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$\overline{\partial}$-cohomology and geometry of the boundary of pseudoconvex domains

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Abstract. In 1958, H. Grauert proved: If D is a strongly pseudoconvex domain in a complex manifold, then D is holomorphically convex. In contrast, various cases occur if the Levi form of the boundary of D is everywhere zero, i.e. if ∂D is Levi flat. A review is given of the results on the domains with Levi flat boundaries in recent decades. Related results on the domains with divisorial boundaries and generically strongly pseudoconvex domains are also presented. As for the methods, it is explained how Hartogs type extension theorems and L^2 finiteness theorem for the $\overline{\partial}$-cohomology are applied.

Introduction. A fact lying at the basis of complex analysis of several variables is that not every odd-dimensional orientable real manifold is realized as the boundary of a Stein space. For instance, a real $(2n-1)$-dimensional torus bounds a Stein space if and only if $n \leq 2$ by a topological reason (cf. [O-2]).

On the other hand, although local differential geometric studies of the boundaries of strongly pseudoconvex domains are well developed (cf. [Tn], [C-M], [W]), relatively few things are known about their global geometric properties. So it makes sense to study global questions of the boundaries of pseudoconvex domains from various viewpoints (cf. [H]).

The purpose of the present note is to give a short expository account on our recent results [O-6–12] as examples of such trials. Here we shall be mainly concerned with certain flatness properties of the boundaries of pseudoconvex domains in complex manifods.

As the research works directly foregoing ours, we would like to mention [D-O-1,2,3], [LN], [Ne] and [U-2,3].

If §1 we shall review some generalizations of the Hodge theory and Kodaira's vanishing theorem to complete Kähler manifolds, and deduce from them Hartogs type extension theorems on weakly 1-complete mani-

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folds. In §2, after recalling the motivating examples and theorems, we shall make an overview of our results with outlines of their proofs based on the facts presented in §1. Finally, in §3 we shall sketch how a finiteness of L^2 $\bar\partial$-cohomology (cf. §1, Theorem 1.3) is applied to show the nonexistence of Levi flat components of certain domains with pseudoconvex boundary.

1. $\bar\partial$-cohomology on complete Kähler manifolds. Let M be a complex manifold of dimension n and let E be a holomorphic vector bundle over M. The rank of E may be restricted to one for the later applications.

Let $C^{p,q}(M,E)$ denote the space of C^∞ E-valued (p,q)-forms on M and let $\bar\partial : C^{p,q}(M,E) \to C^{p,q+1}(M,E)$ be the complex exterior derivative of type $(0,1)$. The E-valued $\bar\partial$-cohomology group of M of type (p,q), which will be denoted by $H^{p,q}(M,E)$, is by definition the kernel of $\bar\partial : C^{p,q}(M,E) \to C^{p,q+1}(M,E)$ modulo the image of $\bar\partial : C^{p,q-1}(M,E) \to C^{p,q}(M,E)$.

We put

$$C_0^{p,q}(M,E) = \{u \in C^{p,q}(M,E) \mid \operatorname{supp} u \text{ is compact}\}$$

and denote the kernel of $\bar\partial : C_0^{p,q}(M,E) \to C_0^{p,q+1}(M,E)$ modulo the image of $\bar\partial : C_0^{p,q-1}(M,E) \to C_0^{p,q}(M,E)$ by $H_0^{p,q}(M,E)$.

Then there exists a canonical long exact sequence

$$\begin{aligned}(1)\qquad \cdots \to H_0^{p,q}(M,E) \to H^{p,q}(M,E) &\to \varinjlim H^{p,q}(M\setminus K,E)\\ &\to H_0^{p,q+1}(M,E) \to H^{p,q+1}(M,E) \to \cdots .\end{aligned}$$

Here $\varinjlim$ denote the inductive limit with respect to the natural restriction homomorphisms

$$H^{p,q}(M\setminus K,E) \to H^{p,q}(M\setminus K',E) \quad (K \subset K')$$

where K runs through the compact subsets of M.

From (1) we infer that the restriction homomorphism

$$H^{p,q}(M,E) \to \varinjlim H^{p,q}(M\setminus K,E)$$

is surjective if and only if the homomorphism $\iota : H_0^{p,q+1}(M,E) \to H^{p,q+1}(M,E)$ induced from the inclusion $C_0^{p,q+1}(M,E) \to C^{p,q+1}(M,E)$ is injective.

It is known that ι is injective if (M,E) satisfies some differential geometric conditions. Such results are important for our purposes.

From now on we assume that M and E are equipped with a complete Kähler metric g and a C^∞ Hermitian fiber metric h. Let ω be the fundamental form of g. By an abuse of language, we shall also call ω a Kähler metric.

By identifying h with a section of the bundle $\mathrm{Hom}(E,\overline{E}^*)$, where $\overline{E}$ denotes the conjugate of E and $\overline{E}^*$ the dual to $\overline{E}$, we put $\partial_h = h^{-1}\circ\partial\circ h$. Here ∂ denotes the complex exterior derivative of type $(1,0)$. Recall that the curvature form Θ_h of h is by definition an element of $C^{1,1}(M, \mathrm{Hom}(E,E))$ which satisfies

$$\Theta_h \wedge u = (\overline{\partial}\partial_h + \partial_h\overline{\partial})u \tag{2}$$

where u runs through the C^∞ E-valued forms on M. (E,h) is said to be *flat* if $\Theta_h = 0$.

Let $T^{1,0}M$ (resp. $T^{0,1}M$) denote the holomorphic (resp. antiholomorphic) tangent bundle of M.

We put

$$\widetilde{\Theta}_h = (\mathrm{Id}_{(T^{0,1}M)^*} \otimes h) \circ \Theta_h$$

by identifying Θ_h with an element of $C^{0,0}(M, \mathrm{Hom}(T^{1,0}M \otimes E, (T^{0,1}M)^* \otimes E))$. Then $\widetilde{\Theta}_h$ is a Hermitian form on the fibers of $T^{1,0}M \otimes E$. We say (E,h) is *Nakano semipositive* (resp. *Nakano seminegative*) if $\widetilde{\Theta}_h$ is semipositive (resp. seminegative) everywhere.

If E is the trivial bundle $M \times \mathbb{C}$ and $h(z)(\zeta,\zeta) = e^{-\psi(z)}|\zeta|^2$ for $(z,\zeta) \in E$, for some real-valued C^∞ function ψ on M, we have $\Theta_h = \partial\overline{\partial}\psi$. In this case (E,h) is flat (resp. Nakano semipositive) if and only if ψ is pluriharmonic (resp. plurisubharmonic). By an abuse of notation, we shall identify $\partial\overline{\partial}\psi$ with the Levi form, i.e. the complex Hessian of ψ.

Let ϑ_h (resp. $\overline{\vartheta}$) denote the formal adjoint of $\overline{\partial}$ (resp. ∂_h) with respect to g and h.

An element $u \in C^{p,q}(M,E)$ is said to be *harmonic* if u is square integrable with respect to g and h and satisfies the equations $\overline{\partial}u = 0$ and $\vartheta_h u = 0$.

We put

$$\mathcal{H}^{p,q}(M,E) = \{u \in C^{p,q}(M,E) \mid u \text{ is harmonic}\}.$$

The space $\mathcal{H}^{p,q}(M,E)$ depends on the choice of g and h in general, but in some cases it is canonically isomorphic to the space $H^{p,q}(M,E)$ by the homomorphism induced from the inclusion map (cf. [O-3]). For our purpose here, the following specialized form is sufficient.

THEOREM 1.1. *Let (M,g) and (E,h) be as above. Suppose that there exists a C^∞ plurisubharmonic exhaustion function φ on M whose Levi form has everywhere at least $n-k+1$ positive eigenvalues outside a compact subset of M. Then there exists a C^∞ convex increasing function λ on $\mathbb{R}$ such that the space $\mathcal{H}^{n,q}(M,E)$ with respect to the metrics g and $he^{-\lambda(\varphi)}$ is canonically isomorphic to $H^{n,q}(M,E)$ if $q \geq k$.*

For the proof of the case where rank $E = 1$ and $\widetilde{\Theta}_h > 0$ outside a compact subset of M, see [O-1]. The general case is similar.

REMARK. A theory with a scope wider than Theorem 1.1 was developed in [O-3, Chap. 2] to describe a condition for the isomorphism $\mathcal{H}^{p,q}(M,E) \simeq$

$H^{p,q}(M,E)$, by combining the methods of Andreotti–Vesentini [A-V] and Hörmander [Hö].

Definition 1.1. M is said to be *weakly* 1-*complete* if M is equipped with a C^∞ plurisubharmonic exhaustion function $\varphi : M \to \mathbb{R}$.

Theorem 1.2. *Let* (M,φ) *be a noncompact and connected weakly* 1*-complete manifold equipped with a complete Kähler metric* g, *and let* (E,h) *be a Nakano semipositive vector bundle over* M. *If* $\partial\overline{\partial}\varphi$ *has everywhere at least* $n-k+1$ *positive eigenvalues outside a compact subset of* M, *then*

$$H^{n,q}(M,E)=0 \quad \text{for } q \geq k. \tag{3}$$

Sketch of proof (see also [G-R] and [T]). By Theorem 1.1 there exists a C^∞ convex increasing function λ such that $H^{n,q}(M,E)$ is isomorphic to $\mathcal{H}^{n,q}(M,E)$ with respect to g and $he^{-\lambda(\varphi)}$. By the Nakano semipositivity of (E,h) and the assumption on $\partial\overline{\partial}\varphi$, it is easy to see from a Bochner–Weitzenböck type formula of Nakano (cf. [N]) that each element α of $\mathcal{H}^{n,q}(M,E)$ $(q \geq k)$ is zero outside a compact subset of M. Since M is noncompact and connected, α is identically zero on M by Aronszajn's unique continuation theorem (cf. [A]).

The equality (3) means that the $\overline{\partial}$-equation $\overline{\partial}u = v$ is solvable for any C^∞ $\overline{\partial}$-closed E-valued (n,q)-form v on M.

If $\dim H^{n,q}(M,E) < \infty$, then for any finitely many $\overline{\partial}$-closed C^∞ E-valued forms v_j $(j=1,\ldots,m)$ with $m > \dim H^{n,q}(M,E)$, there exists a nontrivial linear combination $\sum c_j v_j$ $(c_j \in \mathbb{C})$ for which the equation

$$u = \sum_{j=1}^{m} c_j v_j$$

is solvable. We shall later apply such a weak solvability in a refined form. To state such a refined variant of the finite-dimensionality of $H^{n,q}(M,E)$, let us recall the notion of the L^2 $\overline{\partial}$-cohomology.

Let $L^{p,q}_{(2)}(M,E)$ denote the space of measurable E-valued (p,q)-forms on M which are square integrable with respect to g and h. For any $u \in L^{p,q}_{(2)}(M,E)$ the derivative $\overline{\partial}u$ is defined in the distribution sense. Then the E-valued L^2 $\overline{\partial}$*-cohomology group* $H^{p,q}_{(2)}(M,E)$ is defined as the space of $\overline{\partial}$-closed elements of $L^{p,q}_{(2)}(M,E)$ modulo the subspace

$$\{\overline{\partial}w \in L^{p,q}_{(2)}(M,E) \mid w \in L^{p,q-1}_{(2)}(M,E)\}.$$

Theorem 1.3 (cf. [O-10, Corollary 1.2]). *Let* (M,g) *be a connected complete Kähler manifold of dimension* n *and let* (E,h) *be a holomorphic Hermitian line bundle over* M. *Assume that there exists a compact set* $K \subset M$ *such that on each connected component of* $M \setminus K$ *either* $i\Theta_{dV\otimes h} \geq \omega$ *everywhere* *or* $-i\Theta_h \geq \omega$ *everywhere. Then* $\dim H^{0,q}_{(2)}(M,E) < \infty$ *for* $1 \leq q \leq n-1$. *Here* dV *denotes the volume form of* g *as a fiber metric of the anticanonical bundle* $K_M^* = \bigwedge^n T^{1,0}M$.

Now let us recall the Hartogs type extension theorems.

THEOREM 1.4. *Let* (M,φ) *and* (E,h) *be as in Theorem* 1.3. *Then*

$$H_0^{0,q}(M,E^*) = 0 \quad \text{for } q \leq n-k. \tag{4}$$

Proof. By Serre's duality theorem we obtain (4) from (3).

COROLLARY 1.1. *In the above situation, the natural restriction homomorphisms*

$$H^{0,q}(M,E^*) \to \varinjlim H^{0,q}(M \setminus K, E^*)$$

are surjective for $q \leq n-k-1$.

From now on we assume that (E,h) is a flat vector bundle. Then the exterior multiplication by ω^m induces a homomorphism

$$\omega^m : \mathcal{H}^{p-m,q-m}(M,E) \to \mathcal{H}^{p,q}(M,E)$$

which is bijective if $p+q = n+m$ (the Lefschets isomorphism).

Combining this classical fact with a harmonic representability as in Theorem 1.1, we obtain the following.

THEOREM 1.5 (cf. [O-2], [O-T], [D]). *Let* (M,φ) *be a weakly* 1*-complete manifold of dimension* n *equipped with a Kähler metric* g, *and let* (E,h) *be a flat Hermitian vector bundle over* M. *Assume that* $\partial\overline{\partial}\varphi$ *has at least* $n-k+1$ *positive eigenvalues outside a compact subset of* M. *Then* ω^m *induces an isomorphism between* $H_0^{p-m,q-m}(M,E)$ *and* $H^{p,q}(M,E)$ *for* $1 \leq m \leq p+q-k+1$.

COROLLARY 1.2. *In the above situation, the natural homomorphism*

$$H_0^{p,q}(M,E) \to H^{p,q}(M,E)$$

is surjective if $p+q \geq n+k$.

By the Serre duality we obtain

COROLLARY 1.3. *In the above situation, the natural homomorphism*

$$H_0^{p,q}(M,E^*) \to H^{p,q}(M,E^*)$$

is injective if $p+q \leq n-k$.

Thus we obtain the following Hartogs type extension theorem.

THEOREM 1.6. *Let* (M,φ) *and* (E,h) *be as in Theorem* 1.5. *Then the restriction homomorphism*

$$H^{p,q}(M,E^*) \to \varinjlim H^{p,q}(M \setminus K, E^*)$$

is surjective if $p+q \leq n-k-1$.

2. Domains with Levi flat or divisorial boundary. Let D be a relatively compact domain in a complex manifold X. H. Grauert [G-1] showed that D is holomorphically convex if ∂D is C^2-smooth and everywhere strongly pseudoconvex. From this it follows immediately that D is holomorphically convex if ∂D is the support of an effective divisor A such that $[A]|\partial D$ is ample. Here an *effective divisor* is by definition a formal linear combination of finitely many 1-codimensional subvarieties with nonnegative integral coefficients, and its support is the union of these subvarieties. $[A]$ denotes the holomorphic line bundle associated to the divisor A. (D is actually bimeromorphically equivalent to a Stein space with isolated singularities under such conditions.)

We want to know what happens if, in contrast with the above situation, one of the following conditions is satisfied.

1) ∂D locally admits pluriharmonic defining functions.
2) ∂D is the support of an effective divisor A such that $[A]|\partial D$ is topologically trivial.

A compact real hypersurface $S \subset X$ of class C^2 is said to be *Levi flat* if the holomorphic tangent vectors of S are annihilated by the Levi form of the defining function of S. If S is real-analytic, then it is clear that S is Levi flat if and only if it is locally defined by pluriharmonic functions.

We shall assume throughout this section that ∂D is either a Levi flat real-analytic hypersurface or a complex-analytic subset of codimension one.

Here are some of the motivating examples:

(i) Let X be a complex 2-torus defined as $\mathbb{C}^2$ modulo

$$\mathbb{Z}\begin{pmatrix}1\\0\end{pmatrix}+\mathbb{Z}\begin{pmatrix}0\\1\end{pmatrix}+\mathbb{Z}\begin{pmatrix}i\\i\end{pmatrix}+\mathbb{Z}\begin{pmatrix}\sqrt{2}\,i\\i\end{pmatrix}$$

and let

$$D=\left\{\left[\begin{pmatrix}z\\w\end{pmatrix}\right]\in X\,\middle|\,0<\operatorname{Re}z<1/2\right\}.$$

Then (X, D) satisfies 1). It is easy to see that D admits no nonconstant holomorphic functions (cf. [G-2]).

(ii) Let $X = \mathbb{P}^1 \times (\mathbb{C}^*/\mathbb{Z})$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and the action of $\mathbb{Z}$ on $\mathbb{C}^*$ is defined by $m \cdot z = e^m z$ for $m \in \mathbb{Z}$ and $z \in \mathbb{C}^*$, and let $D = \{(\zeta, [z]) \in X \mid \operatorname{Re}(\zeta z) > 0\}$. Then (X, D) satisfies 1). It is easy to see that D is biholomorphically equivalent to the product of $\mathbb{C}^*$ and the domain $e^{-2\pi^2} < |w| < 1$ in $\mathbb{C}$. In particular, D is a Stein manifold (cf. [O-4]).

(iii) Let C be a compact Riemann surface of genus ≥ 1 and let $B \to C$ be a holomorphic line bundle of degree 0 such that B^m is not trivial for any $m \in \mathbb{Z} \setminus \{0\}$. Let X be the compactification of B obtained by adding the section at infinity, and let $D = B$. Then (X, D) satisfies 2). It is easy to see that D is not holomorphically convex.

(iv) Let C be a compact Riemann surface of genus ≥ 1 and let $L \to C$ be a holomorphic affine line bundle of degree 0 which admits no holomorphic sections. Let X be the compactification of L obtained by adding the section at infinity, and let $D = L$. Then (X, D) satisfies 2). It is not difficult to see that D is a Stein manifold.

Questions arising from these examples are:

(a) Is there any natural criterion to distinguish the Stein and non-Stein (X, D) satisfying 1) or 2)?
(b) How do these examples carry over to more general cases?

As for these questions, an intensive study was done by T. Ueda [U-2,3] for case 2). According to [U-2], D is bimeromorphically equivalent to a Stein space if ∂D is a smooth complex curve whose neighbourhood in X is not formally equivalent to that of the zero section of the normal bundle of ∂D in X. For case 1), although no counterpart of the Ueda theory is known, it was proved that the disc bundles over compact Riemann surfaces arising from noncommutative representations of the fundamental groups to the automorphism group of the disc are bimeromorphically equivalent to Stein spaces (cf. [D-O-2]). Note that holomorphic disc bundles over compact complex manifolds are naturally regarded as domains with Levi flat boundaries in the associated $\mathbb{P}^1$-bundles.

The latter assertion is essentially contained in the following, which may provide some insight into the general case.

THEOREM 2.1 (cf. [D-O-2]). *Every holomorphic disc bundle over a compact Kähler manifold is weakly* 1*-complete.*

Sketch of proof. Let M be a compact Kähler manifold and $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Since the automorphism group of $\mathbb{D}$ contains no nontrivial complex Lie subgroup, every holomorphic $\mathbb{D}$-bundle $\mathcal{D} \to M$ has locally constant transition functions, so that $\mathcal{D}$ is associated with a homomorphism from $\pi_1(M)$, the fundamental group of M, to $\mathrm{Aut}(\mathbb{D})$, say ϱ. Then it is not difficult to see from the theory of Eells and Sampson [E-S] that $\mathcal{D}$ admits a harmonic section s if the image of ϱ is not commutative. Here $\mathbb{D}$ is endowed with the Poincaré metric. By a computation of Siu [S-1] we know that s is pluriharmonic, i.e. the restriction of s to any germ of complex curve in M is harmonic. It is then readily seen that the distance measured fiberwise from $s(M)$ is exhaustive and plurisubharmonic on $\mathcal{D} \setminus s(M)$, which suffices to conclude that $\mathcal{D}$ is weakly 1-complete in this case. If the image of ϱ is commutative, it is easy to see that $\mathcal{D}$ is weakly 1-complete by the Hodge theory on M.

Remark. The Kähler assumption on M cannot be removed, for there exists a $\mathbb{D}$-bundle over a Hopf manifold $\mathcal{H} = \mathbb{C}^n \setminus \{0\}/\mathbb{Z}$ $(n \geq 2)$ which is biholomorphically equivalent to the product of $\mathbb{C}^n \setminus \{0\}$ and an annulus as in the above example (ii) (cf. [D-F-2]).

From Theorem 2.1 and the above remark one might think that the less X is Kählerian, the less D becomes pseudoconvex. However, this is not the case because the Hopf manifold $\mathcal{H}$ contains a Stein domain $\Omega = \{[(z_1, \ldots, z_n)] \in \mathcal{H} \mid \operatorname{Re} z_1 > 0\}$ and the pair $(X, D) = (\mathcal{H}, \Omega)$ satisfies 1). This construction of the domain Ω turned out to be a special case of that of Stein domains in principal complex 1-torus bundles over projective algebraic manifolds (cf. [Ne] and [O-6, Supplement]).

As for domains in Kähler manifolds, we recently proved the following.

Theorem 2.2 (cf. [O-7]). *Let X be a compact Kähler manifold and let $D \subset X$ be a domain such that ∂D is a real-analytic Levi flat hypersurface in X. Then $X \setminus \partial D$ has no C^∞ plurisubharmonic exhaustion function whose Levi form has everywhere at least three positive eigenvalues outside a compact subset of D. In particular, $X \setminus \partial D$ is not Stein if* $\dim X \geq 3$.

Outline of proof. Suppose that there existed such an exhaustion function ψ on $X \setminus \partial D$. Then, since the local defining functions of ∂D can be chosen to be the real part of holomorphic functions by assumption, there exists a holomorphic function of codimension one, on a neighbourhood of ∂D, which is tangent to ∂D. Letting N be the normal bundle of this foliation say $\mathcal{F}$, one can find a holomorphic N-valued 1-form γ on a neighbourhood of ∂D such that the holomorphic tangent bundle of $\mathcal{F}$ is $\operatorname{Ker} \gamma$.

Since N is topologically trivial, we deduce from Corollary 1.1, applied for $(M, \varphi) = (X \setminus \partial D, \psi)$ and $E = (X \setminus \partial D) \times \mathbb{C}$, that there exists a topologically trivial holomorphic line bundle $\widetilde{N}$ over X which extends N.

Since X is Kählerian, $\widetilde{N}$ admits a flat structure. Hence, by applying Theorem 1.6 for $(M, \varphi) = (X \setminus \partial D, \psi)$ and $E = \widetilde{N}^* | X \setminus \partial D$, one can extend γ holomorphically to an $\widetilde{N}$-valued 1-form.

Therefore, $\mathcal{F}$ locally consists of the level sets of holomorphic functions F_α which satisfy the transition relations

$$F_\alpha = e^{i\theta_{\alpha\beta}} F_\beta + c_{\alpha\beta} \quad (\theta_{\alpha\beta} \in \mathbb{R},\ c_{\alpha\beta} \in \mathbb{C}). \tag{5}$$

It follows from (5) that there exist a neighbourhood $U \supset \partial D$ and a neighbourhood V of the diagonal of $U \times U$ such that $|F_\alpha(z) - F_\beta(w)|$ is a well defined continuous function on V, say $d(z, w)$.

Then we put

$$\delta(z) = \inf\{d(z, w) \mid (z, w) \in V \cap (U \times \partial D)\}$$

and choose a sufficiently small positive number ε so that $\delta^{-1}(\varepsilon)$ is compact.

Since ψ is continuous, there exists a point $z_0 \in \delta^{-1}(\varepsilon)$ where $\psi|\delta^{-1}(\varepsilon)$ attains its maximum. If we choose U sufficiently small in advance, then $\psi|F_\alpha^{-1}(F_\alpha(z_0))$ will attain its maximum at z_0.

But this contradicts the assumption that $\partial\overline{\partial}\psi$ has at least three positive eigenvalues near ∂D.

COROLLARY 2.1 (cf. [LN]). *The complex projective space $\mathbb{P}^n$ with $n \geq 3$ admits no real-analytic Levi flat hypersurfaces.*

Proof. If there were such a hypersurface S, then $\mathbb{P}^n \setminus S$ would be a Stein manifold (cf. [F] or [Tk]). But this contradicts Theorem 2.2.

NOTE. In view of the proof of Theorem 2.2, the smoothness assumption on ∂D is actually superfluous. It is easy to see that the same conclusion holds if ∂D is locally the zero of a pluriharmonic function.

REMARK. Similarly to Corollary 2.1, we can show that there exist no real-analytic Levi flat hypersurfaces in X if X is one of the following manifolds:

(i) Grassmann manifold of dimension ≥ 3 (cf. [U-1]).
(ii) A hypersurface of degree ≤ 3 in $\mathbb{P}^n$ with $n \geq 4$ (cf. [O-12]).
(iii) A complete intersection of type $(2,2)$ in $\mathbb{P}^n$ with $n \geq 5$ (cf. [O-12]).

Now we shall pursue an analogue of Theorem 2.2 for those (X, D) which satisfy 2).

First we note that the Kähler condition cannot be removed also in this case.

EXAMPLE. Let $X = (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ $(n \geq 2)$, where two points $z, w \in \mathbb{C}^n \setminus \{0\}$ are identified if and only if

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} e & & & & \\ & \ddots & & 0 & \\ & & e & & \\ & 0 & & e & e \\ & & & 0 & e \end{pmatrix}^m \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

for some $m \in \mathbb{Z}$. Then we put $D = \{[z] \in X \mid z_n = 0\}$. Then $[\partial D]|\partial D = \partial D \times \mathbb{C}$ and D is a Stein manifold because it is a covering space of $(\mathbb{C}^*)^n$ by

$$(z_1, \ldots, z_n) \mapsto (z_1/z_n, \ldots, z_{n-2}/z_n, e^{2\pi i z_{n-1}/z_n}, z_n e^{-z_{n-1}/z_n}).$$

Therefore the counterpart of Theorem 2.2 is

THEOREM 2.3. *Let X be a compact Kähler manifold and let D be a domain in X. Suppose that ∂D is a complex-analytic subset of codimension one and there exists an effective divisor A with support ∂D such that the line* *bundle* $[A]|\partial D$ *is topologically trivial. Then* D *admits no* C^∞ *pluriharmonic exhaustion function whose Levi form has everywhere at least three positive eigenvalues outside a compact subset of* D. *In particular*, D *is not Stein if* $\dim X \geq 3$.

For the proof, see [O-11].

Theorem 2.3 tells us that the complement of an effective divisor with topologically trivial normal bundles can be pseudoconvex only in a very restrictive way. In that sense, the following result of B. Totaro detects a similar phenomenon.

THEOREM 2.4 (cf. [To]). *Let* X *be a smooth complex projective variety. Let* $A_1, \ldots, A_r$, $r \geq 3$, *be connected effective divisors* (*not zero*) *that are pairwise disjoint and whose rational cohomology classes lie in a line in* $H^2(X, \mathbb{Q})$. *Then there is a map* $f : X \to C$ *with connected fibers to a smooth curve* C *such that* $A_1, \ldots, A_r$ *are all positive rational multiples of fibers of* f. *In fact*, *there is only one map* f *with these properties.*

As works done in a similar spirit, we also mention the results of Napier and Ramachandran [N-R-1,2] concerning the ends of weakly 1-complete Kähler manifolds.

Finally, we would like to make additional comments on 2-dimensional domains with Levi flat boundaries.

(i) There exists a Kummer surface which contains a domain D such that ∂D is Levi flat and D is bimeromorphically equivalent to a Stein space (cf. [O-8]).

(ii) It remains open whether there exists no real analytic Levi flat hypersurfaces in $\mathbb{P}^2$. (The proofs that appeared in [O-5], [S-3], [I] and [C-S-W] are not complete.) A similar nonexistence question arises in classifying real-analytic Levi flat hypersurfaces in complex tori which is also open (cf. [O-6,9]). Nonexistence in $\mathbb{P}^2$ will follow from the nonexistence of Levi flat surfaces in 2-tori which do not contain any complex line segments.

3. Generically strongly pseudoconvex domains. Let (X, D) be as in §2. We shall supplement Grauert's theorem and the results in §2 by adding a remark on the intermediate case.

DEFINITION. ∂D is said to be *generically strongly pseudoconvex at* $x_0 \in \partial D$ if ∂D is C^2-smooth at x_0 and x_0 is contained in the closure of the set of strongly pseudoconvex points of ∂D. D is said to be *generically strongly pseudoconvex* if ∂D is everywhere generically strongly pseudoconvex.

Generically strongly pseudoconvex domains arise naturally as the worm domains of Diederich–Fornæss type (cf. [D-F-1], [D-O-1,3]).

We note that D is generically strongly pseudoconvex if ∂D is connected, real-analytic, pseudoconvex and generically strongly pseudoconvex at some point.

In [D-O-1] the following was proved.

THEOREM 3.1. *Let D be a relatively compact domain in a two-dimensional complex manifold X. If ∂D is connected, real-analytic, pseudoconvex and strongly pseudoconvex at some point, then D is holomorphically convex.*

It is easy to see from the proof of Theorem 3.1 that one can replace the assumption by the real-analytic everywhere generic strong pseudoconvexity of ∂D, dropping the connectedness assumption. A question is whether or not ∂D may have both generically pseudoconvex and Levi flat components.

Concerning this point, we recently obtained the following.

THEOREM 3.2. *Let X be a connected complex manifold of dimension 2 which admits a nonconstant meromorphic function, and let $D \subset X$ be a relatively compact domain with real-analytic pseudoconvex boundary. If ∂D is strongly pseudoconvex at some point, then D is generically strongly pseudoconvex.*

Sketch of proof. Suppose that the union of the Levi flat components of ∂D, say $\partial_0 D$, were nonempty. Then, for any nonconstant meromorphic function f on X, the line bundle $L = [f^{-1}(0)]$ admits a fiber metric h whose curvature form Θ is positive on a neighbourhood of $\partial_0 D$.

Similarly to [D-O-1] one can find a C^∞ exhaustion function ψ on D and a C^∞ convex increasing function λ on $\mathbb{R}$ with the following properties.

(i) There exists a complete Hermitian metric g on D whose fundamental form ω coincides with $i(\Theta+\partial\overline{\partial}\psi)$ outside a compact subset of D such that $\omega^2 > \varrho^{-2}(i\Theta)^2$ near $\partial_0 D$ for some defining function ϱ of ∂D.

(ii) There exists a neighbourhood $W \supset \partial D \setminus \partial_0 D$ such that

$$i\Theta_{dV\otimes h^{-1}\exp(-\lambda(\psi))} > \omega \quad \text{and} \quad \int_{W\cap D} e^{-\lambda(\psi)}\omega^2 < \infty.$$

Then, by applying Theorem 1.3 to $E = L^*$ with fiber metric $h^{-1}\exp(-\lambda(\psi))$ near $\partial D \setminus \partial_0 D$ and $h^{-1}e^{\psi}$ near $\partial_0 D$, one has the finite-dimensionality of $H^{0,1}_{(2)}(D, L^*)$. Hence, by exploiting the existence of a strongly pseudoconvex boundary point in $\partial D \setminus \partial_0 D$, one can produce as in [G-1] a nonzero holomorphic section of L^* over D which is square integrable on $U \cap D$ with respect to ω^2 for some neighbourhood $U \supset \partial_0 D$. But this contradicts the estimate $\omega^2 > \varrho^{-2}(i\Theta)^2$.

OPEN QUESTION. Can one remove the assumption on the existence of nonconstant meromorphic functions on X from Theorem 3.2?

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