u_n}$ *is a well defined probability measure.*

Proof. Basically we repeat the arguments from Proposition 1.3 of [GZ2]. It suffices to define the (positive) current

$$-u_k \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-1}}.$$

Since we can write

$$\omega_{u_1} \wedge \cdots \wedge \omega_{u_k} := \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-1}} \wedge \omega + dd^c(u_k \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-1}})$$

it is enough to check that

$$\int_X -u_k \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-1}} \wedge \omega^{n-(k-1)} < \infty.$$

When $k = 1$ this of course holds for all $u \in \mathrm{PSH}(X, \omega)$.

To check that $-u_2 \omega_{u_1}$ is well defined, it is enough to use the same calculations as in [GZ2]. One has to check that

$$\int_X du_i \wedge d^c u_i \wedge \omega^{n-1} < \infty, \qquad i = 1, 2,$$

Observe that $\int_X du_i \wedge d^c u_i \wedge \omega^{n-1} \leq \int_X (-u_i)^{n-2} du_i \wedge d^c u_i \wedge \omega^{n-1} < \infty$, where we have used the definition for the second inequality and the condition (2.1) for the first. Now we can proceed by induction. Indeed, using again an idea from [GZ2] we have

$$\begin{aligned}
\int_X -u_k \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-1}} \wedge \omega^{n-(k-1)} &= \int_X -u_k \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-2}} \wedge \omega^{n-(k-2)} \\
&\quad + \int_X du_k \wedge d^c u_{k-1} \wedge \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-2}} \wedge \omega^{n-(k-1)} \\
&\leq \int_X -u_k \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-2}} \wedge \omega^{n-(k-2)} \\
&\quad + \Big(\int_X du_k \wedge d^c u_k \wedge \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-2}} \wedge \omega^{n-k+1} \Big)^{1/2} \\
&\quad \times \Big(\int_X du_{k-1} \wedge d^c u_{k-1} \wedge \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-2}} \wedge \omega^{n-k+1} \Big)^{1/2}.
\end{aligned}$$

The first integral on the right hand side is bounded by induction (we have $k-1$ functions). For each of the integrals in the product we proceed in the following way:

$$\begin{aligned}
\int_X du_k \wedge d^c u_k \wedge \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-2}} \wedge \omega^{n-(k-1)} \\
&= \int_X -u_k dd^c u_k \wedge \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-2}} \wedge \omega^{n-(k-1)} \\
&= \int_X -u_k \omega_{u_k} \wedge \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-2}} \wedge \omega^{n-(k-1)} \\
&\quad + \int_X u_k \omega_{u_1} \wedge \cdots \wedge \omega_{u_{k-2}} \wedge \omega^{n-(k-2)}
\end{aligned}$$

since we add and subtract a bounded integral. We notice that we have got rid of u_{k-1} in the first term and of u_k in the second. This implies that integrals involving k distinct functions are controlled by integrals with $k-1$ functions (the integrated function also appears in the wedge product). Our goal will be to estimate these integrals by integrals where the integrated function appears in the wedge product at least twice and so on. In the end we get integrals of the type

$$\int_X -u_j\omega_{u_j}^k \wedge \omega^{n-k},$$

which are finite by definition.

Before we proceed further we make slight adjustments. Instead of the functions $u_1, \dots, u_k$ we take their bounded approximants, namely $u_{s,j_s} := \max\{u_s, -j_s\}$. We do so in order to ensure that all integrals are finite and then we show that the estimates are uniform. We drop the indices j_s in what follows. Using the same idea as before we get

$$\begin{aligned}
&\int_X -u_k\omega_{u_k} \wedge \omega_{u_1} \wedge \dots \wedge \omega_{u_{k-2}} \wedge \omega^{n-(k-1)} \\
&\qquad \le \Big[\int_X -u_k\omega_{u_k}^2 \wedge \omega_{u_1} \wedge \dots \wedge \omega_{u_{k-3}} \wedge \omega^{n-(k-1)} \\
&\qquad\quad + \int_X u_k\omega_{u_k} \wedge \omega_{u_1} \wedge \dots \wedge \omega_{u_{k-3}} \wedge \omega^{n-(k-2)}\Big]^{1/2} \\
&\qquad\quad \times \Big[\int_X -u_{k-2}\omega_{u_{k-2}} \wedge \omega_{u_k} \wedge \omega_{u_1} \wedge \dots \wedge \omega_{u_{k-3}} \wedge \omega^{n-(k-1)} \\
&\qquad\quad + \int_X u_{k-2}\omega_{u_k} \wedge \omega_{u_1} \wedge \dots \wedge \omega_{u_{k-3}} \wedge \omega^{n-(k-2)}\Big]^{1/2} \\
&\qquad\quad + \int_X -u_k\omega_{u_k} \wedge \omega_{u_1} \wedge \dots \wedge \omega_{u_{k-3}} \wedge \omega^{n-(k-2)}.
\end{aligned}$$

Now if we denote by $M_l = \max\{\int_X -u_{i_1}\omega_{u_{i_1}}^l \wedge \omega_{u_{i_2}} \wedge \dots \wedge \omega_{u_{i_{k-l}}} \wedge \omega^{n-(k-1)} : i_j \in \{1, \dots, k\},\ i_1 \neq i_j,\ j \neq 1\}$ the maximum over all integrals such that the function we integrate appears in the wedge product l times, the last inequality can be read as

$$M_1 \le C + (M_1 + C)^{1/2}(M_2 + C)^{1/2}$$

for some constant C independent of M_1 and M_2. If M_1 and M_2 are finite (this is the point where we need the approximants!), we get uniform control of M_1 in terms of M_2. Proceeding analogously we get M_l controlled by M_{l+1}. Putting these results together we conclude that the initial integral is bounded, thus proving our claim. ∎

We note that using the above argument one can get the following proposition:

PROPOSITION 2.9. *If* $u_1, \ldots, u_n \in \mathcal{E}^1$, *then* $\mathcal{E}^1 \subset L^1(X, \omega_{u_1} \wedge \cdots \wedge \omega_{u_n})$.

Proof (cf. [GZ2, Proposition 3.2]). Indeed, from Proposition 2.8 it follows that $\mathcal{E}$ and $\mathcal{E}^1$ are convex sets (one can decompose $\omega_{(u+v)/2}$ into $\frac{1}{2}(\omega_u + \omega_v)$ in each term and use the results for mixed terms). But then for every $u_{n+1} \in \mathcal{E}^1$,

$$\int_X -u_{n+1}\omega_{u_1} \wedge \cdots \wedge \omega_{u_n} \le (n+1)^{n+1} \int_X -\frac{u_1 + \cdots + u_{n+1}}{n+1}\, \omega^n_{\frac{u_1+\cdots+u_{n+1}}{n+1}},$$

which is finite. ∎

PROPOSITION 2.10. *Let* $u \in \mathrm{PSH}(X, \omega)$. *Then* $-(-u)^\varepsilon \in \mathcal{E}^p$ *for small* $\varepsilon > 0$.

Proof. An elementary computation shows that $-(-u)^\varepsilon \in \mathrm{PSH}(X, \omega)$ for $\varepsilon < 1$ (we use the initial condition (2.1)!). Now the result follows from its flat analogue (see [Ce3]). Indeed, $-(-u - v)^\varepsilon \in \mathrm{PSH}$ (v is as usual a local potential for ω). Now $-(-u-v)^\varepsilon \le -(-u)^\varepsilon + v$ if we take a negative potential (which is possible, since we can add a constant to the potential), and the result follows from the stability of $\mathcal{E}^p$ under taking maximums. ∎

COROLLARY 2.11. *The measures*

$$\omega_{u_1} \wedge \cdots \wedge \omega_{u_n}$$

put no mass on pluripolar sets for $u_1, \ldots, u_n \in \mathcal{E}^1$.

Proof. Indeed, Propositions 2.9 and 2.10 tell us that $-(-u)^\varepsilon$ is integrable with respect to such a measure for small positive ε. But any pluripolar set is contained in $\{u = -\infty\}$ for some $u \in \mathrm{PSH}(X, \omega)$ (see [GZ1]), hence the measure cannot put any mass on that set. ∎

REMARK. A similar result also holds for the measures

$$du_1 \wedge d^c u_1 \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_n}.$$

3. Main result. In [GZ2] the authors posed the following

PROBLEM. *Let* ϕ *be a* $\mathrm{PSH}(X, \omega)$ *function such that* $\int_X (-\phi)^p \omega_\phi^n$ *is finite* (*for some* $p \ge 1$). *Does there exist a sequence* $\phi_j \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ *decreasing to* ϕ *such that* $\sup_j \int_X (-\phi_j)^p \omega_{\phi_j}^n < \infty$?

We prove that this is the case. We show moreover that we can choose ϕ_j's in such a way that $\lim_{j\to\infty} \int_X (-\phi_j)^p \omega_{\phi_j}^n = \int_X (-\phi)^p \omega_\phi^n$.

REMARK. As in the surface case, the problem is when $p > 1$. For $p = 1$ calculations similar to those in [GZ2] (cf. also Theorem 3.2 below) give us the result. We mention this, because using (2.1) and L^p integrability we get

L^1 integrability. This yields $\phi \in \mathcal{E}^1$ and we can use all the machinery needed from Section 2.

We start with a technical lemma that gives us the main tool for our later study, "integration by parts".

LEMMA 3.1. *Let ϕ be as before and $\phi_j := \max\{\phi, -j\}$. Then for all currents T of the form $\omega_{\phi_j}^m \wedge \omega^s \wedge \omega_\phi^{n-1-m-s}$:*

(a$_m$) *the numbers $\int_X (-\phi)^p \omega_\phi \wedge T$, $\int_X (-\phi)^p \omega \wedge T$ and $\int_X (-\phi)^{p-1} d\phi \wedge d^c\phi \wedge T$ are bounded by a constant independent of j,*

(b$_m$) $\int_X (-\phi)^p dd^c\phi_j \wedge T = \int_X \phi_j dd^c(-\phi)^p \wedge T$.

Proof. We shall use induction on m. We shall prove (a$_0$) and (a$_m$)$\Rightarrow$(b$_m$), (a$_m$)&(b$_m$)$\Rightarrow$(a$_{m+1}$).

Proof of (a$_0$). From the Stokes theorem we have

$$\int_X (-\phi)^p(\omega_\phi - \omega) \wedge T = p \int_X (-\phi)^{p-1} d\phi \wedge d^c\phi \wedge T \geq 0.$$

So it is enough to check the boundedness of the numbers $\int_X (-\phi)^p \omega_\phi \wedge T$. In this special case these numbers are independent of j. Using the above inequality we have

$$\int_X (-\phi)^p \omega^k \wedge \omega_\phi^{n-k} \leq \cdots \leq \int_X (-\phi)^p (\omega_\phi)^n < \infty.$$

Proof of (a$_m$)$\Rightarrow$(b$_m$). We have

$$\int_X (-\phi)^p dd^c\phi_j \wedge T = \lim_{k\to\infty} \int_X (-\phi_k)^p dd^c\phi_j \wedge T$$

($dd^c\phi_j \wedge T$ need not be a positive measure, but can be written as a difference of two positive probability measures, hence we can use monotone convergence). So we have

$$\int_X (-\phi_k)^p dd^c\phi_j \wedge T = \int_X \phi_j dd^c(-\phi_k)^p \wedge T$$

(when both functions are bounded, integration by parts is legitimate). So

$$\begin{aligned}
&\Big| \int_X \phi_j dd^c(-\phi)^p \wedge T - \int_X (-\phi)^p dd^c\phi_j \wedge T \Big| \\
&\quad = \lim_{k\to\infty} \Big| \int_{\phi\leq -k} \phi_j dd^c((-\phi)^p - (-\phi_k)^p) \wedge T \Big| \\
&\quad \leq \limsup_{k\to\infty} \Big(\Big| \int_{\phi\leq -k} -j dd^c(-\phi)^p \wedge T \Big| + \Big| \int_{\phi\leq -k} -j dd^c(-\phi_k)^p \wedge T \Big| \Big) \\
&\quad =: \limsup_{k\to\infty} (A_k + B_k).
\end{aligned}$$

Now we shall estimate A_k (the case of B_k is similar) using the Hölder inequality:

$$
\begin{aligned}
A_k &\le \Big|jp\int_{\phi\le -k}(-\phi)^{p-1}dd^c\phi\wedge T\Big| + \Big|j\int_{\phi\le -k}p(p-1)(-\phi)^{p-2}d\phi\wedge d^c\phi\wedge T\Big| \\
&\le jp\Big(\Big|\int_{\phi\le -k}(-\phi)^{p-1}\omega_\phi\wedge T\Big| + \Big|\int_{\phi\le -k}(-\phi)^{p-1}\omega\wedge T\Big|\Big) \\
&\quad + \Big|j\int_{\phi\le -k}p(p-1)(-\phi)^{p-2}d\phi\wedge d^c\phi\wedge T\Big| \\
&\le jp\Big(\int_{\phi\le -k}(-\phi)^p\omega_\phi\wedge T\Big)^{(p-1)/p}\Big(\int_{\phi\le -k}\omega_\phi\wedge T\Big)^{1/p} \\
&\quad + jp\Big(\int_{\phi\le -k}(-\phi)^p\omega\wedge T\Big)^{(p-1)/p}\Big(\int_{\phi\le -k}\omega\wedge T\Big)^{1/p} \\
&\quad + jp(p-1)\Big(\int_{\phi\le -k}(-\phi)^{p-1}d\phi\wedge d^c\phi\wedge T\Big)^{\frac{p-2}{p-1}}\Big(\int_{\phi\le -k}d\phi\wedge d^c\phi\wedge T\Big)^{1/(p-1)}.
\end{aligned}
$$

In each term the second factor tends to zero (since all these measures vanish on pluripolar sets and we integrate over sets that decrease to $\{\phi=-\infty\}$). The first factors are bounded by (a_m), so we obtain $\limsup_{k\to\infty}A_k=0$. Analogously $B_k\to 0$ and we are done.

Proof of $(\mathrm{a}_m)\&(\mathrm{b}_m)\Rightarrow(\mathrm{a}_{m+1})$. Write T as $T=\omega_{\phi_j}\wedge S$. As in the proof of (a_0), it is enough to estimate the numbers $\int_X(-\phi)^p\omega_\phi\wedge T$. We have

$$
\begin{aligned}
\int_X(-\phi)^p\omega_\phi\wedge T &= \int_X(-\phi)^p\omega_{\phi_j}\wedge\omega_\phi\wedge S \\
&= \int_X(-\phi)^p\omega\wedge\omega_\phi\wedge S + \int_X(-\phi)^p dd^c\phi_j\wedge\omega_\phi\wedge S.
\end{aligned}
$$

Now by (a_m) the first term is bounded. It is easy to check that it is bounded by $\int_X(-\phi)^p(\omega_\phi)^n$. Also by (b_m) we can integrate by parts in the second term to obtain

$$
\begin{aligned}
\int_X(-\phi)^p dd^c\phi_j\wedge\omega_\phi\wedge S &= \int_X\phi_j dd^c(-\phi)^p\wedge\omega_\phi\wedge S \\
&= p\int_X(-\phi_j)(-\phi)^{p-1}dd^c\phi\wedge\omega_\phi\wedge S \\
&\quad + p(p-1)\int_X\phi_j(-\phi)^{p-2}d\phi\wedge d^c\phi\wedge\omega_\phi\wedge S.
\end{aligned}
$$

But $-\phi_j\le\phi$ and $\phi_j<0$, so the second term is nonpositive, and the first can be estimated by

$$p\int_X(-\phi_j)(-\phi)^{p-1}dd^c\phi\wedge\omega_\phi\wedge S\le p\int_X(-\phi)^p\omega_\phi^2\wedge S,$$

which according to (a_m) can be bounded by a constant independent of j. Now it follows from our estimates that we can bound all the initial numbers by $(p+1)^n\int_X(-\phi)^p(\omega_\phi)^n$. ∎

REMARK. Note that in the proof we used some special features of ϕ_j. We conjecture that "integration by parts" holds in a much more general situation (as in the flat case, see [Ce2]). In particular, it seems that integration by parts is legitimate whenever one of the integrals is finite.

Now we are ready to prove that ϕ_j is a sequence solving our problem:

THEOREM 3.2.

$$\sup_j\int_X(-\phi_j)^p\omega_{\phi_j}^n<\infty.$$

Proof. Fix j. We have

$$\int_X(-\phi_j)^p\omega_{\phi_j}^n\le\int_X(-\phi)^p\omega_{\phi_j}^n$$

but by Lemma 3.1(a_n) the last term is estimated by $(p+1)^n\int_X(-\phi)^p(\omega_\phi)^n$, which is finite by assumption. ∎

This argument is in fact the same as in Lemma 4.2 of [GZ2]. To prove our next result we need more delicate estimates. We first prove

LEMMA 3.3. *Let u be an ω-psh function which belongs to $\mathcal{E}$. Then*

$$\omega+dd^c\max(u,-j)\ge\chi_{\{u>-j\}}(\omega+dd^cu)$$

in the sense of currents.

Proof. Analogous results for bounded functions are well known and can be found in [BT2], [K2]. Here we use similar arguments. Let S be a positive form of bidegree $(n-1,n-1)$. We have to prove that

$$(\omega+dd^cu_j)\wedge S\ge\chi_{\{u>-j\}}(\omega+dd^cu)\wedge S.$$

It is enough to have this estimate on compact subsets K of $\{u>-j\}$. Fix $\varepsilon>0$. Using quasicontinuity of $\mathrm{PSH}(X,\omega)$ functions (see [GZ1]) one can find an open set U with $\mathrm{cap}_\omega(U)<\varepsilon$ and $u=u_0$ on $X\backslash U$ for some continuous u_0. Let $u^s\searrow u$ as $s\to\infty$, $u^s\in\mathrm{PSH}(X,\omega)\cap C(X)$, and $V_t:=\{-j<u_0+t\}$ $(t>0)$. We have $\{-j<u^s+t\}$ on $V_t\backslash U$. Take any open V such that $K\subset V\subset V_t\cup U$. Then

$$\int_K(\omega+dd^cu)\wedge S$$

$$\le\liminf_{s\to\infty}\int_{V\cup U}(\omega+dd^cu^s)\wedge S\le\liminf_{s\to\infty}\int_{V\backslash U}(\omega+dd^cu^s)\wedge S+2jC\varepsilon$$

for some constant C (depending on S but not on j), which follows from the CLN inequalities (see [CLN]). Indeed, it is enough to prove the inequality for small compacts contained in coordinate charts and then apply classical CLN inequalities. Now

$$\liminf_{s\to\infty} \int\limits_{V\setminus U} (\omega + dd^c u^s)\wedge S + 2jC\varepsilon \\ \le \liminf_{s\to\infty} \int\limits_{V\setminus U} (\omega + dd^c \max\{u^s+t,-j\})\wedge S + 2jC\varepsilon.$$

Let $V\searrow K$ to get

$$\int\limits_K (\omega+dd^c u)\wedge S \le \int\limits_K (\omega + dd^c\max\{u+t,-j\})\wedge S + 2jC\varepsilon,$$

then let $t\searrow 0$ to end up with

$$\int\limits_K (\omega+dd^c u)\wedge S \le \int\limits_K (\omega + dd^c\max\{u,-j\})\wedge S + 2jC\varepsilon.$$

Now since ε is arbitrary and C depends only on S but not on ε we get the desired conclusion. ∎

THEOREM 3.4. *Let $\phi\in\mathcal{E}\cap L^p(X,\omega_\phi^n)$ and $\phi_j := \max(\phi,-j)$. Then*

$$\lim_{j\to\infty}\int\limits_X (-\phi_j)^p\omega_{\phi_j}^n = \int\limits_X (-\phi)^p\omega_\phi^n.$$

Proof. It follows from standard measure-theoretic arguments that

$$\liminf_{j\to\infty}\int\limits_X (-\phi_j)^p\omega_{\phi_j}^n \ge \int\limits_X (-\phi)^p\omega_\phi^n,$$

so we have to prove that

$$\limsup_{j\to\infty}\int\limits_X (-\phi_j)^p\omega_{\phi_j}^n \le \int\limits_X (-\phi)^p\omega_\phi^n.$$

The proof will be inductive. We shall prove that

$$\limsup_{j\to\infty}\int\limits_X (-\phi_j)^p\omega_{\phi_j}^k\wedge\omega^l\wedge\omega_\phi^{n-k-l} \le \int\limits_X (-\phi)^p\omega_\phi^{n-l}\wedge\omega^l.$$

For $k=n$ and $l=0$ we get the desired result. Let us start with $k=1$ and l arbitrary:

$$\limsup_{j\to\infty}\int\limits_X (-\phi_j)^p\omega_{\phi_j}\wedge\omega^l\wedge\omega_\phi^{n-k-1} \le \int\limits_X (-\phi)^p\omega^l\wedge\omega_\phi^{n-k}.$$

Perhaps much simpler arguments would do, since the function is constant and we have continuity results for decreasing sequences in $\mathcal{E}^1$. We nevertheless perform here some calculations since the main proof also uses similar estimates:

$$\begin{aligned}\limsup_{j\to\infty}\int_X(-\phi)^p\omega_{\phi_j}\wedge\omega^l\wedge\omega_\phi^{n-l-1} &= \int_X(-\phi)^p\omega\wedge\omega^l\wedge\omega_\phi^{n-l-1}\\ &\quad+\limsup_{j\to\infty}\int_X\phi_j dd^c(-\phi)^p\wedge\omega^l\wedge\omega_{\phi_j}^{n-l-1}\\ &= \int_X(-\phi)^p\omega\wedge\omega^l\wedge\omega_\phi^{n-l-1}\\ &\quad+\limsup_{j\to\infty}\Big(\int_X p(p-1)\phi_j(-\phi)^{p-2}d\phi\wedge d^c\phi\wedge\omega^l\wedge\omega_\phi^{n-l-1}\\ &\quad+\int_X p(-\phi_j)(-\phi)^{p-1}dd^c\phi\wedge\omega^l\wedge\omega_\phi^{n-l-1}\Big).\end{aligned}$$

Now by monotone convergence (which is justified as in Lemma 3.1) this is equal to

$$\begin{aligned}&-p(p-1)\int_X(-\phi)^{p-1}d\phi\wedge d^c\phi\wedge\omega^l\wedge\omega_\phi^{n-l-1}+p\int_X(-\phi)^p dd^c\phi\wedge\omega^l\wedge\omega_\phi^{n-l-1}\\ &\qquad+\int_X(-\phi)^p\omega^{l+1}\wedge\omega_\phi^{n-l-1}\\ &\quad= -(p-1)\int_X(-\phi)^p dd^c\phi\wedge\omega^l\wedge\omega_\phi^{n-l-1}+p\int_X(-\phi)^p dd^c\phi\wedge\omega^l\wedge\omega_\phi^{n-l-1}\\ &\qquad+\int_X(-\phi)^p\omega^{l+1}\wedge\omega_\phi^{n-l-1}=\int_X(-\phi)^p\omega^l\wedge\omega_\phi^{n-l},\end{aligned}$$

which was to be proved.

Assume the result holds for $k-1$ and arbitrary l. We shall prove it for k:

$$\begin{aligned}&\limsup_{j\to\infty}\int_X(-\phi_j)^p\omega_{\phi_j}^k\wedge\omega^l\wedge\omega_\phi^{n-k-l}\\ &\quad\le\limsup_{j\to\infty}\int_X\phi_j dd^c(-\phi)^p\wedge\omega_{\phi_j}^{k-1}\wedge\omega^l\wedge\omega_\phi^{n-k-l})\\ &\qquad+\int_X(-\phi)^p\omega_{\phi_j}^{k-1}\wedge\omega^{l+1}\wedge\omega_\phi^{n-k}\\ &\quad\le\limsup_{j\to\infty}\Big[p(p-1)\int_X\phi_j(-\phi)^{p-2}d\phi\wedge d^c\phi\wedge\omega_{\phi_j}^{k-1}\wedge\omega^l\wedge\omega_\phi^{n-k-l}\\ &\qquad-p\int_X\phi_j(-\phi)^{p-1}dd^c\phi\wedge\omega_{\phi_j}^{k-1}\wedge\omega^l\wedge\omega_\phi^{n-k-l}\\ &\qquad+\int_X(-\phi)^p\omega_{\phi_j}^{k-1}\wedge\omega^{l+1}\wedge\omega_\phi^{n-k-l}\Big]\end{aligned}$$

$$\le \limsup_{j\to\infty} \Big[p(p-1)\int_X \phi_j(-\phi)^{p-2} d\phi\wedge d^c\phi\wedge\omega_{\phi_j}^{k-1}\wedge\omega^l\wedge\omega_\phi^{n-k-l}$$
$$+p\int_X \phi_j(-\phi)^{p-1}\omega\wedge\omega_{\phi_j}^{k-1}\wedge\omega^l\wedge\omega_\phi^{n-k-l}$$
$$-p\int_X \phi_j(-\phi)^{p-1}\omega_{\phi_j}^{k-1}\wedge\omega^l\wedge\omega_\phi^{n-k-l+1}+\int_X(-\phi)^p\omega_{\phi_j}^{k-1}\wedge\omega^{l+1}\wedge\omega_\phi^{n-k-l}\Big].$$

Now we can use Lemma 3.3 to bound from above the first two terms on the right with $\chi_{\{u>-j\}}$ times measures independent of j. Indeed,

$$p(p-1)\int_X \phi_j(-\phi)^{p-2} d\phi\wedge d^c\phi\wedge\omega_{\phi_j}^{k-1}\wedge\omega^l\wedge\omega_\phi^{n-k-l}$$
$$+p\int_X \phi_j(-\phi)^{p-1}\omega\wedge\omega_{\phi_j}^{k-1}\wedge\omega^l\wedge\omega_\phi^{n-k-l}$$
$$\le p(p-1)\int_X \phi_j(-\phi)^{p-2}\chi_{\{u>-j\}} d\phi\wedge d^c\phi\wedge\omega_{\phi}^{k-1}\wedge\omega^l\wedge\omega_\phi^{n-k-l}$$
$$+p\int_X \phi_j(-\phi)^{p-1}\chi_{\{u>-j\}}\omega\wedge\omega_{\phi}^{k-1}\wedge\omega^l\wedge\omega_\phi^{n-k-l}.$$

Then we can use monotone convergence (for those terms), and induction hypothesis for the next two. What we get reads

$$-p(p-1)\int_X(-\phi)^{p-1}d\phi\wedge d^c\phi\wedge\omega^l\wedge\omega_\phi^{n-l-1}+p\int_X(-\phi)^p\omega^l\wedge\omega_\phi^{n-l}$$
$$-(p-1)\int_X(-\phi)^p\omega^{l+1}\wedge\omega_\phi^{n-l-1}$$
$$=p\int_X(-\phi)^p\omega^l\wedge\omega_\phi^{n-l}-(p-1)\int_X(-\phi)^p\omega^l\wedge\omega_\phi^{n-l}=\int_X(-\phi)^p\omega^l\wedge\omega_\phi^{n-l},$$

which finishes the proof. ■

We finish this section with an analogous result in the flat theory. Let us recall some terminology. A domain Ω in $\mathbb{C}^n$ is called *hyperconvex* if it admits a negative exhaustion function, i.e a PSH function f such that $\{z\in\Omega \mid f(z)<-c\}\subset\subset\Omega$ for all $c>0$. Let $\mathcal{E}_0$ be the set of bounded exhaustion functions and $\mathcal{E}$ the set of PSH functions for which one can define their Monge–Ampère mass in such a way that the Monge–Ampère operator is still continuous on decreasing sequences (see [Ce2]). Let $\mathcal{F}^p$ be the subclass of $\mathcal{E}$ consisting of those functions g for which there exists a sequence $g_j\in\mathcal{E}_0$ decreasing to g such that

$$\text{(3.1)}\qquad \sup_j\int_\Omega(-g_j)^p(dd^cg_j)^n<\infty,\qquad \sup_j\int_\Omega(dd^cg_j)^n<\infty.$$

Finally, let $\mathcal{F}$ be the subclass of $\mathcal{E}$ consisting of those functions g for which there exists a sequence $g_j \in \mathcal{E}_0$ decreasing to g such that

$$\sup_j \int_\Omega (dd^c g_j)^n < \infty$$

(for more details concerning these topics we refer to [Ce1], [Ce2]).

One can prove the following result:

THEOREM 3.5. *Let $h \in \mathcal{F}$ be a function such that*

$$\int_\Omega (-h)^p (dd^c h)^n < \infty.$$

Then $h \in \mathcal{E}^p$, i.e. there exists a sequence of functions decreasing to h and satisfying (3.1) *and their p-energies tend to $\int_\Omega (-h)^p (dd^c h)^n$.*

Proof. Let $h_j := \max\{h, -j\}$. These functions need not belong to $\mathcal{E}_0$ (they need not tend to 0 on the boundary, but of course they belong to $\mathcal{F}$), but if we take any $w \in \mathcal{E}_0$ then $w_j := \max\{h_j, m_j w\} \in \mathcal{E}_0$ for any positive m_j. Now $w_j \searrow h_j$ as $m_j \to \infty$ (but we keep j fixed!), we can fix m_j so large that $w_{j+1} \le w_j$ and $|\int_\Omega ((-w_j)^p (dd^c w_j)^n - (-h_j)^p (dd^c h_j)^n)| < 1/j$ (here we use the continuity of the Monge–Ampère operator on decreasing sequences). Therefore we can restrict ourselves to the sequence h_j:

$$\begin{aligned}
\int_\Omega (-h_j)^p (dd^c h_j)^n \le \int_\Omega (-h)^p (dd^c h_j)^n &= \int_\Omega h_j dd^c (-h)^p (dd^c h_j)^{n-1} \\
&= p \int_\Omega (-h_j)(-h)^{p-1} (dd^c h_j)^n \\
&\quad + p(p-1) \int_\Omega h_j (-h)^{p-2} dh \wedge d^c h \wedge (dd^c h_j)^{n-1} \\
&\le p \int_\Omega (-h)^p dd^c h \wedge (dd^c h_j)^{n-1} \\
&\quad + p(p-1) \int_\Omega h_j (-h)^{p-2} \chi_{(h>-j)} dh \wedge d^c h \wedge (dd^c h)^{n-1}
\end{aligned}$$

where we have used the "flat" variant of Lemma 3.3 and integration by parts, which is legitimate in $\mathcal{F}$ (see [Ce1] and [Ce2]). Hence

$$\begin{aligned}
\limsup_{j\to\infty} \int_\Omega (-h_j)^p (dd^c h_j)^n &\le \limsup_{j\to\infty} p \int_\Omega (-h)^p dd^c h \wedge (dd^c h_j)^{n-1} \\
&\qquad - p(p-1) \int_\Omega (-h)^{p-1} dh \wedge d^c h \wedge (dd^c h)^{n-1} \\
&\le \cdots \le \sum_{k=1}^n -p^k (p-1) \int_\Omega (-h)^{p-1} dh \wedge d^c h (dd^c h)^{n-1} + p^n \int_\Omega (-h)^p (dd^c h)^n
\end{aligned}$$

$$= -p(p^n-1)\int_{\Omega}(-h)^{p-1}dh\wedge d^ch\wedge(dd^ch)^{n-1}+p^n\int_{\Omega}(-h)^p(dd^ch)^n$$
$$= -(p^n-1)\int_{\Omega}(-h)^p(dd^ch)^n+p^n\int_{\Omega}(-h)^p(dd^ch)^n=\int_{\Omega}(-h)^p(dd^ch)^n. \blacksquare$$

4. Local comparison theorem. In [K1] the author proved the following result:

THEOREM 4.1. *Let u, v be* $\mathrm{PSH}(X,\omega)\cap\mathcal{C}(X)$ *functions on a compact n-dimensional Kähler manifold. Then*

$$\int_{\{u<v\}}\omega_v^n\le\int_{\{u<v\}}\omega_u^n.$$

In that paper the author analyzed only continuous $\mathrm{PSH}(X,\omega)$ functions, nevertheless it was claimed (see the remark after Theorem 2.1 in [K1]) that the continuity assumption is redundant. It was also suggested that the general case of bounded $\mathrm{PSH}(X,\omega)$ functions could be proved by using a quasicontinuity argument. This can also be done by using a recent result from [BK], namely one can approximate any bounded $\mathrm{PSH}(X,\omega)$ function by a decreasing sequence of continuous ω-plurisubharmonic functions.

Here we prove that this result still holds when $u, v\in\mathcal{E}^p$ for all $p\ge 1$. The proof repeats arguments of [Ce1] from the flat context. We shall concentrate only on those points where slight adjustments are made.

LEMMA 4.2. *Let $\phi\in\mathcal{E}^p$ and $\phi_j=\max\{\phi,-j\}$. Then*

$$\int_{\{u<v\}}\omega_\phi^n\le\liminf_{j\to\infty}\int_{\{u<v\}}\omega_{\phi_j}^n.$$

Proof. Note that ϕ_j is a sequence as in the definition of $\mathcal{E}^p$ for ϕ (due to results in Section 3). Now one has to repeat the proof of Lemma 4.3 (first part) in [Ce1]. ■

THEOREM 4.3. *Let $u, v\in\mathcal{E}^p$ be functions on a compact n-dimensional Kähler manifold. Then*

$$\int_{\{u<v\}}\omega_v^n\le\int_{\{u<v\}}\omega_u^n.$$

Proof. Let v_j, u_j be defined as above. Then

$$\int_{\{u<v\}}\omega_v^n\le\liminf_{j\to\infty}\lim_{k\to\infty}\int_{\{u_k<v\}}\omega_{v_j}^n\le\liminf_{j\to\infty}\limsup_{k\to\infty}\int_{\{u_k<v_j\}}\omega_{v_j}^n.$$

Now Theorem 4.1 yields

$$\begin{aligned}\liminf_{j\to\infty}\limsup_{k\to\infty}\int\limits_{\{u_k<v_j\}}\omega_{u_k}^n &\le \limsup_{j\to\infty}\limsup_{k\to\infty}\int\limits_{\{u<v_j\}}\omega_{u_k}^n\\ &\le \limsup_{j\to\infty}\int\limits_{\{u<v_j\}}\omega_u^n=\int\limits_{\{u<v\}}\omega_u^n,\end{aligned}$$

where we have used monotone convergence for the last equality. ∎

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References

[BT1] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge–Ampère operator*, Invent. Math. 37 (1976), 1–44.

[BT2] —, —, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.

[Bl] Z. Błocki, *The domain of definition of the complex Monge–Ampère operator*, Amer. J. Math. 128 (2006) 519–530.

[BK] Z. Błocki and S. Kołodziej, *On regularization of plurisubharmonic functions on manifolds*, Proc. Amer. Math. Soc., to appear.

[Ce1] U. Cegrell, *Pluricomplex energy*, Acta Math. 180 (1998), 187–217.

[Ce2] —, *The general definition of the complex Monge–Ampère operator*, Ann. Inst. Fourier (Grenoble) 54 (2004), 159–179.

[Ce3] —, *Explicit calculation of a Monge–Ampère measure*, in: Actes des Recontres d'Analyse Complexe (Poitiers, 1999), Atlantique, Poitiers, 2002, 39–42.

[CLN] S. S. Chern, H. I. Levine and L. Nirenberg, *Intrinsic norms on a complex manifold*, in: Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, 119–139.

[GZ1] V. Guedj and A. Zeriahi, *Intrinsic capacities on compact Kähler manifolds*, J. Geom. Anal. 15 (2005), 607–639.

[GZ2] —, —, *Monge–Ampère operators on compact Kähler surfaces*, arXiv/math.CV/0504234.

[K1] S. Kołodziej, *The Monge–Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. J. 52 (2003), 667–686.

[K2] —, *The complex Monge–Ampère equation and pluripotential theory*, Mem. Amer. Math. Soc. 178 (2005), no. 840.

[Kli] M. Klimek, *Pluripotential Theory*, Clarendon Press, 1991.

Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: slawomir.dinew@im.uj.edu.pl

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