On the five-point theorems due to Lappan

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Abstract. By using an extension of the spherical derivative introduced by Lappan, we obtain some results on normal functions and normal families, which extend Lappan's five-point theorems and Marty's criterion, and improve some previous results due to Li and Xie, and the author. Also, another proof of Lappan's theorem is given.

1. Introduction. Let $\Delta = \{z : |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$. A function f meromorphic in Δ is called *normal*, in the sense of Lehto and Virtanen [5], if there exists a constant $M > 0$ such that

$$(1) \qquad\qquad (1 - |z|^2)f^\#(z) \le M$$

for each $z \in \Delta$, where $f^\#(z) = |f'(z)|/(1+|f(z)|^2)$ is the spherical derivative of f.

Let D be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined in D. The family $\mathcal{F}$ is said to be *normal* in D, in the sense of Montel, if any sequence in $\mathcal{F}$ has a subsequence that converges spherically locally uniformly in D to a meromorphic function or ∞ (see [1, 8, 10]).

Marty's criterion [7] asserts that a family $\mathcal{F}$ of meromorphic functions defined in D is normal if and only if for each compact subset K of D there exists a constant $M(K) > 0$ such that

$$(2) \qquad\qquad f^\#(z) \le M(K)$$

for each $f \in \mathcal{F}$ and each $z \in K$.

Lappan [2, 4] reduced drastically the set on which inequality (1) or (2) is required, and obtained the following two results, which are called the *Lappan five-point theorems* relating to normal functions and normal families respectively.

THEOREM A ([2]). *Let f be a meromorphic function in Δ. If there exist a subset E of $\mathbb{C} \cup \{\infty\}$ containing at least five distinct points and a constant*

$M(E) > 0$ *such that*

$$(1 - |z|^2)f^{\#}(z) \leq M(E)$$

for each $z \in \Delta \cap f^{-1}(E)$, *then* f *is a normal function.*

THEOREM B ([4]). *Let* $\mathcal{F}$ *be a family of meromorphic functions defined in a domain* D, *and let* E *be a subset* E *of* $\mathbb{C} \cup \{\infty\}$ *containing at least five distinct points. If for each compact set* K *of* D *there exists a constant* $M(K)$ *such that*

$$f^{\#}(z) \leq M(K) \quad \text{for each } f \in \mathcal{F} \text{ and } z \in K \cap f^{-1}(E),$$

then $\mathcal{F}$ *is normal in* D.

Let k be a positive integer. The expression

$$\frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}}$$

can be viewed as an extension of the spherical derivative of f, which is introduced by Lappan (see [3]). This expression proves useful in connection with normal functions and normal families. In [3], Lappan proved

THEOREM C ([3]). *If* f *is a normal meromorphic function in* Δ, *then, for each positive integer* k, *there exists a constant* M_k *such that*

$$(1 - |z|^2)^k \frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}} \leq M_k \quad \text{for each } z \in \Delta.$$

For normal families, Li and Xie [6] obtained

THEOREM D ([6]). *Let* k *be a positive integer, and let* $\mathcal{F}$ *be a family of meromorphic functions defined in a domain* D, *all of whose zeros have multiplicity at least* k. *Then* $\mathcal{F}$ *is normal in* D *if and only if for each compact set* K *of* D *there exists a constant* $M(K)$ *such that*

$$\frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}} \leq M(K) \quad \text{for each } f \in \mathcal{F} \text{ and } z \in K.$$

REMARK 1. In fact, the condition "all zeros of $f \in \mathcal{F}$ have multiplicity at least k" is not required in the necessity part of Theorem D, which can be seen from the proof in [6]. The proofs (of Theorem C in [3] and Theorem D in [6]) are completely different. In Section 3 below, we shall give another proof of Theorem C by borrowing an idea from Li and Xie [6].

Lappan [3, Question 3] also asked: *Is the converse of Theorem C true, that is, if there exist* $k > 1$ *and a constant* M *such that*

$$(1 - |z|^2)^k \frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}} \leq M \quad \text{for each } z \in \Delta,$$

is f *a normal meromorphic function?*

The case $k = 1$ was excluded since the answer is obviously affirmative by the definition of a normal function.

In [9], the present author obtained a partial answer to the above question, as follows.

THEOREM E ([9]). *Let k be a positive integer with $k \geq 2$, and let f be a meromorphic function in Δ such that all zeros of f are of multiplicity at least k. If there exists a constant M such that*

$$(1 - |z|^2)^k \frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}} \leq M \quad \text{for each } z \in \Delta,$$

then f is a normal function.

In this paper, by using a different method, we prove the following stronger result.

THEOREM 1. *Let k be a positive integer, let f be a meromorphic function in Δ, and suppose that there exists $M > 0$ such that $\max_{1 \leq i \leq k-1} |f^{(i)}(z)| \leq M$ whenever $f(z) = 0$. If there exist a subset E of $\mathbb{C} \cup \{\infty\}$ containing at least $k + 4$ distinct points and a constant $P(E)$ such that*

$$(1 - |z|^2)^k \frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}} \leq P(E) \quad \text{for each } z \in \Delta \cap f^{-1}(E),$$

then f is a normal function.

For normal families, we get

THEOREM 2. *Let k be a positive integer, let E be a subset of $\mathbb{C} \cup \{\infty\}$ containing at least $k + 4$ distinct points, let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, and suppose that for each $f \in \mathcal{F}$ there exists $M > 0$ such that $\max_{1 \leq i \leq k-1} |f^{(i)}(z)| \leq M$ whenever $f(z) = 0$. If for each compact subset K of D there exists a constant $P(K)$ such that*

$$\frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}} \leq P(K) \quad \text{for each } f \in \mathcal{F} \text{ and } z \in K \cap f^{-1}(E),$$

then $\mathcal{F}$ is normal in D.

REMARK 2. For the case $k = 1$, the condition "$\max_{1 \leq i \leq k-1} |f^{(i)}(z)| \leq M$ whenever $f(z) = 0$" holds naturally. So, Theorem 1 is an extension of Theorem A, and Theorem 2 is an extension of Theorem B and Marty's criterion. Also, Theorems 1 and 2 improve Theorem E and the sufficiency part of Theorem D, respectively.

REMARK 3. For the case $k \geq 2$, the condition "there exists $M > 0$ such that $\max_{1 \leq i \leq k-1} |f^{(i)}(z)| \leq M$ whenever $f(z) = 0$" in Theorem 2 cannot be omitted, as is shown by the following example. We conjecture that this condition is also necessary in Theorem 1, but so far we have not found an appropriate counter-example.

EXAMPLE. Let $k \geq 2$, $\Delta = \{z : |z| < 1\}$, and

$$\mathcal{F} = \{f_n(z) = nz^{k-1} : n = 1, 2, \ldots\}.$$

It is easy to see that $f_n(z) = 0 \Rightarrow \max_{1 \leq i \leq k-1} |f_n^{(i)}(z)| = (k-1)!n \to \infty$ as $n \to \infty$. This means that the condition "for each $f \in \mathcal{F}$ there exists $M > 0$ such that $\max_{1 \leq i \leq k-1} |f^{(i)}(z)| \leq M$ whenever $f(z) = 0$" is not satisfied. For each $f_n \in \mathcal{F}$ and each $z \in \Delta$, we have

$$\frac{|f_n^{(k)}(z)|}{1 + |f_n(z)|^{k+1}} = 0 \leq 1.$$

But $\mathcal{F}$ is not normal in Δ.

REMARK 4. In both Theorem 1 and Theorem 2, there is nothing special about the value 0, so these results are valid for any fixed value replacing 0.

2. Lemmas. To prove our results, we need some lemmas. Let f be a nonconstant meromorphic function in $\mathbb{C}$. We shall use the following standard notation of value distribution theory (see [1, 8, 10]):

$$T(r, f), \; m(r, f), \; N(r, f), \; \bar{N}(r, f), \; \ldots.$$

We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\}$$

as $r \to \infty$, possibly outside a set of finite measure. We use $\bar{N}_{(2}(r, f)$ to denote the Nevanlinna counting function of the poles of f with multiplicity ≥ 2.

LEMMA 1 ([1, 10]). *Let f be a nonconstant meromorphic function in $\mathbb{C}$, and let $a_1, \ldots, a_q$ ($q \geq 3$) be distinct complex numbers (one of these can take value ∞). Then*

$$(q - 2)T(r, f) \leq \sum_{i=1}^{q} \bar{N}\left(r, \frac{1}{f - a_i}\right) + S(r, f).$$

LEMMA 2 ([1, 10]). *Let f be a nonconstant meromorphic function in $\mathbb{C}$, and k be a positive integer. Then*

$$T(r, f^{(k)}) \leq (k + 1)T(r, f) + S(r, f).$$

The following is a local version of the well-known Zalcman lemma.

LEMMA 3 ([11]). *Let $\mathcal{F}$ be a family of meromorphic functions in D. If $\mathcal{F}$ is not normal at a point $z_0 \in D$, then there exists a sequence of functions $f_n \in \mathcal{F}$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that $f_n(z_n + \rho_n\zeta)$ spherically and uniformly converges to a nonconstant meromorphic function on each compact subset of $\mathbb{C}$.*

The next lemma reveals the close relationship between normal functions and normal families; it can be found in [5].

LEMMA 4 *Let $\Delta = \{z : |z| < 1\}$ be the unit disc in $\mathbb{C}$. A meromorphic function f in Δ is normal if and only if the family $\mathcal{F} = \{f(g(z)) : g \in \mathrm{Aut}(\Delta)\}$ is normal in Δ, where $\mathrm{Aut}(\Delta)$ is the collection of all conformal mappings of Δ into itself.*

3. Proof of theorems

Proof of Theorem 1. Suppose that f is not a normal function. Then we can find a sequence $\{z_n\}$ in Δ such that
$$(1 - |z_n|^2)f^{\#}(z_n) \to \infty$$
as $n \to \infty$. Define the family
$$\mathcal{G} = \{g_n(z) = f(z_n + (1 - |z_n|^2)z) : z \in \Delta\}.$$
Since
$$g_n^{\#}(0) = (1 - |z_n|^2)f^{\#}(z_n) \to \infty$$
as $n \to \infty$, $\mathcal{G}$ is not normal at $z = 0$. By Lemma 3, there exist functions $g_n \in \mathcal{G}$, points $\xi_n \in \Delta$, $\xi_n \to 0$ and positive numbers $\rho_n \to 0$ such that

(3) $\quad G_n(\zeta) = g_n(\xi_n + \rho_n\zeta) = f(z_n + (1 - |z_n|^2)\xi_n + (1 - |z_n|^2)\rho_n\zeta) \to G(\zeta)$

spherically uniformly on compact subsets of $\mathbb{C}$, where $G(\zeta)$ is a nonconstant meromorphic function on $\mathbb{C}$. From (3), we have

(4) $\quad G_n^{(i)}(\zeta) = ((1-|z_n|^2)\rho_n)^i f^{(i)}(z_n+(1-|z_n|^2)\xi_n+(1-|z_n|^2)\rho_n\zeta) \to G^{(i)}(\zeta)$

uniformly on compact subsets of $\mathbb{C}$ avoiding the poles of G.

Suppose that $G(\zeta_0) = 0$. Hurwitz's theorem implies that there exist ζ_n with $\zeta_n \to \zeta_0$ such that
$$f(z_n + (1 - |z_n|^2)\xi_n + (1 - |z_n|^2)\rho_n\zeta_n) = 0.$$
Obviously, $z_n + (1 - |z_n|^2)\xi_n + (1 - |z_n|^2)\rho_n\zeta_n \in \Delta$ for sufficiently large n. Then, by the assumption of Theorem 1, we have

(5) $\quad \max_{1 \le i \le k-1} |f^{(i)}(z_n + (1 - |z_n|^2)\xi_n + (1 - |z_n|^2)\rho_n\zeta_n)| \le M.$

It follows from (4) and (5) that $G^{(i)}(\zeta_0) = 0$ for $i = 1, \ldots, k-1$. This means that all zeros of G, if any, have multiplicity at least k. Hence $G^{(k)} \not\equiv 0$.

Without loss of generality, we may assume $E = \{a_1, \ldots, a_{k+4}\}$. Let $G(\zeta_1) - a_i = 0$ for some i (for otherwise, by Lemma 1, G is a constant, a contradiction). If a_i is finite, by (3) and Hurwitz's theorem, there exists a sequence of points ζ_n with $\zeta_n \to \zeta_1$ such that

(6) $\quad f(z_n + (1 - |z_n|^2)\xi_n + (1 - |z_n|^2)\rho_n\zeta_n) - a_i = 0.$

If $a_i = \infty$, noting $1/G_n \to 1/G$ and using Hurwitz's theorem, we still have (6). For brevity, set $\widehat{\zeta}_n = z_n + (1 - |z_n|^2)\xi_n + (1 - |z_n|^2)\rho_n\zeta_n$. Clearly, for sufficiently large n, $\widehat{\zeta}_n \in \Delta$, and so $\widehat{\zeta}_n \in \Delta \cap f^{-1}(E)$. According to the assumption of the theorem, for sufficiently large n, we have

$$(1 - |\widehat{\zeta}_n|^2)^k \frac{|f^{(k)}(\widehat{\zeta}_n)|}{1 + |f(\widehat{\zeta}_n)|^{k+1}} \le P(E).$$

It follows that

$$(7) \qquad \frac{|G_n^{(k)}(\zeta_n)|}{1 + |G_n(\zeta_n)|^{k+1}} = \rho_n^k(1 - |z_n|^2)^k \frac{|f^{(k)}(\widehat{\zeta}_n)|}{1 + |f(\widehat{\zeta}_n)|^{k+1}}$$

$$\le \rho_n^k P(E)\left(\frac{1 - |z_n|^2}{1 - |\widehat{\zeta}_n|^2}\right)^k.$$

In view of $(1 - |z_n|^2)/(1 - |\widehat{\zeta}_n|^2) \to 1$ as $n \to \infty$, we deduce from (7) that

$$\frac{|G^{(k)}(\zeta_1)|}{1 + |G(\zeta_1)|^{k+1}} = 0.$$

Noting that $G^{(k)} \not\equiv 0$, we conclude that ζ_1 is either a multiple pole of $G(\zeta)$ or a zero of $G^{(k)}(\zeta)$. We have thus proved that if ζ_1 is a zero of $G(\zeta) - a_i$, then ζ_1 is either a multiple pole of $G(\zeta)$ (for the case $a_i = \infty$) or a zero of $G^{(k)}(\zeta)$ (for the case $a_i \in \mathbb{C}$).

By Lemma 1, we have

$$(8) \qquad (k + 2)T(r, G) \le \sum_{i=1}^{k+4} \bar{N}\left(r, \frac{1}{G - a_i}\right) + S(r, G).$$

If all a_i $(i = 1, \ldots, k + 4)$ are finite, then by the above discussion, we have

$$\sum_{i=1}^{k+4} \bar{N}\left(r, \frac{1}{G - a_i}\right) \le \bar{N}\left(r, \frac{1}{G^{(k)}}\right).$$

Substituting this in (8), and using Lemma 2, we get

$$(k + 2)T(r, G) \le \bar{N}(r, 1/G^{(k)}) + S(r, G) \le T(r, G^{(k)}) + S(r, G)$$
$$\le (k + 1)T(r, G) + S(r, G),$$

that is, $T(r, G) < S(r, G)$. We arrive at a contradiction since G is nonconstant.

If one of a_i is infinite, say $a_1 = \infty$, then

$$\overline{N}\left(r, \frac{1}{G - a_1}\right) \le \bar{N}_{(2}(r, G),$$

and

$$\sum_{i=2}^{k+4} \bar{N}\left(r, \frac{1}{G - a_i}\right) \le \bar{N}\left(r, \frac{1}{G^{(k)}}\right).$$

Substituting the above in (8), we have

$$(k + 2)T(r, G) < \bar{N}_{(2}(r, G) + \bar{N}(r, 1/G^{(k)}) + S(r, G)$$
$$\le \tfrac{1}{2}N(r, G) + T(r, G^{(k)}) + S(r, G)$$
$$\le \tfrac{1}{2}N(r, G) + (k + 1)T(r, G) + S(r, G)$$
$$\le (k + 3/2)T(r, G) + S(r, G),$$

that is, $\tfrac{1}{2}T(r, G) < S(r, G)$, a contradiction. Theorem 1 is thus proved. ∎

Proof of Theorem 2. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. Then applying Lemma 3 directly, there exist functions $f_n \in \mathcal{F}$, points $z_n \to z_0$ and positive numbers $\rho_n \to 0$ such that

$$g_n(\zeta) = f_n(z_n + \rho_n\zeta) \to g(\zeta)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\zeta)$ is a nonconstant meromorphic function in $\mathbb{C}$.

Using f_n, g_n, and g as in the proof of Theorem 1, we can derive the same contradiction as in that proof by essentially the same argument. We omit the details. ∎

Next, by borrowing an idea from Li and Xie [6], we give another proof of Theorem C.

Another proof of Theorem C. If $k = 1$, there is nothing to prove. Suppose that the conclusion of Theorem C is not valid. Then, for a positive integer $k \ge 2$, we can find $\{z_n\} \subset \Delta$ such that

$$(9) \qquad (1 - |z_n|^2)^k \frac{|f^{(k)}(z_n)|}{1 + |f(z_n)|^{k+1}} \to \infty$$

as $n \to \infty$. Define the family

$$\mathcal{G} = \{g_n(z) = f(z_n + (1 - |z_n|)z) : z \in \Delta\}.$$

Obviously, $h_{z_n}(z) = z_n + (1 - |z_n|)z$ is a conformal mapping of Δ into itself for each n. Since f is normal in Δ, Lemma 4 implies that $\mathcal{G}$ is a normal family in Δ. Thus there exists a subsequence of $\{g_n\}$ (denoted also by $\{g_n\}$) such that $\{g_n(z)\}$ converges spherically locally uniformly in Δ to a meromorphic function $h(z)$ (possibly identically infinite).

We distinguish two cases.

CASE 1: $h(0) \ne \infty$. Then there exists $1 > \delta > 0$ such that $h(z)$ is holomorphic in $\overline{\Delta}_\delta = \{z : |z| \le \delta\}$, and hence $g_n(z)$ (for sufficiently large n)

are holomorphic in $\overline{\Delta}_\delta$, and $g_n(z) \to h(z)$ uniformly in $\overline{\Delta}_\delta$. It follows that

$$\frac{|g_n^{(k)}(z)|}{1 + |g_n(z)|^{k+1}} \to \frac{|h^{(k)}(z)|}{1 + |h(z)|^{k+1}}$$

uniformly in $\overline{\Delta}_\delta$. Obviously, there exists $Q > 0$ such that

$$\max_{z \in \overline{\Delta}_\delta} \frac{|h^{(k)}(z)|}{1 + |h(z)|^{k+1}} \leq Q.$$

Then for sufficiently large n,

$$\max_{z \in \overline{\Delta}_\delta} \frac{|g_n^{(k)}(z)|}{1 + |g_n(z)|^{k+1}} \leq Q + 1.$$

In particular, for sufficiently large n,

$$\frac{|g_n^{(k)}(0)|}{1 + |g_n(0)|^{k+1}} = (1 - |z_n|)^k \frac{|f^{(k)}(z_n)|}{1 + |f(z_n)|^{k+1}} \leq Q + 1,$$

which contradicts (9).

CASE 2: $h(0) = \infty$. There exists $1 > \delta_1 > 0$ such that $1/h(z)$ and $1/g_n(z)$ (for sufficiently large n) are holomorphic in $\overline{\Delta}_{\delta_1} = \{z : |z| \leq \delta_1\}$, and $1/g_n(z) \to 1/h(z)$ uniformly in $\overline{\Delta}_{\delta_1}$. Let $Q_1 = 1 + \max_{z \in \overline{\Delta}_{\delta_1}} |1/h(z)|$. Then, for sufficiently large n,

$$(10) \qquad \max_{z \in \overline{\Delta}_{\delta_1}} \frac{1}{|g_n(z)|} \leq Q_1.$$

By Cauchy's formula for derivatives, for sufficiently large n, we get

$$(11) \qquad \left| \left(\frac{1}{g_n(z)} \right)^{(l)} \right| = \left| \frac{l!}{2\pi i} \int_{|z| = \delta_1} \frac{\frac{1}{g_n(\zeta)}}{(\zeta - z)^{l+1}} \, d\zeta \right|$$

$$\leq \frac{l! Q_1}{(\delta_1 - \delta_2)^l} \qquad (l = 1, \ldots, k)$$

for each $z \in \overline{\Delta}_{\delta_2} = \{z : |z| \leq \delta_2\}$, where $0 < \delta_2 < \delta_1$. An elementary calculation yields

$$(12) \qquad \frac{g_n^{(k)}}{g_n^{k+1}} = -\frac{1}{g_n^{k-1}} \left(\frac{1}{g_n} \right)^{(k)} + P\left(\frac{g_n'}{g_n^2}, \frac{g_n''}{g_n^3}, \ldots, \frac{g_n^{(k-1)}}{g_n^k} \right),$$

where $P(w_1, \ldots, w_{k-1})$ is a polynomial in $w_1, \ldots, w_{k-1}$ with integer coefficients. From (10)–(12) we conclude that, for sufficiently large n and each $z \in \overline{\Delta}_{\delta_2}$,

$$\frac{|g_n^{(k)}(z)|}{1 + |g_n(z)|^{k+1}} \leq \left| \frac{g_n^{(k)}(z)}{g_n(z)^{k+1}} \right| \leq C(k, \delta_1, \delta_2) Q_1,$$

where $C(k, \delta_1, \delta_2)$ is a constant depending only on k, δ_1 and δ_2.

Letting $z = 0$ and noting that

$$\frac{|g_n^{(k)}(0)|}{1 + |g_n(0)|^{k+1}} = (1 - |z_n|^2)^k \frac{|f_n^{(k)}(z_n)|}{1 + |f_n(z_n)|^{k+1}},$$

we arrive at a contradiction of (9). This completes the proof of Theorem C. ∎

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