On prolongations of projectable connections

by Jan Kurek (Lublin) and Włodzimierz M. Mikulski (Kraków)

Dedicated to Professor Ivan Kolář on the occasion of his 75th birthday with respect and gratitude

Abstract. We extend the concept of $r$-order connections on fibred manifolds to the one of $(r,s,q)$-order projectable connections on fibred-fibred manifolds, where $r,s,q$ are arbitrary non-negative integers with $s \geq r \leq q$. Similarly to the fibred manifold case, given a bundle functor $F$ of order $r$ on $(m_1,m_2,n_1,n_2)$-dimensional fibred-fibred manifolds $Y \to M$, we construct a general connection $F(\Gamma,\Lambda) : FY \to J^1FY$ on $FY \to M$ from a projectable general (i.e. $(1,1,1)$-order) connection $\Gamma : Y \to J^{1,1,1}Y$ on $Y \to M$ by means of an $(r,r,r)$-order projectable linear connection $\Lambda : TM \to J^{r,r,r}TM$ on $M$.

In particular, for $F = J^{1,1,1}$ we construct a general connection $J^{1,1,1}(\Gamma,\nabla) : J^{1,1,1}Y \to J^{1,1,1}J^{1,1,1}Y$ on $J^{1,1,1}Y \to M$ from a projectable general connection $\Gamma$ on $Y \to M$ by means of a torsion-free projectable classical linear connection $\nabla$ on $M$. Next, we observe that the curvature of $\Gamma$ can be considered as $R_{\Gamma} : J^{1,1,1}Y \to T^*M \otimes VJ^{1,1,1}Y$. The main result is that if $m_1 \geq 2$ and $n_2 \geq 1$, then all general connections $D(\Gamma,\nabla) : J^{1,1,1}Y \to J^{1,1,1}J^{1,1,1}Y$ on $J^{1,1,1}Y \to M$ canonically depending on $\Gamma$ and $\nabla$ form the one-parameter family $J^{1,1,1}(\Gamma,\nabla) + tR_{\Gamma}$, $t \in \mathbb{R}$. A similar classification of all general connections $D(\Gamma,\nabla) : J^1Y \to J^1J^1Y$ on $J^1Y \to M$ from $(\Gamma,\nabla)$ is presented.

1. Introduction. Higher order jets in the sense of C. Ehresmann (see [E2]) constitute a powerful tool in differential geometry and in many areas of mathematical physics. They globalize the theory of differential systems and play an important role in the calculus of variations (see [S], [V]). Higher order connections were first introduced on groupoids by C. Ehresmann (see [E1]) and next on arbitrary fibred manifolds by I. Kolář (see [K1]). Roughly speaking, higher order connections are sections of bundles of higher order jets. Higher order connections play an important role in the theory of higher order absolute differentiation (see [K1]). The theory of jets and connections is closely related to the theory of natural operations in differential...
geometry (see [KMS]). The theory of jets and (principal) connections constitutes the geometrical background for field theories and theoretical physics (see [LR], [MM]).

In the present paper, an r-order connection on a fibred manifold \( Y \to M \) is a section \( \Theta : Y \to J^rY \) of the r-jet prolongation \( J^rY \to Y \) of \( Y \to M \). For \( r = 1 \), we obtain the concept of general connections on \( Y \to M \). A general connection \( \Gamma : Y \to J^1Y \) on \( Y \to M \) can be equivalently defined as the corresponding lifting map \( \Gamma : Y \times_M TM \to TY \). An r-order linear connection on a vector bundle \( Y \to M \) is an r-order connection on \( Y \to M \) which is additionally a vector bundle morphism \( \Theta : Y \to J^rY \) covering the identity map \( \text{id}_M \) of \( M \). An r-order linear connection on a manifold \( M \) is an r-order linear connection on the tangent bundle \( TM \to M \) of \( M \). A classical linear connection on \( M \) is a first order linear connection on \( M \). A classical linear connection \( \nabla : TM \to J^1TM \) on \( M \) can be equivalently defined as the corresponding covariant derivative \( \nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \). A more detailed notion of connection can be found in the fundamental monograph [KMS].

In [K3] (see also [KMS, Section 45.1]), given a bundle functor \( F \) of order \( r \) on \( (m, n) \)-dimensional fibred manifolds \( Y \to M \), I. Kolář constructed a general connection \( \mathcal{F}(\Gamma, \Lambda) : FY \to J^1FY \) on \( FY \to M \) from a general connection \( \Gamma : Y \to J^1Y \) on \( Y \to M \) by means of an r-order linear connection on \( Y \). In particular, for \( F = J^1 \) he obtained a general connection \( \mathcal{J}^1(\Gamma, \nabla) : J^1Y \to J^1J^1Y \) on \( J^1Y \to M \) from a general connection \( \Gamma \) on \( Y \to M \) by means of a torsion-free classical linear connection \( \nabla \) on \( M \). In [KMS] Sections 45.7–8, the authors presented another general connection \( P(\Gamma, \nabla) : J^1Y \to J^1J^1Y \) on \( J^1Y \to M \) and deduced that all general connections \( D(\Gamma, \nabla) : J^1Y \to J^1J^1Y \) on \( J^1Y \to M \) canonically depending on \( \Gamma \) and \( \nabla \) form the one-parameter family \( tJ^1(\Gamma, \nabla) + (1 - t)P(\Gamma, \nabla), \ t \in \mathbb{R} \), where the first jet prolongation \( J^1Z \to Z \) of a fibred manifold \( Z \to M \) (in particular of \( Z = J^1Y \to M \)) is always endowed with the well-known affine bundle structure with the corresponding vector bundle \( T^*M \otimes VZ \).

In Section 2 of the present paper we observe that the curvature tensor \( \mathcal{R}_\Gamma : Y \to \bigwedge^2 T^*M \otimes VY \) of \( \Gamma \) can be interpreted as the corresponding fibred map \( \mathcal{R}_\Gamma : J^1Y \to J^1T^*M \otimes VJ^1Y \) covering the identity map of \( J^1Y \). So, all general connections \( D(\Gamma, \nabla) : J^1Y \to J^1J^1Y \) on \( J^1Y \to M \) canonically depending on \( \Gamma \) and \( \nabla \) form the one-parameter family \( J^1(\Gamma, \nabla) + t\mathcal{R}_\Gamma, \ t \in \mathbb{R} \).

In [M1], the second author defined a fibred-fibred manifold to be a fibred surjective submersion \( Y \to M \) between fibred manifolds \( Y \) and \( M \) such that the restrictions of it to fibres are submersions. Moreover, he defined the so-called \((r, s, q)\)-jet prolongation \( J^{r,s,q}Y \to Y \) of \( Y \to M \). In [K2],
I. Kolář observed that a fibred-fibred manifold can be defined as a fibred square in the sense of J. Pradines (see [P]), and generalized the concept of general connections on fibred manifolds to the one of $(1,1,1)$-order square connections on fibred squares. He also defined linear square connections of order $(r,s,q)$ on fibred manifolds.

In Section 3 of the present paper, we extend the concept of $(1,1,1)$-order square connections to the one of $(r,s,q)$-order projectable connections $\Theta : Y \to J^{r,s,q}Y$ on fibred-fibred manifolds (also called $(r,s,q)$-order square connections on fibred squares), where $r,s,q$ are arbitrary non-negative integers with $s \geq r \leq q$. Next, we generalize the above-mentioned “construction” $\mathcal{F}(\Gamma, \nabla)$. Namely, given a bundle functor $F$ of order $r$ on $(m_1, m_2, n_1, n_2)$-dimensional fibred-fibred manifolds $Y \to M$, we construct a general connection $\mathcal{F}(\Gamma, \Lambda) : FY \to J^1FY$ on $FY \to M$ from a projectable general (i.e. $(1,1,1)$-order square) connection $\Gamma : Y \to J^{1,1,1}Y$ on $Y \to M$ by means of an $(r,r,r)$-order projectable linear connection $\Lambda : TM \to J^{r,r,r}TM$ on $M$ (i.e. linear square connection of order $(r,r,r)$ on the fibred manifold $M$). In particular, for $F = J^{1,1,1}$ we obtain a general connection $\mathcal{J}^{1,1,1}(\Gamma, \nabla) : J^{1,1,1}Y \to J^{1,1,1}J^{1,1,1}Y$ on $J^{1,1,1}Y \to M$ from a projectable general connection $\Gamma$ on $Y \to M$ by means of a torsion-free projectable classical linear connection $\nabla$ on $M$. Moreover, we observe that the curvature tensor $\mathcal{R}_\Gamma : Y \to \bigwedge^2 T^*M \otimes VY$ can be interpreted as the corresponding fibred map $\mathcal{R}_\Gamma : J^{1,1,1}Y \to T^*M \otimes VJ^{1,1,1}Y$ covering the identity map of $J^{1,1,1}Y$.

In Section 4, we formulate and prove the main result of the present paper saying that if $m_1 \geq 2$ and $n_2 \geq 1$ then all general connections $D(\Gamma, \nabla) : J^{1,1,1}Y \to J^{1,1,1,1}Y$ on $J^{1,1,1}Y \to M$ canonically depending on a projectable general connection $\Gamma : Y \to J^{1,1,1}Y$ on an $(m_1, m_2, n_1, n_2)$-dimensional fibred-fibred manifold $Y \to M$ and a torsion-free projectable classical linear connection $\nabla$ on the fibred manifold $M$ form the one-parameter family $\mathcal{J}^{1,1,1}(\Gamma, \nabla) + t\mathcal{R}_\Gamma$, $t \in \mathbb{R}$. A similar classification of all general connections $D(\Gamma, \nabla) : J^{1}Y \to J^{1}J^{1}Y$ on $J^{1}Y \to M$ canonically depending on $\Gamma : Y \to J^{1,1,1}Y$ and $\nabla$ is also presented.

All manifolds and maps in the present paper are assumed to be of class $C^\infty$.

2. On constructions on connections on fibred manifolds. Let $\Gamma : Y \to J^{1}Y$ be a general connection on a fibred manifold $p : Y \to M$ and $\nabla$ be a torsion-free classical linear connection on $M$. Let $\mathcal{J}^{1}(\Gamma, \nabla) : J^{1}Y \to J^{1}J^{1}Y$ be the induced general connection on $J^{1}Y \to M$ (see Introduction). Using $\mathcal{J}^{1}(\Gamma, \nabla)$ one can produce the following family of general connections on $J^{1}Y \to M$. 


Example 2.1. It is well-known that \( p^1 : J^1Y \to Y \) (the target jet projection) is (canonically) an affine bundle with the corresponding vector bundle \( T^*M \otimes VY \). Then \( p^1 : J^1J^1Y \to J^1Y \) is (canonically) an affine bundle with the corresponding vector bundle \( T^*M \otimes VJ^1Y \). The curvature tensor \( R_{\Gamma} : Y \to \bigwedge^2 T^*M \otimes VY \) of \( \Gamma : Y \to J^1Y \) can be (in an obvious way) considered as a fibred map \( R_{\Gamma} : J^1Y \to T^*M \otimes T^*M \otimes VY \subset T^*M \otimes VJ^1Y \) covering the identity map \( \text{id}_{J^1Y} \), where the inclusion is induced by the injection \( T^*M \otimes VY \to VJ^1Y \) from the known exact sentence \( 0 \to T^*M \otimes VY \to VJ^1Y \to VY \to 0 \) of vector bundles over \( J^1Y \) (the obvious pull-backs are not indicated). So, for any \( t \in \mathbb{R} \) we have the general connection \( D_t(\Gamma, \nabla) := J^1(\Gamma, \nabla) + tR_{\Gamma} : J^1Y \to J^1J^1Y \) on \( J^1Y \to M \).

Remark 2.2. The most general concept of natural operators can be found in [KMS]. In particular, an \( \mathcal{FM}_{m,n} \)-natural operator \( D : J^1 \times Q_{\tau}(B) \hookrightarrow J^1(J^1 \to B) \) transforming general connections \( \Gamma \) on fibred manifolds \( Y \to M \) and torsion-free classical linear connections \( \nabla \) on \( M \) into general connections \( D(\Gamma, \nabla) : J^1Y \to J^1J^1Y \) on \( J^1Y \to M \) is a family of \( \mathcal{FM}_{m,n} \)-invariant regular operators

\[
D : \text{Con}(Y \to M) \times Q_{\tau}(M) \to \text{Con}(J^1Y \to M)
\]

for all \( \mathcal{FM}_{m,n} \)-objects \( Y \to M \), where \( \text{Con}(Y \to M) \) is the set of all general connections on \( Y \to M \) and \( Q_{\tau}(M) \) is the set of all torsion-free classical linear connections on \( M \). The \( \mathcal{FM}_{m,n} \)-invariance means that \( D(\Gamma, \Lambda) \) is \( J^1f \)-related to \( D(\Gamma_1, \Lambda_1) \) for any \( \Gamma \in \text{Con}(Y \to M), \Gamma_1 \in \text{Con}(Y_1 \to M_1), \nabla \in Q_{\tau}(M) \) and \( \nabla_1 \in Q_{\tau}(M_1) \) such that \( \Gamma \) is \( f \)-related to \( \Gamma_1 \) by an \( \mathcal{FM}_{m,n} \)-map \( f : Y \to Y_1 \) covering \( \overline{f} : M \to M_1 \) (i.e. \( J^1f \circ \Gamma = \Gamma_1 \circ f \)) and \( \nabla \) is \( f \)-related (or more precisely \( Tf \)-related) to \( \nabla_1 \) (i.e. \( J^1Tf \circ \nabla = \nabla_1 \circ Tf \)). The regularity means that \( D \) transforms smoothly parametrized families of connections into smoothly parametrized ones.

Thus (because of the canonical character of the construction of \( D_t(\Gamma, \nabla) \) in Example 2.1) we have the corresponding \( \mathcal{FM}_{m,n} \)-natural operator \( D_t : J^1 \times Q_{\tau}(B) \hookrightarrow J^1(J^1 \to B) \) for any \( t \in \mathbb{R} \).

We see that the classification result [KMS, Proposition 45.8] mentioned in Introduction can be immediately reformulated as follows.

Proposition 2.3. All \( \mathcal{FM}_{m,n} \)-natural operators \( D : J^1 \times Q_{\tau}(B) \hookrightarrow J^1(J^1 \to B) \) form the one-parameter family \( D_t := J^1 + tR, t \in \mathbb{R} \).

3. On constructions on connections on fibred-fibred manifolds.

In this section we extend the results presented in the previous section to fibred-fibred manifolds instead of fibred manifolds.
A fibred-fibred manifold is a fibred surjective submersion \( p = (p, p) : (p_Y : Y \to Y) \to (p_M : M \to M) \) between fibred manifolds \( p_Y : Y \to Y \) and \( p_M : M \to M \) covering \( p : Y \to M \) such that the restrictions of \( p \) to the fibres are submersions, or (equivalently) it is a fibred square \( p = (p, p_Y, p_M, p) \), i.e. a commutative square diagram with arrows being surjective submersions \( p : Y \to M, p_Y : Y \to Y, p_M : M \to M \) and \( p : Y \to Y \) such that the system \( (p, p_Y) : Y \to M \times_M Y \) of maps \( p \) and \( p_Y \) is a submersion. If \( p^1 = (p^1, p^1_Y) : (p^1_Y : Y^1 \to Y^1) \to (p^1_M : M^1 \to M^1) \) is another fibred-fibred manifold then a fibred-fibred map \( f : Y \to Y^1 \) is a system \( f = (f, f_1, f_2, f) \) of maps \( f : Y \to Y^1, f_1 : Y \to Y^1, f_2 : M \to M^1 \) and \( f : M \to M^1 \) such that the obvious cubic diagram is commutative.

A fibred-fibred manifold \( p = (p, p) : (p_Y : Y \to Y) \to (p_M : M \to M) \) of dimension \( (m_1, m_2, n_1, n_2) \) if \( \dim(Y) = m_1 + m_2 + n_1 + n_2 \), \( \dim(M) = m_1 + m_2 \), \( \dim(Y) = m_1 + n_1 \) and \( \dim(M) = m_1 \). The fibred-fibred manifolds of dimension \( (m_1, m_2, n_1, n_2) \) and their local fibred-fibred diffeomorphisms form a local admissible category over manifolds (in the sense of [KMS] Section 18), which will be denoted by \( \mathcal{F}_M \). Any \( \mathcal{F}_M \)-object is locally isomorphic to the trivial fibred square (denoted by \( \mathbb{R}^{m_1, m_2, n_1, n_2} \)) with vertices \( \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \), \( \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \), \( \mathbb{R}^{m_1} \times \mathbb{R}^{n_1} \) and \( \mathbb{R}^{m_1} \) and arrows being obvious projections.

Let \( r, s, q \) be non-negative integers with \( s \geq r \leq q \). Let \( p = (p, p) : (p_Y : Y \to Y) \to (p_M : M \to M) \) be a fibred-fibred manifold of dimension \( (m_1, m_2, n_1, n_2) \). According to [KMS] Section 12.19, two fibred sections \( \sigma_1, \sigma_2 : (p_M : M \to M) \to (p_Y : Y \to Y) \) of \( p : Y \to M \) (i.e. fibred maps with \( p \circ \sigma_i = \text{id}_M \)) covering sections \( \sigma_1, \sigma_2 : M \to Y \) of \( p : Y \to M \) have the same \( (r, s, q) \)-jet \( J^{r,s,q}_x \sigma_1 = J^{r,s,q}_x \sigma_2 \) at \( x \in M \) iff \( J^{r,s}_x \sigma_1 = J^{r,s}_x \sigma_2 \), \( J^{s}_x \sigma_1 = J^{s}_x \sigma_2 \), \( J^{s}_x \sigma_1 = J^{s}_x \sigma_2 \), where \( M_x \) is the fibre of \( M \) over \( x = p_M(x) \). The space \( J^{r,s,q}_x Y \) of \( (r, s, q) \)-jets of fibred sections \( M \to Y \) of \( p : Y \to M \) is a fibred manifold over \( Y \) with respect to the target projection \( p^{r,s,q} : J^{r,s,q}_x Y \to Y \). If \( f = (f, f_1, f_2, f_3) : Y \to Y^1 \) is an \( \mathcal{F}_M \)-object, then we have the fibred map \( J^{r,s,q} f : J^{r,s,q}_x Y \to J^{r,s,q}_x Y \) covering \( f \) given by \( J^{r,s,q} f(j^{r,s,q}_x \sigma) = j^{r,s,q}_{f_2}(f \circ \sigma \circ j^{r,s}_1) \). The correspondence \( J^{r,s,q}_x \mathcal{F}_M \) is a (regular) bundle functor in the sense of [KMS], which is called the \( (r, s, q) \)-jet prolongation functor. This functor \( J^{r,s,q}_x \) was first introduced by the second author in [M].

The space \( J^{r,s,q}_x Y \) is also a fibred manifold over \( J^{q}_x Y \) with respect to the projection \( p^{r,s,q}_q : J^{r,s,q}_x Y \to J^{q}_x Y \) given by \( p^{r,s,q}_q(j^{r,s,q}_x \sigma) = j^{q}_x \sigma \). Consequently, \( J^{r,s,q}_x Y \) can be considered as a fibred-fibred manifold \( p^{r,s,q} = (p^{r,s,q}_r, p^{r,s,q}_q) : (p^{r,s,q}_r : J^{r,s,q}_x Y \to J^{q}_x Y) \to (p_Y : Y \to Y^1) \), where \( p^{r,s,q}_r : J^{q}_x Y \to Y \) is the target projection of the \( q \)-jet prolongation of the fibred manifold \( p : Y \to M \).
The fibred-fibred manifold $p^{r,s,q}$ is called the \((r,s,q)\)-jet prolongation of the fibred-fibred manifold $p$.

The concept of higher order connections on fibred manifolds can be extended to the one of higher order projectable connections on fibred-fibred manifolds as follows.

**Definition 3.1.** Let $r, s, q$ be non-negative integers with $s \geq r \leq q$. An \((r,s,q)\)-order projectable connection on a fibred-fibred manifold $p = (p,p)$: $(p_Y : Y \to Y) \to (p_M : M \to M)$ is a fibred section $\Theta : (p_Y : Y \to Y) \to (p_Y^{r,s,q} : J^{r,s,q}Y \to J^qY)$ (or briefly a fibred section $\Theta : Y \to J^{r,s,q}Y$) of $p^{r,s,q} = (p^{r,s,q}, p^q) : (p_Y^{r,s,q} : J^{r,s,q}Y \to J^qY) \to (p_Y : Y \to Y)$ (or briefly of $p^{r,s,q} : J^{r,s,q}Y \to Y$) covering a section $\Theta : Y \to J^qY$ of $p^q : J^qY \to Y$, where $p^{r,s,q}$ is the \((r,s,q)\)-jet prolongation of $p$.

A *projectable general connection* on a fibred-fibred manifold $p$ is a \((1,1,1)\)-order projectable connection $\Gamma : Y \to J^{1,1,1}Y$ on $p$, or (equivalently) it is a square connection in the sense of [K2] on the fibred square $p$ (i.e. a pair of general connections $\Gamma : Y \times_M TM \to TY$ and $\Gamma : Y \times_M TM \to TY$ on the fibred manifolds $p : Y \to M$ and $p : Y \to M$ (respectively) such that $\Gamma \circ (p_Y \times_{id_M} TP_M) = Tp_Y \circ \Gamma$). If $p = (p_Y : Y \to Y) \to (p_M : M \to M)$ is a fibred-fibred vector bundle (i.e. a fibred-fibred manifold such that $p : Y \to M$ and $p : Y \to M$ are vector bundles and $p_Y : Y \to Y$ is a vector bundle map covering $p_M : M \to M$), then an \((r,s,q)\)-order projectable linear connection on $p$ is by definition an \((r,s,q)\)-order projectable connection $\Theta : Y \to J^{r,s,q}Y$ on the fibred-fibred manifold $p$ such that $\Theta : (p : Y \to M) \to (p \circ p^{r,s,q} : J^{r,s,q}Y \to M)$ is a vector bundle map covering $id_M$ and (consequently $\Theta : (p : Y \to M) \to (p \circ p^q : J^qY \to M)$ is a vector bundle map covering $id_M$). An \((r,s,q)\)-order projectable linear connection on a fibred manifold $p_M : M \to M$ is an \((r,s,q)\)-order projectable linear connection $\Lambda : TM \to J^{r,s,q}TM$ on the fibred-fibred vector tangent bundle $p_M^T = (p_M^T, p_M^T) : (TP_M : TM \to TM) \to (p_M : M \to M)$, or (equivalently) it is a linear square connection of order \((r,s,q)\) in the sense of [K2] on $p_M$. A *projectable classical linear connection* on a fibred manifold $p_M : M \to M$ is a \((1,1,1)\)-order projectable linear connection $\nabla$ on $p_M : M \to M$, or (equivalently) a classical linear connection $\nabla$ on the manifold $M$ such that there is a (unique) $p_M$-related (to $\nabla$) classical linear connection $\nabla$ on $M$.

Let $F : \mathcal{F}^2M_{m_1,m_2,n_1,n_2} \to \mathcal{F}M$ be a (regular) bundle functor of order $r$ in the sense of [KMS]. The construction $\mathcal{F}(\Gamma, \Lambda)$ from [K3] (mentioned in Introduction) can be adapted to the fibred-fibred manifold situation as follows.
Example 3.2. Let $\Gamma : Y \to J^{1,1,1}Y$ be a projectable general connection on an $(m_1, m_2, n_1, n_2)$-dimensional fibred-fibred manifold $p = (p, p) : (p_Y : Y \to Y) \to (p_M : M \to M)$ and let $\Lambda : TM \to J^{r,r,r}TM$ be an $(r, r, r)$-order projectable linear connection on the fibred manifold $p_M : M \to M$. Let us recall that a projectable-projectable vector field on $p$ is a vector field $X \in \mathcal{X}(Y)$ on $Y$ such that there exist underlying vector fields $X_M \in \mathcal{X}(M)$, $X_Y \in \mathcal{X}(Y)$ and $X_M \in \mathcal{X}(M)$ such that $X$ is $p$-related to $X_M$, $X$ is $p_Y$-related to $X_Y$, $X_M$ is $p_M$-related to $X_M$ and $X_Y$ is $p$-related to $X_M$. Or (equivalently) the flow $\text{Exp}(tX)$ of $X$ is formed by $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$-morphisms. So, similarly to the fibred manifold case, the flow operator $\mathcal{F}$ of $F$ lifting projectable-projectable vector fields $X$ on $p$ into vector fields $\mathcal{F}X := \frac{\partial}{\partial t}|_{t=0} F(\text{Exp}(tX))$ on $FY$ (we can apply $F$ as $\text{Exp}(tX)$ is an $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$-map) is of order $r$, and then it can be interpreted as the flow morphism $\mathcal{F} : FY \times_Y J^r \mathcal{M}_{\text{proj}, \text{proj}}Y \to TFY$, $\mathcal{F}(v, j_y^*X) = \mathcal{F}X(v), v \in F_y Y, y \in Y, X \in \mathcal{X}_{\text{proj}, \text{proj}}(p : Y \to M)$. Since the general connection $\Gamma : Y \times_M TM \to TY$ on $p$ is projectable, the $\Gamma$-horizontal lift $\mathcal{X}^\Gamma$ of a projectable vector field $\mathcal{X}$ on $p_M$ (defined by $\mathcal{X}^\Gamma|_z = \Gamma(z, \mathcal{X}|_z)$, $z \in Y$) is a projectable-projectable vector field on $p$. Then (as in the fibred manifold case) we have $\tilde{\mathcal{F}} \Gamma : FY \times_M J^r \mathcal{M}_{\text{proj}, \text{proj}}M \to TFY$, $\tilde{\mathcal{F}} \Gamma(v, j_y^*X) = \mathcal{F}(v, j_y^*(\mathcal{X}^\Gamma)), v \in F_y Y, y \in Y, x \in M, \mathcal{X} \in \mathcal{X}_{\text{proj}}(p_M : M \to M)$. So, applying $\Lambda : TM \to J^{r,r,r}TM = J^r \mathcal{M}_{\text{proj}}M$, we get a general connection $\mathcal{F}(\Gamma, \Lambda) =: \tilde{\mathcal{F}} \Gamma \circ (\text{id}_{FY} \times \Lambda) : FY \times_M TM \to TFY$ on $FY \to M$.

In particular, if $F = J^{1,1,1}$ we have the general connection $\mathcal{J}^{1,1,1}(\Gamma, \nabla) : J^{1,1,1}Y \to J^1 J^{1,1,1}Y$ on $J^{1,1,1}Y \to M$ for any projectable general connection $\Gamma : Y \rightarrow J^{1,1,1}Y$ on the fibred-fibred manifold $p$ and a (torsion-free) projectable-projectable classical linear connection $\nabla$ on the fibred manifold $p_M$. Now, quite similarly to Section 2, using $\mathcal{J}^{1,1,1}(\Gamma, \nabla)$ one can produce the following family of general connections on $J^{1,1,1}Y \to M$ from a projectable general connection $\Gamma : Y \rightarrow J^{1,1,1}Y$ on $p$ by means of a (torsion-free) projectable classical linear connection $\nabla$ on $p_M$.

Example 3.3. In Example 2.1, we observed that the curvature tensor $\mathcal{R}_\Gamma : Y \to \bigwedge^2 T^* M \otimes VY$ of $\Gamma$ (treated as a general connection on the fibred manifold $p : Y \to M$) can be considered as the fibred map $\mathcal{R}_\Gamma : J^1 Y \to T^* M \otimes V J^1 Y$. Now (see Remark 4.3 in the next section), using the “special” coordinates from Lemma 4.2 and the characterization (4.1) (see the next section) of $V J^{1,1,1}Y$ (the vertical bundle of $J^{1,1,1}Y \to M$) and recalling what is the curvature of $\Gamma$ (e.g. from [KMS]), one can rather easily verify that (in our situation of projectable $\Gamma$) $\mathcal{R}_\Gamma$ restricts to a fibred map $\mathcal{R}_\Gamma : J^{1,1,1}Y \to T^* M \otimes V J^{1,1,1}Y$ covering $\text{id}_{J^{1,1,1}Y}$. On the other hand,
Proof. Let $\Gamma, \nabla$ be the usual fibred-fibred coordinates on the trivial bundle $\mathbb{R}^{m_1, m_2, n_1, n_2}$ of the fibred manifold $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, and $Y^q = dy^q$, $\Gamma = dy^i$, $\delta Y^q = dy^q$, $\delta Y^i = dy^i$, $\delta \Gamma = dy^i$, be the essential coordinates on the vertical bundle $VJ^1\mathbb{R}^{m_1, m_2, n_1, n_2}$ of $J^1\mathbb{R}^{m_1, m_2, n_1, n_2} \rightarrow \mathbb{R}^{m_1+m_2}$, $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$, $q = 1, \ldots, n_1$, $s = 1, \ldots, n_2$.

The $(1, 1, 1)$-jet prolongation of the fibred-fibred manifold $\mathbb{R}^{m_1, m_2, n_1, n_2}$ can be characterized as the subset $J^1\mathbb{R}^{m_1, m_2, n_1, n_2} \subset J^1\mathbb{R}^{m_1, m_2, n_1, n_2}$ satisfying the equalities $y^q_{[j]} = 0$, $q = 1, \ldots, n_1$, $j = 1, \ldots, m_2$. Similarly, the vertical bundle of $J^1\mathbb{R}^{m_1, m_2, n_1, n_2} \rightarrow \mathbb{R}^{m_1+m_2}$ is the subset $VJ^1\mathbb{R}^{m_1, m_2, n_1, n_2} \subset VJ^1\mathbb{R}^{m_1, m_2, n_1, n_2}$ satisfying (on $J^1\mathbb{R}^{m_1, m_2, n_1, n_2}$) the equalities

\begin{equation}
Y^q_{[j]} = 0, \quad q = 1, \ldots, n_1, \quad j = 1, \ldots, m_2.
\end{equation}
Consequently, on $J^{1,1,1} \mathbb{R}^{m_1,m_2,n_1,n_2}$ we have the additional coordinates $y_i^q = \frac{\partial y^q}{\partial x^i}$, $y_i^{(s)} = \frac{\partial y^{(s)}}{\partial x^i}$, and on $V J^{1,1,1} \mathbb{R}^{m_1,m_2,n_1,n_2}$ we have the essential coordinates $Y^q = dy^q, Y^{(s)} = dy^{(s)}$, $Y_i^q = dy_i^q, Y_i^{(s)} = dy_i^{(s)}$, $q = 1, \ldots, n_1$, $s = 1, \ldots, n_2$, $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$.

The following lemma can be treated as a fibred-fibred manifold version of [M2] Proposition 2.2(a) for $r = 1$.

**Lemma 4.2.** Let $\tilde{\Gamma} : Y \times_M TM \to TY$ be a projectable general connection on an $\mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2}$-object $p = (p, p) : (p_Y : Y \to Y) \to (p_M : M \to M)$ and $\nabla$ be a torsion-free projectable classical linear connection on $p_M : M \to M$. Let $y_o \in Y$ and $x_o = p(y_o) \in M$. Then there exists an $\mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2}$-chart $\psi$ on $Y$ covering a $\nabla$-normal fibred coordinate system on $M$ with centre $x_o$ such that $\psi(y_o) = (0, 0, 0, 0)$ and $j^1_{(0,0,0,0)}(\psi \circ \tilde{\Gamma}) = j^1_{(0,0,0,0)}$, where $\Gamma$ is of the form

\[
\Gamma = \sum_{i=1}^{m_1} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{j=1}^{m_2} dx^j \otimes \frac{\partial}{\partial x^j}
\]

\[
+ \sum_{i_1,i_2=1}^{m_1} \sum_{q=1}^{n_1} A^q_{i_1 i_2} x^{i_1} dx^{i_2} \otimes \frac{\partial}{\partial y^q} + \sum_{i_1,i_2=1}^{m_1} \sum_{s=1}^{n_2} B^s_{i_1 i_2} x^{i_1} dx^{i_2} \otimes \frac{\partial}{\partial y^{(s)}}
\]

\[
+ \sum_{i,j=1}^{m_2} \sum_{s=1}^{n_2} C^s_{i j} x^{i} dx^{j} \otimes \frac{\partial}{\partial y^q} + \sum_{i,j=1}^{m_2} \sum_{s=1}^{n_2} D^s_{i j} x^{i} dx^{j} \otimes \frac{\partial}{\partial y^{(s)}}
\]

\[
+ \sum_{j_1,j_2=1}^{m_2} \sum_{s=1}^{n_2} E^s_{j_1 j_2} x^{[j_1]} dx^{[j_2]} \otimes \frac{\partial}{\partial y^{(s)}}
\]

for some real numbers $A^q_{i_1 i_2}, B^s_{i_1 i_2}, C^s_{i j}, D^s_{i j}, E^s_{j_1 j_2}$ satisfying

\[
A^q_{i_1 i_2} = -A^q_{i_2 i_1}, \quad B^s_{i_1 i_2} = -B^s_{i_2 i_1}, \quad C^s_{i j} = -D^s_{i j}, \quad E^s_{j_1 j_2} = -E^s_{j_2 j_1}
\]

for $i, i_1, i_2 = 1, \ldots, m_1$, $j, j_1, j_2 = 1, \ldots, m_2$, $q = 1, \ldots, n_1$, $s = 1, \ldots, n_2$.

If $\psi$ is a chart having the above properties, then so is $(A \times B) \circ \psi$ for any $A \in \text{GL}(m_1, m_2)$ (= the group of fibred linear isomorphisms $(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_1}) \to (\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_1}))$ and $B \in \text{GL}(n_1, n_2)$.

**Proof.** Choose an $\mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2}$-chart $\varphi$ on $Y$ covering a $\nabla$-normal fibred coordinate system on $M$ with centre $x_o \in M$ such that $\varphi(y_o) = (0, 0, 0, 0)$. Replacing $(\tilde{\Gamma}, \nabla)$ by $\psi \circ (\tilde{\Gamma}, \nabla)$, we can additionally assume that $Y = \mathbb{R}^{m_1,m_2,n_1,n_2}$, $y_o = (0, 0, 0, 0)$ and that the identity map on $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ is a $\nabla$-normal fibre coordinate system with centre $(0, 0)$. So, one can write

\[
j^1_{(0,0,0,0)}(\tilde{\Gamma}) = j^1_{(0,0,0,0)} \left( \sum_{i=1}^{m_1} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{j=1}^{m_2} dx^j \otimes \frac{\partial}{\partial x^j} + \ldots \right)
\]
with the dots denoting

\[
(4.4) \quad \sum_{i_1, i_2 = 1}^{m_1} \sum_{i_1, i_2 = 1}^{n_1} \sum_{s=1}^{n_2} A^{q}_{i_1 i_2} x^{i_1} d^{x^{i_2} \otimes \frac{\partial}{\partial y^{s}}} + \sum_{i_1, i_2 = 1}^{m_1} \sum_{s=1}^{n_2} B^{s}_{i_1 i_2} x^{i_1} d^{x^{i_2} \otimes \frac{\partial}{\partial y^{s}}} + \sum_{i_1, i_2 = 1}^{m_1} \sum_{j=1}^{m_2} \sum_{s=1}^{n_2} C^{i_1 j} x^{i_1} d^{x^{j} \otimes \frac{\partial}{\partial y^{s}}} + \sum_{i_1, i_2 = 1}^{m_1} \sum_{s=1}^{n_2} D^{s}_{j_1 i_2} x^{i_1} d^{x^{j} \otimes \frac{\partial}{\partial y^{s}}} + \sum_{i_1, i_2 = 1}^{m_1} \sum_{j=1}^{m_2} \sum_{s=1}^{n_2} E^{s}_{j_1 j_2} x^{i_1} d^{x^{j} \otimes \frac{\partial}{\partial y^{s}}} + \sum_{i_1, i_2 = 1}^{m_1} \sum_{s=1}^{n_2} F^{s}_{j_1 i_2} x^{i_1} d^{x^{j} \otimes \frac{\partial}{\partial y^{s}}}
\]

for some real numbers $A^{q}_{i_1 i_2}, \ldots, h^{s}_{j}$ (because of the projectability of $\tilde{\Gamma}$).

Next, replacing $\tilde{\Gamma}$ by $(\psi_1)_*$ $\tilde{\Gamma}$, where $\psi_1 : \mathbb{R}^{m_1, m_2, n_1, n_2} \to \mathbb{R}^{m_1, m_2, n_1, n_2}$ is an $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$-map such that (defined by)

\[
\psi_1(v, w) = \left( v, \left( w^{q} - \sum_{i=1}^{m_1} f^{q}_{i} v^{i} \right) \right)_{q=1}^{n_1}, \left( w^{s} - \sum_{j=1}^{m_2} h^{s}_{j} w^{[j]} - \sum_{j=1}^{m_2} g^{s}_{i} v^{i} \right)_{s=1}^{n_2}
\]

for any $v = (v^{i}, v^{[j]}) \in \mathbb{R}^{m_1 + m_2}$ and $w = (w^{q}, w^{(s)}) \in \mathbb{R}^{n_1 + n_2}$, we can additionally assume that in [4.4] we have $f^{q}_{i} = 0$ and $g^{s}_{i} = 0, h^{s}_{j} = 0$.

Next, replacing $\tilde{\Gamma}$ by $(\psi_2)_* \tilde{\Gamma}$, where $\psi_2 : \mathbb{R}^{m_1, m_2, n_1, n_2} \to \mathbb{R}^{m_1, m_2, n_1, n_2}$ is a local $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$-map such that (defined by)

\[
\psi_2(v, w) = \left( v, \left( w^{q} - \sum_{i=1}^{m_1} \sum_{q_1=1}^{n_1} a^{q}_{q_1 i} v^{i} w^{q_1} \right) \right)_{q=1}^{n_1}, \left( w^{(s)} - \cdots \right)_{s=1}^{n_2}
\]

with the dots denoting

\[
\sum_{j=1}^{m_2} \sum_{s_1=1}^{n_2} e^{s}_{s_1 j} w^{(s_1)} v^{[j]} + \sum_{j=1}^{m_2} \sum_{q=1}^{n_1} d^{q}_{q_1 j} v^{i} w^{q} + \sum_{i=1}^{m_1} \sum_{s_1=1}^{n_1} c^{s}_{s_1 i} v^{i} w^{(s_1)} + \sum_{i=1}^{m_1} \sum_{q=1}^{n_1} b^{q}_{q_1 i} v^{i} w^{q}
\]

for any $v = (v^{i}, v^{[j]}) \in \mathbb{R}^{m_1 + m_2}$ and $w = (w^{q}, w^{(s)}) \in \mathbb{R}^{n_1 + n_2}$, we can
additionally assume that in (4.4) we have $e^{s}_{s_{1}j} = 0$, $d^{s}_{qj} = 0$, $c^{s}_{s_{1}i} = 0$, $b^{s}_{qi} = 0$ and $a^{q}_{q_{1}i} = 0$.

Finally, replacing $\tilde{\Gamma}$ by $(\psi_{3})_{*}\tilde{\Gamma}$, where $\psi_{3} : \mathbb{R}^{m_{1},m_{2},n_{1},n_{2}} \to \mathbb{R}^{m_{1},m_{2},n_{1},n_{2}}$ is a local $\mathcal{F}^{2}\mathcal{M}_{m_{1},m_{2},n_{1},n_{2}}$-map such that (defined by)

$$\psi_{3}(v, w) = \left( v, \left( w^{q} - \frac{1}{2} \sum_{i_{1},i_{2}=1}^{m_{1}} (A^{q}_{i_{1}i_{2}} + A^{q}_{i_{2}i_{1}}) v^{i_{1}} v^{i_{2}} \right)^{n_{1}}_{q=1}, \left( w^{[s]} - \frac{1}{2} (\cdots) \right)^{n_{2}}_{s=1} \right)$$

with the dots denoting

$$\sum_{i_{1},i_{2}=1}^{m_{1}} (B^{s}_{i_{1}i_{2}} + B^{s}_{i_{2}i_{1}}) v^{i_{1}} v^{i_{2}} + \sum_{i=1}^{m_{1}} \sum_{j=1}^{m_{2}} (C^{s}_{ij} + D^{s}_{ji}) v^{i} v^{[j]} + \sum_{j_{1},j_{2}=1}^{m_{2}} (E^{s}_{j_{1}j_{2}} + E^{s}_{j_{2}j_{1}}) v^{[j_{1}]} v^{[j_{2}]},$$

for any $v = (v^{i}, v^{[j]}) \in \mathbb{R}^{m_{1} + m_{2}}$ and $w = (w^{q}, w^{(s)}) \in \mathbb{R}^{n_{1} + n_{2}}$, we can additionally assume that in (4.4) we have $A^{q}_{i_{1}i_{2}} = -A^{q}_{i_{2}i_{1}}$, $B^{s}_{i_{1}i_{2}} = -B^{s}_{i_{2}i_{1}}$, $C^{s}_{ij} = -D^{s}_{ji}$ and $E^{s}_{j_{1}j_{2}} = -E^{s}_{j_{2}j_{1}}$.

Thus the proof of the main part of the lemma is complete.

The last sentence of the lemma is a simple observation. ■

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Put $\Delta(\Gamma, \nabla) := D(\Gamma, \nabla) - J^{1,1,1}(\Gamma, \nabla) : J^{1,1,1}Y \to T^{*}M \otimes VJ^{1,1,1}Y$. As $D(\Gamma, \nabla)$ is determined by $\Delta(\Gamma, \nabla)$, it suffices to study the $\mathcal{F}^{2}\mathcal{M}_{m_{1},m_{2},n_{1},n_{2}}$-natural operator $\Delta$ corresponding to the construction $\Delta(\Gamma, \nabla)$.

Using the invariance of $\Delta$ with respect to the homotheties $t \text{id}_{\mathbb{R}^{m_{1},m_{2},n_{1},n_{2}}}$ for $t > 0$, the non-linear Peetre theorem (see [KMS]) and the homogeneous function theorem one can easily observe that $\Delta$ is of order 1 in $\Gamma$ and of order 0 in $\nabla$. Then (using Lemma 4.2, the invariance of $\Delta$ with respect to $\mathcal{F}\mathcal{M}_{m_{1},m_{2},n_{1},n_{2}}$-charts, the regularity of $\Delta$ and the density of respective $\text{GL}(m_{1},m_{2}) \times \text{GL}(n_{1},n_{2})$th orbits) one can rather standardly deduce that $\Delta$ is determined by the values (contractions)

$$(4.5) \quad \left\langle Y^{(n_{2})}_{m_{1}}, \left\langle \Delta(\Gamma, \nabla^{o})(\rho), \frac{\partial}{\partial x^{m_{1}-1}} \right|_{(0,0)} \right\rangle \right\rangle \in \mathbb{R}$$

and

$$(4.6) \quad \left\langle Y^{(n_{2})}_{\rho}, \left\langle \Delta(\Gamma, \nabla^{o})(\rho), \frac{\partial}{\partial x^{m_{1}-1}} \right|_{(0,0)} \right\rangle \right\rangle \in \mathbb{R}$$

for all $\rho \in (J^{1,1,1,1,1}M_{m_{1},m_{2},n_{1},n_{2}})_{(0,0,0,0)}$ and all projectable general connections $\Gamma$ on $\mathbb{R}^{m_{1},m_{2},n_{1},n_{2}}$ of the form (4.2) with coefficients satisfying (4.3) ($Y^{(n_{2})}_{m_{1}}$ and $\frac{\partial}{\partial x^{m_{1}-1}}$ exist as $m_{1} \geq 2$ and $n_{2} \geq 1$), where $\nabla^{o}$ is the flat projectable classical linear connection on the trivial bundle $\mathbb{R}^{m_{1}} \times \mathbb{R}^{m_{2}} \to \mathbb{R}^{m_{1}}$. 


One can easily see that the (local) $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-map $\psi : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ given by

$$\psi^{-1}(v, w) = (v, (w^{q})_{q=1}^{n_1}, w^{(l)}, \ldots, w^{(n_2-1)}, w^{(n_2)} + (w^{(n_2)})^2),$$

where $v = (v^i, v^j) \in \mathbb{R}^{m_1+m_2}$, $w = (w^q, w^s) \in \mathbb{R}^{n_1+n_2}$, preserves

$$j_{(0,0,0,0)}^\Gamma, \nabla^0$$

and sends $Y^{(n_2)}_{m_1}$ into $Y^{(n_2)}_{m_1} + 2y^{(n_2)}_{m_1}Y^{(n_2)}$ over $(0, 0, 0, 0) \in \mathbb{R}^{m_1,m_2,n_1,n_2}$ (we have $y^{(n_2)} = 0$ over $(0, 0, 0, 0)$). Then (by the invariance of $\Delta$ with respect to $\psi$) the values (4.6) for all $\Gamma$ satisfying (4.2) and (4.3) and all $\rho$ as above are determined by the values (4.5) for all $\Gamma$ satisfying (4.2) and (4.3) and all $\rho$ as above.

Consequently, $\Delta$ is uniquely determined by the values (4.5) for all $\Gamma$ satisfying (4.2) and (4.3) and all $\rho$ as above.

On the other hand, by the invariance of $\Delta$ with respect to the “homotheties” $\psi_{t, \tau} : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ (for all $t = (t_1, t_2) \in \mathbb{R}^{m_1+m_2}$ and $\tau = (\tau_1, \tau_2) \in \mathbb{R}^{n_1+n_2}$) given by

$$\psi_{t, \tau}(v, w) = \left(\left(\frac{1}{t_1}v^i, \frac{1}{t_2}v^j\right), (\tau_1w^q, \tau_2w^s)\right),$$

$v = (v^i, v^j) \in \mathbb{R}^{m_1+m_2}$, $w = (w^q, w^s) \in \mathbb{R}^{n_1+n_2}$, we deduce (using the homogeneous function theorem) that the value (4.5) for $\Gamma$ satisfying (4.2) and (4.3) and $\rho$ as above is a constant multiple of $B^{n_2}_{m_1(m_1-1)} = -B^{n_2}_{m_1(m_1-1)}$.

Therefore the vector space of all $\Delta$ (as above) is of dimension $\leq 1$.

The proof of Theorem 4.1 is complete. \[\square\]

Remark 4.3. In Example 3.3 we used the inclusion $\text{im}(\mathcal{R}_\Gamma) \subset T^*M \otimes VJ^{1,1,1}$. We can prove this inclusion as follows. We see that $\mathcal{R}_\Gamma$ is of first order in $\Gamma$. Then (because of Lemma 4.2 and equalities (4.11)) it suffices to observe that

$$\left\langle Y^q_{[j]}, \left( \mathcal{R}_\Gamma(\rho), \frac{\partial}{\partial x^i}(0,0) \right) \right\rangle = 0$$

and

$$\left\langle Y^q_{[j]}, \left( \mathcal{R}_\Gamma(\rho), \frac{\partial}{\partial x^{[j]}}(0,0) \right) \right\rangle = 0$$

for any $\Gamma$ of the form (4.2) with coefficients satisfying (4.3), $j_1 = 1, \ldots, m_2$, $q = 1, \ldots, n_1$, $i = 1, \ldots, m_1$ and any $\rho \in \left( J^{1,1,1} \mathbb{R}^{m_1,m_2,n_1,n_2} \right)_{(0,0,0,0)}$. To show (4.8) and (4.9) we use the invariance of the operator $\mathcal{R}$ with respect to the homotheties (4.7) and then apply the homogeneous function theorem.

Example 4.4. Considering $(m_1, m_2, n_1, n_2)$-dimensional fibred-fibred manifolds $p = (p, p) : (pY : Y \to Y) \to (pM : M \to M)$ as $(m_1 + m_2, n_1 + n_2)$-dimensional fibred manifolds $p : Y \to M$ we have the “inclusion”
$\mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2} \rightarrow \mathcal{F} \mathcal{M}_{m_1+m_2,n_1+n_2}$ (the “forgetting” functor being injective on morphisms). So, we have the “restriction” $J^1 : \mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2} \rightarrow \mathcal{F} \mathcal{M}$ of $J^1 : \mathcal{F} \mathcal{M}_{m_1,m_2,n_1+n_2} \rightarrow \mathcal{F} \mathcal{M}$. Any projectable general connection on an $\mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2}$-object $p$ is also a general connection on the $\mathcal{F} \mathcal{M}_{m_1+m_2,n_1,n_2}$-object $p$. Any torsion-free projectable classical linear connection on the fibred manifold $M$ is also a torsion-free classical linear connection on the manifold $M$. So (because of Example 2.1), for any $t \in \mathbb{R}$ we have the $\mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator $J^1 + t \mathcal{R} : J^1_{\text{proj}} \times Q_{\tau,\text{proj}}(\mathcal{B}) \leadsto J^1(J^1 \rightarrow \mathcal{B})$ producing general connections $\mathcal{J}^1(\Gamma, \nabla) + t \mathcal{R}_\Gamma : J^1 Y \rightarrow J^1 J^1 Y$ on $J^1 Y \rightarrow M$ from projectable general connections $\Gamma$ on $\mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2}$-objects $p = (p, \bar{p}) : (p_Y : Y \rightarrow \mathcal{Y}) \rightarrow (p_M : M \rightarrow M)$ by means of torsion-free projectable classical linear connections $\nabla$ on $p_M : M \rightarrow M$.

Quite similarly to Theorem 4.1 one can prove the following one.

**Theorem 4.5.** If $m_1 \geq 2$ and $n_2 \geq 1$ then all $\mathcal{F} \mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D : J^1_{\text{proj}} \times Q_{\tau,\text{proj}}(\mathcal{B}) \leadsto J^1(J^1 \rightarrow \mathcal{B})$ form the one-parameter family $J^1 + t \mathcal{R}, \ t \in \mathbb{R}$.

**References**


Jan Kurek
Institute of Mathematics
Maria Curie-Skłodowska University
Pl. M. Curie-Skłodowskiej 1
20-031 Lublin, Poland
E-mail: kurek@hektor.umcs.lublin.pl

Włodzimierz M. Mikulski
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: Wlodzimierz.Mikulski@im.uj.edu.pl

Received 8.7.2010
and in final form 16.9.2010