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A survey on geometric properties of holomorphic self-maps in some domains of $\mathbb{C}^n$

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Abstract. In this survey we give geometric interpretations of some standard results on boundary behaviour of holomorphic self-maps in the unit disc of $\mathbb{C}$ and generalize them to holomorphic self-maps of some particular domains of $\mathbb{C}^n$.

1. Introduction and metric properties of holomorphic self-maps. We start by considering holomorphic self-maps of Riemann surfaces. This topic is of great interest and has had several approaches (see [9, 21, 22, 28, 32, 35]). A Riemann surface is defined to be a connected one-dimensional complex manifold. In particular, any Riemann surface is an orientable two-dimensional real manifold and it can be proved (see e.g. [12, 20]) that on any orientable connected two-dimensional real manifold one can always introduce a complex structure which agrees with the real structure and the resulting complex manifold is a Riemann surface. Two Riemann surfaces S and T are *conformally* or *biholomorphically equivalent* if there exists an invertible holomorphic map $f : S \to T$ with holomorphic inverse. Such a map is also said to be a *biholomorphism* or a *conformal transformation.* Up to conformal equivalence, there exist only three Riemann surfaces which are simply-connected (see [4], [12] or [13] for the (difficult) proof):

Theorem 1 (Riemann's uniformization theorem). *Any simply-connected Riemann surface is biholomorphic either to the unit disc* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ *of* $\mathbb{C}$, *or to the complex plane* $\mathbb{C}$, *or to the Riemann sphere* $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1 = \widehat{\mathbb{C}}$.

Since any Riemann surface X admits a universal covering $\widetilde{X}$, it is possible to lift the structure of Riemann surface of X onto $\widetilde{X}$ in such a way that the projection $\pi_X : \widetilde{X} \to X$ is holomorphic. A Riemann surface X is called

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elliptic (respectively, *parabolic* or *hyperbolic*) if its universal covering $\widetilde{X}$ is $\widehat{\mathbb{C}}$ (respectively, $\mathbb{C}$ or Δ).

If X is a Riemann surface, the group of *automorphisms* (or *transformations*) of X (i.e. the set of all invertible holomorphic self-maps of X, which is a group with the usual operation of composition of functions) will be denoted by $\mathrm{Aut}(X)$; the set of holomorphic functions $f : X \to Y$ will be denoted by $\mathrm{Hol}(X, Y)$. It has also to be remarked (see [1], [12]) that from the study of the actions of the subgroups of automorphisms which act freely and properly discontinuously on the three (possible) simply-connected Riemann surfaces it is possible to show that:

(i) the unique elliptic Riemann surface is the Riemann sphere $\widehat{\mathbb{C}}$;
(ii) the only non-simply-connected examples of parabolic Riemann surfaces are the punctured plane $\mathbb{C} \setminus \{0\} = \mathbb{C}^*$ and the tori $\mathbb{C}/\Gamma_\tau$.

All such Riemann surfaces have abelian fundamental groups; hence any Riemann surface with non-abelian fundamental group has to be hyperbolic. In particular, almost all plane domains are hyperbolic Riemann surfaces:

PROPOSITION 2. *Every domain $D \subset \widehat{\mathbb{C}}$ such that $\widehat{\mathbb{C}} \setminus D$ contains at least three points is a hyperbolic Riemann surface.*

Indeed, $\widehat{\mathbb{C}}$ minus three points, having non-abelian fundamental group, is hyperbolic; since D is contained in $\widehat{\mathbb{C}}$ minus three points, it is possible to embed D holomorphically and univalently into $\widehat{\mathbb{C}}$ minus three points. If then D were not hyperbolic, the lifting to the universal covering of this immersion (and hence the immersion itself) would be constant, because of Liouville's theorem (see [3, 37]). The class of hyperbolic Riemann surfaces is thus quite large.

The standard metric $|dz|$ of $\mathbb{C}$ corresponds, under the stereographic projection of the unit sphere $S^2 \simeq \widehat{\mathbb{C}}$ of $\mathbb{R}^3$, to the so-called *spherical metric*

$$ds = \frac{2|dz|}{1 + |z|^2},$$

which has constant Gaussian curvature $+1$ and is defined in such a way that the point ∞ has finite distance from any other point of $\widehat{\mathbb{C}}$. Observe that even though the map $z \mapsto 1/z$ is an isometry for this metric, it is not true that every conformal self-map of $\widehat{\mathbb{C}}$ is an isometry.

We now recall how one can determine all the holomorphic automorphisms of Δ. To do this we essentially need the following lemma, whose importance, however, goes far beyond this purpose.

LEMMA 3 (Schwarz lemma). *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be such that $f(0) = 0$. Then*

$$|f(z)| \le |z| \quad \forall z \in \Delta$$

and

$$|f'(0)| \leq 1.$$

In particular, if there exists a $z_0 \in \varDelta \setminus \{0\}$ *such that* $|f(z_0)| = |z_0|$, *or if* $|f'(0)| = 1$, *then* $f(z) = e^{i\theta} z$ *for some real* θ *and* $f'(0) = e^{i\theta}$.

A generalization of the Schwarz lemma, due to Ahlfors, can be found for instance in [28]. The Schwarz lemma is also very useful for the following description of the automorphisms of $\varDelta$:

PROPOSITION 4. *Every holomorphic automorphism* γ *of* $\varDelta$ *into itself is of the form*

$$\gamma(z) = e^{i\theta}\,\frac{z - z_0}{1 - \bar{z}_0 z}$$

with $z_0 \in \varDelta$ *and* $\theta \in \mathbb{R}$.

The automorphisms of $\varDelta$ are also called *Möbius transformations*.

It is clear from the form of such automorphisms that they act transitively on $\varDelta$ and on $\partial\varDelta$. It is also evident that the automorphisms of $\varDelta$ extend holomorphically on $\partial\varDelta$. This leads one to consider their fixed points in $\overline{\varDelta}$.

PROPOSITION 5. *Let* $\gamma \in \mathrm{Aut}(\varDelta)$, $\gamma \neq \mathrm{Id}_{\varDelta}$. *Then either*

(i) γ *has a unique fixed point in* $\varDelta$, *or*
(ii) γ *has a unique fixed point in* $\partial\varDelta$, *or*
(iii) γ *has two distinct fixed points in* $\partial\varDelta$.

An automorphism γ of $\varDelta$, different from the identity, is called *elliptic* if it has a (unique) fixed point in $\varDelta$, *parabolic* if it has a unique fixed point in $\partial\varDelta$, and *hyperbolic* if it has two distinct fixed points on $\partial\varDelta$.

REMARK 6. Unfortunately the terms *elliptic*, *parabolic* and *hyperbolic* have different meanings in different fields. These terms apparently came into use for different historical reasons. In order to avoid confusion we will always refer to a specific meaning by adjoining a (hopefully) clarifying term when necessary.

It is sometimes useful to study the automorphisms of the upper half plane $H^+ = \{w \in \mathbb{C} : \mathrm{Im}\, w > 0\}$, which is conformally equivalent to $\varDelta$ by means of the Cayley transformation $C : \varDelta \to H^+$, $C(z) = i(1+z)/(1-z)$, with inverse $C^{-1} : H^+ \to \varDelta$, $C^{-1}(w) = (w-i)/(w+i)$.

Once the so-called *hyperbolic* or *Poincaré metric*

$$ds = \frac{2|dz|}{1 - |z|^2}$$

is introduced in $\varDelta$, it becomes an example of a Riemann surface with negative constant Gaussian curvature, which can be easily calculated to be -1.

The (analogous) hyperbolic metric in H^+ is

$$ds = \frac{|dw|}{\operatorname{Im} w}.$$

If this metric is integrated (see, e.g., [49]), one gets the corresponding *hyperbolic* or *Poincaré distance*

$$\omega_\Delta(z_1, z_2) = \frac{1}{2} \log \frac{1 + \left|\dfrac{z_1 - z_2}{1 - \overline{z}_1 z_2}\right|}{1 - \left|\dfrac{z_1 - z_2}{1 - \overline{z}_1 z_2}\right|}.$$

The Poincaré distance has the particular property to be contracted by holomorphic self-maps of Δ:

LEMMA 7 (Schwarz–Pick lemma). *Let* $f \in \operatorname{Hol}(\Delta, \Delta)$. *Then*, *for any* $z_1, z_2 \in \Delta$,

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)}\right| \leq \left|\frac{z_1 - z_2}{1 - \overline{z}_1 z_2}\right|,$$

and for any $z \in \Delta$,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Moreover, *the above inequalities are actually equalities for some* $z_1, z_2 \in \Delta$ $(z_1 \neq z_2)$, *or for some* $z \in \Delta$, *if and only if* $f \in \operatorname{Aut}(\Delta)$.

The Schwarz–Pick lemma is a generalized version of the Schwarz lemma once one notices that if we take any $z_0 \in \Delta$ and define $\gamma_{z_0}(z) = (z + z_0)/(1 + \overline{z}_0 z)$ and $\gamma_{f(z_0)}(z) = (z - f(z_0))/(1 - \overline{f(z_0)} z)$, then the function $g = \gamma_{f(z_0)} \circ f \circ \gamma_{-z_0}$ belongs to $\operatorname{Hol}(\Delta, \Delta)$ and is such that $g(0) = 0$.

Since the map $t \mapsto \log \frac{1+t}{1-t}$ is increasing for $0 \leq t < 1$, it is clear that the Schwarz–Pick lemma implies that $\omega_\Delta(f(z), f(w)) \leq \omega_\Delta(z, w)$ for all $z, w \in \Delta$. It also follows that each automorphism of Δ is an isometry for the Poincaré distance. It can actually be proved (see, e.g., [1]) that the group of all isometries for the Poincaré distance consists of all holomorphic and antiholomorphic automorphisms of Δ.

We want to transfer the Poincaré distance ω_Δ from Δ to any hyperbolic Riemann surface. Let X be a hyperbolic Riemann surface and denote by $\pi_X : \Delta \to X$ the universal covering map of X. Defining

$$\forall z, w \in X \quad \omega_X(z, w) = \inf\{\omega_\Delta(\widetilde{z}, \widetilde{w}) : \widetilde{z} \in {\pi_X}^{-1}(z),\ \widetilde{w} \in {\pi_X}^{-1}(w)\},$$

we get a complete hyperbolic distance on X, which induces the standard topology (see, e.g., [1]).

The main property of this hyperbolic distance on an arbitrary hyperbolic Riemann surface is the analogue of the Schwarz–Pick lemma for the Poincaré distance in Δ:

PROPOSITION 8. *Let X and Y be two hyperbolic Riemann surfaces and $f : X \to Y$ be a holomorphic function. Let ω_X and ω_Y be the* (*induced*) *hyperbolic distances on X and on Y. Then*

$$\forall z, w \in X \quad \omega_Y(f(z), f(w)) \leq \omega_X(z, w).$$

More generally, let X be a complex manifold. A *complex chain* α connecting two points z_0 and w_0 in X is a sequence of points $(\xi_0, \ldots, \xi_m, \eta_0, \ldots, \eta_m)$ in Δ and holomorphic maps $\varphi_0, \ldots, \varphi_m \in \mathrm{Hol}(\Delta, X)$ such that $\varphi_0(\xi_0) = z_0$, $\varphi_j(\eta_j) = \varphi_{j+1}(\xi_{j+1})$ for $j = 0, \ldots, m-1$ and $\varphi_m(\eta_m) = w_0$. The *length* of α is defined to be

$$\omega(\alpha) = \sum_{j=0}^{m} \omega_\Delta(\xi_j, \eta_j).$$

We can define the *Kobayashi* (*pseudo*) *distance* k_X by putting

$$\forall z, w \in X \quad k_X(z, w) = \inf_\alpha \{\omega(\alpha)\}$$

(see, e.g., [14, 25, 29, 33, 34]).

Since X is connected, $k_X(z, w)$ is always finite and in general is a pseudo-distance, that is, the infimum might be zero even though $z \neq w$. However (see [1]), we have

PROPOSITION 9. *If D is a bounded domain of $\mathbb{C}^n$ then k_D is a distance.*

The topology induced by the Kobayashi distance is completely described by the following

PROPOSITION 10. *If k_D is a distance*, *then k_D defines on D the relative topology of D as a subset of $\mathbb{C}^n$.*

DEFINITION 11. A domain D of $\mathbb{C}^n$ is *hyperbolic* if k_D is a distance.

Furthermore, the following can be shown easily (see, e.g., [14]):

PROPOSITION 12. *If X is a hyperbolic Riemann surface then the Kobayashi and Poincaré distances coincide. If X is a Riemann surface but is not hyperbolic*, *then $k_X \equiv 0$. Finally*, *if $d : \mathbb{C}^n \to \mathbb{R}^+$ is any norm in $\mathbb{C}^n$*, *then for any z in the unit ball B_d for the norm d*,

$$k_{B_d}(0, z) = \omega(0, d(z)).$$

Therefore any hyperbolic domain in $\mathbb{C}$ is a hyperbolic Riemann surface. From the definition it follows immediately that if $f : X \to Y$ is a holomorphic map between complex manifolds, then for all $x, w \in X$,

$$k_Y(f(z), f(w)) \leq k_X(z, w).$$

In particular we have

DEFINITION 13. Any $\varphi \in \mathrm{Hol}(\Delta, X)$ which is an isometry for the Poincaré and Kobayashi distance will be called a *complex geodesic*.

With the notations of Proposition 12, one can prove that if $x \in B_d \setminus \{0\}$ then the map $\varphi(\xi) = \xi \cdot (x/d(x))$ is a complex geodesic. The existence and uniqueness of complex geodesics in a domain is in general a difficult problem and has been investigated by many authors (see, e.g., [19, 31, 48]).

Another important property of the Kobayashi (pseudo) distance is the following: let X and Y be two domains of $\mathbb{C}^n$; then (see [30]) for any $z, u \in X$ and $w, v \in Y$,

$$k_{X\times Y}((z,w);(u,v)) \geq \max\{k_X(z,u); k_Y(w,v)\}.$$

In particular, if $X \simeq Y \simeq \Delta$ then the inequality becomes an equality.

2. Boundary Schwarz lemmas in $\mathbb{B}^n$. Suppose that $f \in \mathrm{Hol}(\Delta, \Delta)$. If f has a *fixed point* $z_0 \in \Delta$, then the Schwarz–Pick lemma implies that f maps every disc for the Poincaré metric, centred at z_0, into itself. If, instead, f has no fixed points in Δ, then—as we will see shortly—the Wolff lemma states the existence of a unique point on the boundary of Δ, the "Wolff point", which plays the role of a "fixed point" on $\partial\Delta$. Since a map $f \in \mathrm{Hol}(\Delta, \Delta)$ and its derivative need not be continuous in $\overline{\Delta}$, we have to explain the meaning of "fixed point on the boundary" and "derivative of f at a point on the boundary" of Δ.

DEFINITION 14. Take $x \in \partial\Delta$ and $M > 1$. The set

$$K(x,M) = \left\{ z \in \Delta : \frac{|x-z|}{1-|z|} < M \right\}$$

is called the *Stolz region of vertex x and amplitude M*.

The Stolz region $K(x,M)$ is an "angular region" with vertex at x and "opening" less than π. Stolz regions are used to give the following

DEFINITION 15. Let $f : \Delta \to \mathbb{C}$ be a (holomorphic) function. We say that c is the *non-tangential limit* (or *angular limit*) of f at $x \in \partial\Delta$ if $f(z) \to c$ as z tends to x within $K(x,M)$, for all $M > 1$. We shall also write $\text{K-lim}_{z\to x} f(z) = c$.

We can equivalently say that:

DEFINITION 16. A function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ has *non-tangential limit* $L \in \mathbb{C}$ at a point $x \in \partial\Delta$ if

$$f(\sigma(t)) \to L \quad \text{as } t \to 1^-$$

for any curve $\sigma : [0,1) \to \Delta$ such that $\sigma(t) \to x$ non-tangentially as $t \to 1^-$.

REMARK 17. The letter K is generally used for Stolz regions since the analogous set in the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, for $x \in \partial\mathbb{B}^n$, namely

$$K(x,M) = \left\{ z \in \mathbb{B}^n : \frac{|1-\langle x,z\rangle|}{1-\|z\|} < M \right\},$$

with $\langle\cdot,\cdot\rangle$ the scalar product of $\mathbb{C}^n$, is called the *Korányi region* of vertex x and amplitude M.

We recall (see [1]) that the automorphism group of $\mathbb{B}^n$ consists of the maps $\gamma = U\gamma_a$ where U is an element of the unitary group of $\mathbb{C}^n$ and

$$\gamma_a(z) = \frac{a - P_a(z) - (1-|a|^2)^{1/2} Q_a(z)}{1 - \langle z, a\rangle}$$

with

$$P_a(z) = \frac{\langle z, a\rangle}{\langle a, a\rangle} a \quad \text{and} \quad Q_a(z) = z - P_a(z). \tag{2.1}$$

Any automorphism of $\mathbb{B}^n$ extends continuously to a homeomorphism of $\overline{\mathbb{B}^n}$ and acts transitively on $\mathbb{B}^n$ and doubly transitively on $\overline{\mathbb{B}^n}$. To sketch the shape of $K(x,M)$ we may therefore assume, without loss of generality, that $x = (1,0,\dots,0) = e_1 \in \partial\mathbb{B}^n$. Then the intersection of $K(e_1, M)$ with the complex subspace generated by e_1 is the Stolz region of vertex 1 in Δ,

$$K(1,M) = \left\{ z_1 \in \Delta : \frac{|1 - z_1|}{1 - |z_1|} < M \right\},$$

while the intersection of the Korányi region $K(x,M)$ with the copy of $\mathbb{R}^{2n-1}$ obtained by setting $\operatorname{Im} z_1 = 0$ contains the ball

$$\left(\operatorname{Re} z_1 - \frac{1}{M}\right)^2 + \|z'\|^2 < \left(1 - \frac{1}{M}\right)^2,$$

where $z' = (z_2, \dots, z_n)$, which is tangent to $\partial\mathbb{B}^n$. Therefore, since Korányi regions are tangent to $\partial\mathbb{B}^n$ along complex tangential directions, the definition of non-tangential limit in $\mathbb{B}^n$ requires a more subtle approach (see [1]):

Definition 18. Let $x \in \partial\mathbb{B}^n$. An *$x$-curve* is a (continuous) curve $\sigma : [0,1) \to \mathbb{B}^n$ such that $\sigma(t) \to x$ as $t \to 1^-$.

To every x-curve we associate its orthogonal projection $\sigma_x = \langle \sigma, x\rangle \cdot x$ in $\mathbb{C}x$. Therefore $(\sigma - \sigma_x) \perp \sigma_x$, so that

$$\|\sigma\|^2 = \|\sigma - \sigma_x\|^2 + \|\sigma_x\|^2$$

and hence

$$\frac{\|\sigma - \sigma_x\|^2}{1 - \|\sigma_x\|^2} < 1.$$

Definition 19. An *x-curve* σ in $\mathbb{B}^n$ is *special* if

$$\lim_{t\to 1} \frac{\|\sigma(t) - \sigma_x(t)\|^2}{1 - \|\sigma_x(t)\|^2} = 0.$$

An *x-curve* σ in $\mathbb{B}^n$ is *restricted* if it is special and there exists $M > 0$ such that for all $t \in [0,1)$,

$$\frac{\|\sigma(t) - x\|}{1 - \|\sigma_x(t)\|} \le M.$$

Notice that a special curve σ is restricted if and only if its projection σ_x is non-tangential.

DEFINITION 20. A map $f : \mathbb{B}^n \to \mathbb{C}$ has *restricted K-limit* $L \in \mathbb{C}$ at $x \in \partial\mathbb{B}^n$ if $f(\sigma(t)) \to L$ as $t \to 1^-$ for any restricted special x-curve σ. If this is the case we write

$$\widetilde{\text{K}}\text{-}\lim_{w\to x} f(w) = L.$$

REMARK 21. Notice that

$$\text{K-}\lim_{w\to x} f(w) = L \;\Rightarrow\; \widetilde{\text{K}}\text{-}\lim_{w\to x} f(w) = L.$$

The converse is false (see counterexamples in [1], [41]).

We also say that f is *K-bounded* at $x \in \partial\mathbb{B}^n$ if it is bounded inside any Korányi region of vertex x. Using these definitions, we can state the following result which gives a relationship for limits of maps when approaching a boundary point and can be considered as a generalization of the Lindelöf theorem (see [1]) in Δ.

THEOREM 22 (Chirka's theorem). *Given $x \in \partial\mathbb{B}^n$, let $f : \mathbb{B}^n \to \mathbb{C}$ be a bounded holomorphic map and assume that there exists a special x-curve σ^0 such that*

$$\lim_{t\to 1^-} f(\sigma^0(t)) = L \in \mathbb{C}.$$

Then f has restricted K-limit L at x.

Going back to the unit disc Δ, we have

DEFINITION 23. A point $\tau \in \partial\Delta$ is a *fixed point of f on the boundary of Δ* if $\text{K-}\lim_{z\to\tau} f(z) = \tau$; analogously we define the *derivative of f at* $\tau \in \partial\Delta$ to be the value of $\text{K-}\lim_{z\to\tau} f'(z)$ if it exists and is finite.

Before stating one of the main results for boundary fixed points, namely the Wolff lemma, let us recall the following

DEFINITION 24. Let $\tau \in \partial\Delta$. Then for any $R > 0$ the open (Euclidean) disc of Δ tangent to $\partial\Delta$ at τ defined as

$$E(\tau, R) = \left\{ z \in \Delta : \frac{|\tau - z|^2}{1 - |z|^2} < R \right\}$$

is called a *horocycle* of centre τ and radius R.

Geometrically the horocycle $E(\tau, R)$ in Δ is the Euclidean disc in Δ of radius $R/(R+1)$ tangent to $\partial\Delta$ at τ.

REMARK 25. If we denote by

$$P(\xi) = \frac{1 - |\xi|^2}{|1 - \xi|^2}$$

the Poisson kernel in Δ, the horocycle of centre 1 and radius R can be equivalently defined as

$$E(1, R) = \{z \in \Delta : P(\xi) > 1/R\}.$$

(We refer the interested reader to [23] for developments of these relations.)

Furthermore, the horocycle $E(\tau, R)$ can be obtained (see [1]) as a limit of a sequence $\{B_n = B(z_n, R_n)\}_{n\in\mathbb{N}}$ of Poincaré discs whose centres and radii are such that

$$z_n \to \tau \quad \text{and} \quad \frac{1 - |z_n|}{1 - \tanh R_n} = R \neq 0, \infty.$$

We can then state the following

LEMMA 26 (Wolff lemma). *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be without interior fixed points. Then there is a unique $\tau \in \partial\Delta$ such that for all $z \in \Delta$,*

$$\frac{|\tau - f(z)|^2}{1 - |f(z)|^2} \leq \frac{|\tau - z|^2}{1 - |z|^2}, \tag{2.2}$$

that is,

$$f(E(\tau, R)) \subseteq E(\tau, R) \quad \forall R > 0, \tag{2.3}$$

where $E(\tau, R)$ is the horocycle of centre τ and radius $R > 0$. Moreover, *the equality* (2.2) *holds at one point* (*and hence at all points*) *if and only if f is a* (*parabolic*) *automorphism of Δ leaving τ fixed.*

DEFINITION 27. If $f \in \mathrm{Hol}(\Delta, \Delta)$ has a fixed point in Δ (and $f \neq \mathrm{id}_\Delta$), then we denote this fixed point by $\tau(f)$. Otherwise, $\tau(f)$ denotes the point obtained in Wolff's lemma. In both cases $\tau(f)$ is called the *Wolff point* of f.

The notion of Wolff point of a holomorphic map $f \in \mathrm{Hol}(\Delta, \Delta)$ is deeply related to the behaviour of the sequence of iterates of f and commuting or *permutable* holomorphic functions. For the sake of completeness, we only mention here the main results and basic facts on this subject and we refer the interested reader to [6, 11, 36, 45, 46, 47, 50, 51].

DEFINITION 28. Let S and T be two Riemann surfaces. A sequence of holomorphic maps $f_n : S \to T$ is said to *converge on compact sets* to the limit map $g : S \to T$ if for every compact subset $K \subset S$ the sequence $\{f_n|_K\}_{n\in\mathbb{N}}$ converges uniformly to $g|_K$.

DEFINITION 29. Let S and T be two Riemann surfaces. A sequence of holomorphic maps $f_n : S \to T$ is said to be *compactly divergent* if for every pair of compact sets $K_1 \subset S$ and $K_2 \subset T$ there is an $n_0 \in \mathbb{N}$ such that $f_n(K_1) \cap K_2 = \emptyset$ for every $n > n_0$.

A well known theorem of Weierstrass (see for instance [40]) asserts that if a sequence of holomorphic functions converges uniformly on compact sets, then so does the sequence of their derivatives. Moreover, as a consequence of Morera's theorem (see, e.g., [40]), the limit function is holomorphic.

THEOREM 30 (Wolff–Denjoy theorem). *If $f \in \mathrm{Hol}(\varDelta, \varDelta)$ is neither an elliptic automorphism nor the identity, then the sequence of iterates $\{f^{\circ k}\}_{k\in\mathbb{N}}$ converges, uniformly on compact sets, to the Wolff point τ of f.*

The following result has been obtained independently by the two authors (see [7, 44]).

THEOREM 31 (Behan–Shields theorem). *Let $f, g \in \mathrm{Hol}(\varDelta, \varDelta) \setminus \mathrm{id}_{\varDelta}$ be such that $f \circ g = g \circ f$. Let $\tau(f)$ and $\tau(g)$ be their Wolff points.*

(i) *If f is not a hyperbolic automorphism of $\varDelta$, then $\tau(f) = \tau(g)$.*
(ii) *Otherwise, g is also a hyperbolic automorphism of $\varDelta$, with the same fixed point set as f, and either $\tau(f) = \tau(g)$ or $\tau(f^{-1}) = \tau(g)$.*

We have already encountered fruitful applications of the Schwarz and Schwarz–Pick lemmas; the next lemma is a first step towards "boundary generalizations" of these lemmas.

LEMMA 32 (Julia lemma). *Given $f \in \mathrm{Hol}(\varDelta, \varDelta)$, let $\sigma \in \partial\varDelta$ be such that*

$$\liminf_{z\to\sigma} \frac{1-|f(z)|}{1-|z|} = \beta < \infty.$$

Then there exists a unique $\tau \in \partial\varDelta$ such that

$$f(E(\sigma, R)) \subseteq E(\tau, \beta R) \quad \forall R > 0. \tag{2.4}$$

Furthermore, there exists $z_0 \in \partial E(\sigma, R)$ such that $f(z_0) \in \partial E(\tau, \beta R)$ if and only if f is an automorphism of $\varDelta$.

Given $f : \varDelta \to \varDelta$ holomorphic and $\sigma, \tau \in \partial\varDelta$, the behaviour of the images of the horocycles at σ under the action of f is described by means of

$$\beta_f(\sigma, \tau) = \sup_{z\in\varDelta} \left\{ \frac{|\tau - f(z)|^2}{1 - |f(z)|^2} \Big/ \frac{|\sigma - z|^2}{1 - |z|^2} \right\}$$

and of the *boundary dilatation coefficient* of f at σ, defined

$$\beta_f(\sigma) = \inf_{\tau\in\partial\varDelta} \beta_f(\sigma, \tau).$$

It can be proved that for any $f \in \mathrm{Hol}(\varDelta, \varDelta)$ and $\sigma \in \partial\varDelta$ there exists at most one point $\tau \in \partial\varDelta$ such that $\beta_f(\sigma, \tau)$ is finite and that actually $\beta_f(\sigma) = \beta$ as in the Julia lemma. It is also very easy to verify that the relation between the boundary dilatation coefficient $\beta_f^{\varDelta}$ of a self-map f, holomorphic in $\varDelta$, at a certain point τ and the corresponding coefficient $\beta_F^{H^+}$ for the conjugated map F in H^+ at the corresponding point is given by $\beta_F^{H^+} = [\beta_f^{\varDelta}]^{-1}$. In particular, $\beta_F^{H^+}$ is a finite real number, but possibly zero.

The definitions of non-tangential limit and of boundary dilatation coefficient are also used in another classical result, which can be considered another "boundary version" of the Schwarz lemma (see, e.g., [8, 52, 53]).

THEOREM 33 (Julia–Wolff–Carathéodory theorem). *Let* $f \in \mathrm{Hol}(\Delta, \Delta)$, *and let* τ, σ *be any two points in* $\partial\Delta$. *Then*

$$\operatorname*{K-lim}_{z \to \sigma} \frac{\tau - f(z)}{\sigma - z} = \tau\overline{\sigma}\beta_f(\sigma, \tau).$$

If this non-tangential limit is finite, then

$$\operatorname*{K-lim}_{z \to \sigma} f(z) = \tau$$

and

$$\operatorname*{K-lim}_{z \to \sigma} f'(z) = \operatorname*{K-lim}_{z \to \sigma} \frac{\tau - f(z)}{\sigma - z} = \tau\overline{\sigma}\beta_f(\sigma, \tau).$$

In particular, if $\tau = \sigma$ *then the non-tangential limit of* f' *at* σ *is a strictly positive real number.*

REMARK 34. The Julia–Wolff–Carathéodory theorem provides a handy condition ensuring the existence of the non-tangential limit of both f and its derivative at a point of $\partial\Delta$. Furthermore, the theorem states that the derivative of a holomorphic map f at a fixed point τ on the boundary is a positive real number $f'(\tau)$. The Wolff lemma implies that, in particular, if τ is the Wolff point of f, then τ is a fixed point of f on the boundary of Δ and $f'(\tau)$ is bounded from above by 1.

The fact that the derivative of a self-map f, holomorphic in Δ, at a fixed point τ on the boundary is a positive real number implies, among other things, that f is univalent near τ, within a Stolz region of vertex τ (see [38]) and the value of the derivative of a holomorphic map f at its Wolff point gives important information on the behaviour of the sequence of points $f^{\circ n}(z_0)$ (with $z_0 \in \Delta$): namely (see [10]) if $f'(\tau) < 1$ any such sequence converges non-tangentially to τ, whereas if $f'(\tau) = 1$ it is not always the case that a sequence of iterates of f at a point converges to τ tangentially, as the following example shows:

$$z \mapsto \frac{1 + 3z^2}{3 + z^2}.$$

Finally, the Julia–Wolff–Carathéodory theorem also gives a geometric characterization of conformality at the Wolff point of a holomorphic map, with conformality defined in the sense of [39].

It would be desirable to extend the notions so far introduced in Δ to several complex variables. If one takes as a generalization of Δ the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ then it is not difficult (1) to prove the following analogue of the Schwarz lemma.

(1) We refer the reader to [1] for the proofs of the following theorems in this section.

THEOREM 35. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be such that $f(0) = 0$. Then*

$$\|f(z)\| \le \|z\| \qquad \forall z \in \mathbb{B}^n$$

and, for any $v \in \mathbb{C}^n$,

$$\|df_0(v)\| \le \|v\|.$$

Notice that, in $\mathbb{B}^n$, equalities in the Schwarz lemma imply neither the linearity of f nor its invertibility, as can be immediately seen by taking $f : \mathbb{B}^2 \to \mathbb{B}^2$ given

$$f(z_1, z_2) = (z_1 + \tfrac{1}{2} z_2^2, 0).$$

However, it can be shown (see [1]) that for $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ with $f(0) = 0$, if $z \neq 0$, then

$$\|f(z)\| = \|z\| \;\Leftrightarrow\; \|df_0(z)\| = \|z\|;$$

in particular,

$$f(z) = z \;\Leftrightarrow\; df_0(z) = z.$$

Also the Schwarz–Pick lemma has a version in $\mathbb{B}^n$:

THEOREM 36. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$. Then for every $z, w \in \mathbb{B}^n$ we have*

$$\frac{|1 - \langle f(z), f(w)\rangle|^2}{(1 - \|f(z)\|^2)(1 - \|f(w)\|^2)} \le \frac{|1 - \langle z, w\rangle|^2}{(1 - \|z\|^2)(1 - \|w\|^2)}, \tag{2.5}$$

and for every $z \in \mathbb{B}^n$ and $w \in \mathbb{C}^n$ we have

$$\frac{|\langle df_z(v), f(z)\rangle|^2 + (1 - \|f(z)\|^2)\|df_z(v)\|^2}{(1 - \|f(z)\|^2)^2} \le \frac{|\langle v, z\rangle|^2 + (1 - \|z\|^2)\|v\|^2}{(1 - \|z\|^2)^2}. \tag{2.6}$$

In particular, if f is an automorphism of $\mathbb{B}^n$, then both (2.5) *and* (2.6) *are equalities.*

As in the case of the unit disc, (2.6) suggests introducing a differential metric and a distance in $\mathbb{B}^n$. After identification of the tangent space of $\mathbb{B}^n$ at a point x with $\mathbb{C}^n$, if $u, v \in \mathbb{C}^n$ we define

$$d\kappa_a(u, v) = \frac{1}{(1 - \|a\|^2)^2} \cdot [\langle u, a\rangle \cdot \langle a, v\rangle + (1 - \|a\|^2) \cdot \langle u, v\rangle],$$

so that (2.6) becomes

$$d\kappa_{f(z)}(df(v), df(v)) \le d\kappa_z(v, v).$$

This differential metric (which coincides in Δ with the Poincaré metric) is called the *Bergman metric* and when integrated it provides the so-called *Bergman distance*. According to Proposition 12, it immediately follows that in $\mathbb{B}^n$ the Bergman and the Kobayashi distances coincide.

As a consequence of Theorem 35, one can prove the following result which gives a first insight into the differences between $\mathrm{Hol}(\Delta, \Delta)$ and $\mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$.

Proposition 37. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$. Then the fixed point set of f is either empty or an affine subset of $\mathbb{B}^n$.*

However, as a natural generalization of the horocycles we give the following

Definition 38. The *horosphere* of centre $x \in \partial\mathbb{B}^n$ and radius $R > 0$ is the set

$$E(x,R) = \left\{ z \in \mathbb{B}^n : \frac{|1 - \langle z, z\rangle|^2}{1 - \|z\|^2} < R \right\}.$$

An easy computation shows that horospheres are not spheres but ellipsoids, namely

$$E(x,R) = \left\{ z \in \mathbb{C}^n : \frac{\|P_x(z) - a\|^2}{r^2} + \frac{\|Q_x(z)\|^2}{r^2} < 1 \right\},$$

where $r = R/(1+R)$, $a = (1-r)\cdot x$ and P_x, Q_x are as in (2.1). Despite this different shape, the main results for horocycles in Δ can be generalized for horospheres.

Lemma 39 (Julia lemma in $\mathbb{B}^n$). *Given $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$, let $x \in \partial\mathbb{B}^n$ be such that*

$$\liminf_{z \to x} \frac{1 - \|f(z)\|}{1 - \|z\|} = \alpha < \infty.$$

Then there exists a unique $y \in \partial\mathbb{B}^n$ such that

$$f(E(x,R)) \subseteq E(y, \alpha R) \quad \forall R > 0. \tag{2.7}$$

Lemma 40 (Wolff lemma in $\mathbb{B}^n$). *Given $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ without fixed points in $\mathbb{B}^n$, there exists a unique $x \in \partial\mathbb{B}^n$ such that*

$$f(E(x,R)) \subseteq E(x,R) \quad \forall R > 0. \tag{2.8}$$

Remark 41. As in the case of $\mathrm{Hol}(\Delta, \Delta)$, the unique point $x \in \partial\mathbb{B}^n$ defined in the Wolff lemma for f will be called the *Wolff point* of f.

Since for maps in $\mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ the behaviour of the radial component is different from that of the tangential component, a generalization of the Julia–Wolff–Carathéodory theorem can only be partial:

Theorem 42 (Julia–Wolff–Carathéodory theorem in $\mathbb{B}^n$). *Given $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$, let $x \in \partial\mathbb{B}^n$ be such that*

$$\liminf_{z \to x} \frac{1 - \|f(z)\|}{1 - \|z\|} = \alpha < \infty.$$

Then f has K-limit $y \in \partial\mathbb{B}^n$ at x and the following functions are bounded in any Korányi region:

(1) $(1 - \langle f(z), y\rangle)/(1 - \langle z, x\rangle)$,
(2) $Q_y(f(z))/(1 - \langle z, x\rangle)^{1/2}$,

(3) $\langle df_x x, y\rangle$,
(4) $(1-\langle z, x\rangle)^{1/2} Q_y(df_x x)$,
(5) $\langle df_x x^{\perp}, y\rangle/(1-\langle z, x\rangle)^{1/2}$,
(6) $Q_y(df_x x^{\perp})$,

where $x^{\perp}$ *is any non-zero vector orthogonal to* x *and* $Q_y(x) = z - \langle x, y\rangle \cdot y$ *is the orthogonal projection on the orthogonal complement of* $\mathbb{C}y$. *Furthermore*, *the functions* (1) *and* (3) *have restricted* K*-limit* α *at* x, *while the functions* (2), (4) *and* (5) *have restricted* K*-limit* 0 *at* x.

3. Generalizations of horospheres and boundary versions of the Schwarz lemma. When trying to extend the results obtained for self-maps of $\mathbb{B}^n$, the major difficulty is to find an equivalent definition of the objects which have been involved so far, namely horospheres and angular regions. For the horospheres, the correct approach has to follow the guidelines given by

PROPOSITION 43. *Let* $\tau \in \partial\Delta$ *and* $R > 0$. *Then the horocycle of centre* τ *and radius* R *can be equivalently defined as follows*:

$$E(\tau, R) = \left\{ z \in \Delta : \lim_{w\to\tau} [\omega_\Delta(z, w) - \omega_\Delta(0, w)] < \frac{1}{2}\log R \right\}.$$

Since the horospheres in $\mathbb{B}^n$ also have the (analogous) characterization described in the previous proposition (see [1]), we are led to

DEFINITION 44. Let $z_0 \in \mathbb{B}^n$. A *horosphere* of centre $x \in \partial\mathbb{B}^n$ and radius $R > 0$ is defined as

$$E_{\mathbb{B}^n}(x, z_0, R) = \left\{ z \in \mathbb{B}^n : \lim_{w\to x} [k_{\mathbb{B}^n}(z, w) - k_{\mathbb{B}^n}(z_0, w)] < \frac{1}{2}\log R \right\},$$

where $k_{\mathbb{B}^n}$ is the Kobayashi distance in $\mathbb{B}^n$. The point z_0 is called the *pole* of the horosphere.

REMARK 45. The above definition and the results contained in the previous section can also be stated for strongly convex domains with smooth boundary (see [41]).

3.1. *The case of polydiscs.* The polydisc $\Delta^n = \Delta \times \cdots \times \Delta$ is a convex domain, not biholomorphic to $\mathbb{B}^n$, of considerable interest: it was used as a first natural generalization of the unit disc to the case of several complex variables (see e.g. Rudin [42]). Notice that if $z \in \Delta^n$ then $e^{i\vartheta} z \in \Delta^n$ for all ϑ, so that Δ^n is an example of a circular domain.

In this subsection we state some results about the boundary behaviour of holomorphic self-maps of the polydisc. To avoid technical complications, without loss of generality we restrict ourselves to the case of the bidisc.

Let us denote by Δ^2 the unit bidisc of $\mathbb{C}^2$ and by $\partial\Delta^2$ its boundary. Let $x \in \partial\Delta^2$ and let $\Psi \in \mathrm{Hol}(\Delta, \Delta^2)$ be a complex geodesic passing through x.

Definition 46. The *Busemann function* $B^\Psi(z)$ associated to the geodesic Ψ is defined by

$$B^\Psi(z) := \lim_{r\to 1^-} [k_{\Delta^2}(z, \Psi(r)) - k_{\Delta^2}(\Psi(0), \Psi(r))]$$

(see e.g. [5, p. 23]).

Definition 47. The *Busemann sublevel set* of centre $x \in \partial\Delta^2$ and radius $R > 0$ of the function $B^\Psi(z)$ is the set

$$\mathbb{B}^\Psi(y, R) := \left\{x \in \Delta^2 : B_\Psi(x) \le \frac{1}{2}\log R\right\}. \tag{3.1}$$

In [17], the above definition is used to generalize the notion of horosphere in the polydisc (see also [2, 16]). In particular, with this approach, any Busemann sublevel set is a product of horocycles. Namely given $y = (y_1, y_2) \in \partial\Delta^2$ (with no restriction, we may assume $|y_1| = 1$), let $\varphi(z) = (z, g(z))$ be a complex geodesic in Δ^2 passing through y (such a complex geodesic with this particular parametrization exists in Δ^2 but may not be unique, see [1]) and consider the associated Busemann sublevel set $\mathbb{B}^\varphi(y, R)$ of radius $R > 0$ and centre y. If $|y_2| = 1$ (i.e. if y belongs to the Shilov boundary of Δ^2), then (see [17])

$$\mathbb{B}^\varphi(y, R) = E(y_1, R) \times E(y_2, \lambda_g \cdot R) \tag{3.2}$$

where λ_g is the boundary dilatation coefficient of g at y_1, with the convention that if λ_g is not finite, then $E(y_2, \lambda_g \cdot R) = \Delta$. Otherwise, if $|y_2| < 1$, then

$$\mathbb{B}^\varphi(y, R) = E(y_1, R) \times \Delta. \tag{3.3}$$

We will denote by $\mathbb{B}_{(\lambda_1,\lambda_2)}(y, R)$ (with $\lambda_1, \lambda_2 > 0$, possibly ∞) the Busemann sublevel set given by $E(y_1, \lambda_1 R) \times E(y_2, \lambda_2 R)$.

Notice that, with the above notation, if $\varphi_0(z) = z\cdot y$ is a complex geodesic in Δ^2 then the corresponding Busemann sublevel set $\mathbb{B}^{\varphi_0}(y, R)$ is nothing else than the small horosphere centred at y and of radius R, according to [2]; indeed, in [2] the *small horosphere* of centre y and radius $R > 0$ is defined in Δ^n as the set

$$E(y, R) = \left\{z \in \Delta^n : \limsup_{w\to y} [k_{\Delta^n}(z, w) - k_{\Delta^n}(0, w)] < \frac{1}{2}\log R\right\}.$$

In the same paper, and for other purposes related to dynamics, the author also introduces the *big horosphere* of centre y and radius R as the set

$$F(y, R) = \left\{z \in \Delta^n : \liminf_{w\to y} [k_{\Delta^n}(z, w) - k_{\Delta^n}(0, w)] < \frac{1}{2}\log R\right\}.$$

In this setting, in [17] the following version of the Julia lemma is proved:

Proposition 48. *Let* $f = (f_1, f_2) \in \mathrm{Hol}(\Delta^2, \Delta^2)$. *Let* $x = (x_1, x_2) \in \partial(\Delta \times \Delta) = \partial\Delta^2$ *and let (for example)* $\varphi_g(z) = (z, g(z))$ *be a complex geodesic passing through* x. *Let*

$$\frac{1}{2}\log\lambda_j := \lim_{t\to 1^-}[k_{\Delta^2}(0, \varphi_g(tx_1)) - \omega(0, f_j(\varphi_g(tx_1))], \qquad j = 1, 2.$$

Suppose that either $\lambda_1 < \infty$ *or* $\lambda_2 < \infty$. *Then there exists a point* $y = (y_1, y_2) \in \partial\Delta^2$ *such that for all* $R > 0$,

$$f(\mathbb{B}_{(1,\lambda_g)}(x, R)) \subseteq \mathbb{B}_{(\lambda_1,\lambda_2)}(y, R).$$

This approach is useful also in the generalization of the notion of non-tangential limit (see [17]):

Definition 49. Let $x \in \partial\Delta^2$ and $M > 1$. The *g-Korányi region* $H_{\varphi_g}(x, M)$ of vertex x and amplitude M is

$$H_{\varphi_g}(x, M) := \{z \in \Delta^2 : \lim_{r\to 1^-}[k_{\Delta^2}(z, \varphi_g(r)) - k_{\Delta^2}(\varphi_g(0), \varphi_g(r))] + k_{\Delta^2}(\varphi_g(0), z) < \log M\}.$$

A holomorphic function $f \in \mathrm{Hol}(\Delta^2, \Delta)$ has K_g*-limit* equal to $L \in \mathbb{C}$ if f approaches L inside any g-Korányi region.

Moreover it is also possible to generalize the definition of non-tangential limit along curves (see Definition 16). In order to do this we need some preliminary notation (see [17]). Let $x \in \partial\Delta^2$. A continuous curve $\sigma : (0, 1) \to \Delta^2$ is called an x-curve if $\sigma(t) \to x$ as $t \to 1^-$. Let $\varphi_g : \Delta \to \Delta^2$ be a complex geodesic passing through x and parameterized by $z \mapsto (z, g(z))$ with $g \in \mathrm{Hol}(\Delta, \Delta)$. A holomorphic function $\widetilde{\pi}_g : \Delta^2 \to \Delta$ such that $\widetilde{\pi}_g \circ \varphi_g = \mathrm{id}_\Delta$ is called a *g-left inverse* of φ_g. The composition $\pi_g := \varphi_g \circ \widetilde{\pi}_g : \Delta^2 \to \Delta^2$ (such that $\pi_g \circ \varphi_g = \varphi_g$, and $\widetilde{\pi}_g \circ \widetilde{\pi}_g = \widetilde{\pi}_g$) is called a *g-holomorphic retraction*. The pair (φ_g, π_g) is a *g-projection device*. In this setting, in [17] the following definition is given:

Definition 50. Let $\sigma : (0, 1) \to \Delta^2$ be an x-curve.

- The curve σ is *g-special* if $k_{\Delta^2}(\sigma(t), \pi_g(\sigma(t))) \to 0$ as $t \to 1^-$.
- The curve σ is *g-restricted* if $\widetilde{\pi}_g(\sigma(t)) \to \widetilde{\pi}_g(x)$ non-tangentially as $t \to 1^-$.

Moreover, if $h : \Delta^n \to \mathbb{C}$ is holomorphic we say that h has *restricted* K_g*-limit* equal to $L \in \mathbb{C}$ if h has limit L along any curve which is g-special and g-restricted, and we write

$$\widetilde{\mathrm{K}}_g\text{-}\lim_{z\to x} h(z) = L.$$

Finally, the following Julia–Wolff–Carathéodory type theorem can be proved (see [17]):

THEOREM 51. *Let $f \in \mathrm{Hol}(\Delta^2, \Delta^2)$ and $x \in \partial\Delta^2$. Let φ_g be any complex geodesic passing through x and parameterized by $\varphi_g(z) = (z, g(z))$, with $g \in \mathrm{Hol}(\Delta, \Delta)$. Let $\widetilde{\pi}_g : \Delta^2 \to \Delta$ be the g-left inverse of φ_g given by $\widetilde{\pi}_g(z_1, z_2) = z_1$. Suppose that for $j = 1, 2$,*

$$\frac{1}{2}\log\lambda_j = \lim_{t\to 1}[k_{\Delta^2}(0, \varphi_g(tx_1)) - \omega_\Delta(0, f_j(\varphi_g(tx_1)))] < \infty.$$

Then there exists a point $y = (y_1, y_2) \in \partial\Delta^2$ such that the restricted K_g-limit of f_j at x is y_j for $j = 1, 2$, and

$$\widetilde{\mathrm{K}}_g\text{-}\lim_{z\to x} \frac{y_j - f_j(z)}{1 - \widetilde{\pi}_g(z)} = \lambda_j \min\{1, \lambda_g\},$$

$$\widetilde{\mathrm{K}}_g\text{-}\lim_{z\to x} \frac{y_j - f_j(z)}{1 - z_2} = \frac{\lambda_j}{\max\{1, \lambda_g\}}.$$

3.2. *The case of analytic polyhedra.* Let Ω be an analytic polyhedron in $\mathbb{C}^n$, that is, a subset Ω of $\mathbb{C}^n$ defined as follows:

$$\Omega = \{z \in \mathbb{C}^n : |\varrho_j(z)| < 1,\ j = 1, \ldots, n\},$$

where $\varrho_j : \mathbb{C}^n \to \mathbb{C}$ are holomorphic ([2]) in an open neighbourhood of $\overline{\Omega}$. Notice that every function ϱ_j defines a decomposition of Ω into level sets $L_{\varrho_j}(t) = \{z : \varrho_j(z) = t\} \cap \Omega$. The set of all connected components of these level sets is called the *characteristic decomposition*, denoted by ω_{ϱ_j}.

Special cases of analytic polyhedra are polydiscs, and recent results on the relationships between particular analytic polyhedra in $\mathbb{C}^2$ and the bidisc can be found in [18, 26, 27, 43]. In general an analytic polyhedron is not holomorphically equivalent to a strictly pseudoconvex domain (see [24]).

With the above notations, an analytic polyhedron Ω as a subset of $\mathbb{C}^n$ can also be regarded as the intersection of the sets $\Omega_j = \{z \in \mathbb{C}^n : |\varrho_j(z)| < 1\}$. Therefore the *defining functions* ϱ_j for the analytic polyhedron Ω can be considered as components of the map $\varrho : \Omega \to \Delta^n$, where $\varrho(z) = (\varrho_1(z), \ldots, \varrho_n(z))$.

As in the case of polydiscs, we say that a point $x \in \partial P$ belongs to the *Shilov boundary* of Ω if $\varrho_j(x) = 1$ for $j = 1, \ldots, n$.

In [15] a subclass $\mathcal{P}'$ of analytic polyhedra is introduced.

DEFINITION 52. A polyhedron Ω belongs to the class $\mathcal{P}'$ if

1. Ω is simply-connected;

([2]) One can easily extend the definition to analytic polyhedra in a complex manifold of dimension n.

2. each characteristic decomposition ω_i of Ω can be defined by a function ϱ_i analytic on a neighbourhood of $\overline{\Omega}$ and satisfying the following conditions:
   (a) for any $z', z'' \in \overline{\Omega}$ there is an i such that $\varrho_i(z') \neq \varrho_i(z'')$;
   (b) for each i and any $t \in \varrho_i(\overline{\Omega})$ the level set $\{z : \varrho_i(z) = t\} \cap \overline{\Omega}$ is connected.

It is then shown that, given two analytic polyhedra Ω and Ω' of class $\mathcal{P}'$, for any proper holomorphic map $F : \Omega \to \Omega'$ there exists a proper holomorphic map $f : \Delta^s \to \Delta^s$ such that $\varrho' \circ F = f \circ \varrho$. Furthermore F extends to a continuous map $\widetilde{F} : \overline{\Omega} \to \overline{\Omega'}$. Consider a holomorphic self-map F of an analytic polyhedron Ω of class $\mathcal{P}'$. Define $D := \varrho(\Omega)$. Notice that $\varrho : \Omega \to D$ is a biholomorphism. Clearly $D = \varrho_1(\Omega) \times \cdots \times \varrho_n(\Omega)$, and, from the definition, since $\varrho_j(\Omega) \simeq \Delta$ for all $j = 1, \ldots, n$, $D \simeq \Delta^n$ can be regarded as a product space.

Take a point x of the Shilov boundary of Ω and assume ([3]) that there exists a complex geodesic $\varphi : \Delta \to \Omega$ passing through x. Without restriction we may assume that $\varrho_j(x) = 1$ for all $j = 1, \ldots, n$. We will say that $z \in \mathbb{B}_\varphi(x, R)$ if and only if

$$\lim_{r\to 1^-} [k_\Omega(z, \varphi(r)) - k_\Omega(\varphi(0), \varphi(r))] < \tfrac{1}{2} \log R. \tag{3.4}$$

Since φ is a complex geodesic the limit in (3.4) exists and is equivalent to

$$\lim_{r\to 1^-} [k_\Omega(z, \varphi(r)) - \omega_\Delta(0, r)] < \tfrac{1}{2} \log R. \tag{3.5}$$

PROPOSITION 53. *With the notations and assumptions given above*,

$$\mathbb{B}_\varphi(x, R) = \prod_{j=1}^{n} E(1, \beta_{\varrho_j\circ\varphi} R),$$

where $\beta_{\varrho_j\circ\varphi}$ is the boundary dilatation coefficient of $\varrho_j \circ \varphi : \Delta \to \Delta$.

Proof. From the definitions of the objects involved and since $\varrho_j(\Omega) \simeq \Delta$, we infer that $z \in \mathbb{B}_\varphi(x, R)$ if and only if

$$\begin{aligned}
\frac{1}{2}\log R &> \lim_{r\to 1^-} [k_\Omega(z, \varphi(r)) - \omega_\Delta(0, r)] > \lim_{r\to 1^-} [k_\Omega(z, \varphi(r)) - \omega_\Delta(0, r)] \\
&= \lim_{r\to 1^-} \{\max_j [\omega_{\varrho_j(\Omega)}(\varrho_j(z), \varrho_j(\varphi(r)))] - \omega_\Delta(0, r)\} \\
&= \lim_{r\to 1^-} [\omega_\Delta(\varrho_j(z), \varrho_j(\varphi(r))) - \omega_\Delta(0, \varrho_j(\varphi(r))) + \omega_\Delta(0, \varrho_j(\varphi(r))) - \omega_\Delta(0, r)] \\
&= \lim_{w\to 1} [\omega_\Delta(\varrho_j(z), w) - \omega_\Delta(0, w)] - \frac{1}{2}\log \beta_{\varrho_j\circ\varphi}.
\end{aligned}$$

([3]) Such a complex geodesic exists if for instance Ω is convex (see [1]).

In other words, for any $j = 1, \dots, n$,

$$\frac{1}{2}\,\beta_{\varrho_j\circ\varphi}\log R > \lim_{w\to 1}\omega_\Delta[(\varrho_j(z), w) - \omega_\Delta(0, w)],$$

or

$$\varrho_j(z) \in E(1, \beta_{\varrho_j\circ\varphi}R),$$

that is,

$$\varrho(z) \in \prod_{j=1}^{n} E(1, \beta_{\varrho_j\circ\varphi}R). \ \blacksquare$$

Notice that if $z \in \mathbb{B}_\varphi(x, R)$ then $\varrho_j(z) \in E(1, \beta_{\varrho_j\circ\varphi}R)$, but not conversely, since in general $\varrho_j(\Omega)$ is not biholomorphic to Δ.

The crucial correspondence between Busemann sublevel sets in an analytic polyhedron and the product of horocycles in Δ^n found in Proposition 53 allows one to immediately transfer to polyhedra most of the results found for polydiscs. In particular, since a holomorphic self-map of an analytic polyhedron of class $\mathcal{P}'$ maps a Busemann sublevel set into another Busemann sublevel set, a version of the Julia lemma holds in this setting. Analogously a version of Theorem 51 can also be stated.

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