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On prolongations of projectable connections

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Dedicated to Professor Ivan Kolář on the occasion of his 75th birthday with respect and gratitude

Abstract. We extend the concept of r-order connections on fibred manifolds to the one of (r,s,q)-order projectable connections on fibred-fibred manifolds, where r,s,q are arbitrary non-negative integers with $s \geq r \leq q$. Similarly to the fibred manifold case, given a bundle functor F of order r on (m_1,m_2,n_1,n_2)-dimensional fibred-fibred manifolds $Y \to M$, we construct a general connection $\mathcal{F}(\Gamma,\Lambda) : FY \to J^1FY$ on $FY \to M$ from a projectable general (i.e. $(1,1,1)$-order) connection $\Gamma : Y \to J^{1,1,1}Y$ on $Y \to M$ by means of an (r,r,r)-order projectable linear connection $\Lambda : TM \to J^{r,r,r}TM$ on M.

In particular, for $F = J^{1,1,1}$ we construct a general connection $\mathcal{J}^{1,1,1}(\Gamma,\nabla)$: $J^{1,1,1}Y \to J^1J^{1,1,1}Y$ on $J^{1,1,1}Y \to M$ from a projectable general connection Γ on $Y \to M$ by means of a torsion-free projectable classical linear connection ∇ on M. Next, we observe that the curvature of Γ can be considered as $\mathcal{R}_\Gamma : J^{1,1,1}Y \to T^*M \otimes VJ^{1,1,1}Y$. The main result is that if $m_1 \geq 2$ and $n_2 \geq 1$, then all general connections $D(\Gamma,\nabla)$: $J^{1,1,1}Y \to J^1J^{1,1,1}Y$ on $J^{1,1,1}Y \to M$ canonically depending on Γ and ∇ form the one-parameter family $\mathcal{J}^{1,1,1}(\Gamma,\nabla) + t\mathcal{R}_\Gamma$, $t \in \mathbb{R}$. A similar classification of all general connections $D(\Gamma,\nabla) : J^1Y \to J^1J^1Y$ on $J^1Y \to M$ from (Γ,∇) is presented.

1. Introduction. Higher order jets in the sense of C. Ehresmann (see [E2]) constitute a powerful tool in differential geometry and in many areas of mathematical physics. They globalize the theory of differential systems and play an important role in the calculus of variations (see [S], [V]). Higher order connections were first introduced on groupoids by C. Ehresmann (see [E1]) and next on arbitrary fibred manifolds by I. Kolář (see [K1]). Roughly speaking, higher order connections are sections of bundles of higher order jets. Higher order connections play an important role in the theory of higher order absolute differentiation (see [K1]). The theory of jets and connections is closely related to the theory of natural operations in differential

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geometry (see [KMS]). The theory of jets and (principal) connections constitutes the geometrical background for field theories and theoretical physics (see [LR], [MM]).

In the present paper, an *r-order connection* on a fibred manifold $Y \to M$ is a section $\Theta : Y \to J^rY$ of the r-jet prolongation $J^rY \to Y$ of $Y \to M$. For $r = 1$, we obtain the concept of *general connections* on $Y \to M$. A general connection $\Gamma : Y \to J^1Y$ on $Y \to M$ can be equivalently defined as the corresponding lifting map $\Gamma : Y \times_M TM \to TY$. An *r-order linear connection* on a vector bundle $Y \to M$ is an r-order connection on $Y \to M$ which is additionally a vector bundle morphism $\Theta : Y \to J^rY$ covering the identity map id_M of M. An r-order linear connection on a manifold M is an r-order linear connection on the tangent bundle $TM \to M$ of M. A *classical linear connection* on M is a first order linear connection on M. A classical linear connection $\nabla : TM \to J^1TM$ on M can be equivalently defined as the corresponding covariant derivative $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$. A more detailed notion of connection can be found in the fundamental monograph [KMS].

In [K3] (see also [KMS, Section 45.1]), given a bundle functor F of order r on (m, n)-dimensional fibred manifolds $Y \to M$, I. Kolář constructed a general connection $\mathcal{F}(\Gamma, \Lambda) : FY \to J^1FY$ on $FY \to M$ from a general connection $\Gamma : Y \to J^1Y$ on $Y \to M$ by means of an r-order linear connection on M. In particular, for $F = J^1$ he obtained a general connection $\mathcal{J}^1(\Gamma, \nabla) : J^1Y \to J^1J^1Y$ on $J^1Y \to M$ from a general connection Γ on $Y \to M$ by means of a torsion-free classical linear connection ∇ on M. In [KMS, Sections 45.7–8], the authors presented another general connection $P(\Gamma, \nabla) : J^1Y \to J^1J^1Y$ on $J^1Y \to M$ and deduced that all general connections $D(\Gamma, \nabla) : J^1Y \to J^1J^1Y$ on $J^1Y \to M$ canonically depending on Γ and ∇ form the one-parameter family $t\mathcal{J}^1(\Gamma, \nabla) + (1 - t)P(\Gamma, \nabla)$, $t \in \mathbb{R}$, where the first jet prolongation $J^1Z \to Z$ of a fibred manifold $Z \to M$ (in particular of $Z = J^1Y \to M$) is always endowed with the well-known affine bundle structure with the corresponding vector bundle $T^*M \otimes VZ$.

In Section 2 of the present paper we observe that the curvature tensor $\mathcal{R}_\Gamma : Y \to \bigwedge^2 T^*M \otimes VY$ of Γ can be interpreted as the corresponding fibred map $\mathcal{R}_\Gamma : J^1Y \to T^*M \otimes VJ^1Y$ covering the identity map of J^1Y. So, all general connections $D(\Gamma, \nabla) : J^1Y \to J^1J^1Y$ on $J^1Y \to M$ canonically depending on Γ and ∇ form the one-parameter family $\mathcal{J}^1(\Gamma, \nabla) + t\mathcal{R}_\Gamma$, $t \in \mathbb{R}$.

In [M1], the second author defined a *fibred-fibred manifold* to be a fibred surjective submersion $Y \to M$ between fibred manifolds Y and M such that the restrictions of it to fibres are submersions. Moreover, he defined the so-called (r, s, q)-jet prolongation $J^{r,s,q}Y \to Y$ of $Y \to M$. In [K2],

I. Kolář observed that a fibred-fibred manifold can be defined as a fibred square in the sense of J. Pradines (see [P]), and generalized the concept of general connections on fibred manifolds to the one of $(1,1,1)$-order square connections on fibred squares. He also defined linear square connections of order (r,s,q) on fibred manifolds.

In Section 3 of the present paper, we extend the concept of $(1,1,1)$-order square connections to the one of (r,s,q)-order projectable connections $\Theta : Y \to J^{r,s,q}Y$ on fibred-fibred manifolds (also called (r,s,q)-order square connections on fibred squares), where r,s,q are arbitrary non-negative integers with $s \geq r \leq q$. Next, we generalize the above-mentioned "construction" $\mathcal{F}(\Gamma,\nabla)$. Namely, given a bundle functor F of order r on (m_1,m_2,n_1,n_2)-dimensional fibred-fibred manifolds $Y \to M$, we construct a general connection $\mathcal{F}(\Gamma,\Lambda) : FY \to J^1FY$ on $FY \to M$ from a projectable general (i.e. $(1,1,1)$-order square) connection $\Gamma : Y \to J^{1,1,1}Y$ on $Y \to M$ by means of an (r,r,r)-order projectable linear connection $\Lambda : TM \to J^{r,r,r}TM$ on M (i.e. linear square connection of order (r,r,r) on the fibred manifold M). In particular, for $F = J^{1,1,1}$ we obtain a general connection $\mathcal{J}^{1,1,1}(\Gamma,\nabla) : J^{1,1,1}Y \to J^1J^{1,1,1}Y$ on $J^{1,1,1}Y \to M$ from a projectable general connection Γ on $Y \to M$ by means of a torsion-free projectable classical linear connection ∇ on M. Moreover, we observe that the curvature tensor $\mathcal{R}_\Gamma : Y \to \bigwedge^2 T^*M \otimes VY$ can be interpreted as the corresponding fibred map $\mathcal{R}_\Gamma : J^{1,1,1}Y \to T^*M \otimes VJ^{1,1,1}Y$ covering the identity map of $J^{1,1,1}Y$.

In Section 4, we formulate and prove the main result of the present paper saying that if $m_1 \geq 2$ and $n_2 \geq 1$ then all general connections $D(\Gamma,\nabla) : J^{1,1,1}Y \to J^1J^{1,1,1}Y$ on $J^{1,1,1}Y \to M$ canonically depending on a projectable general connection $\Gamma : Y \to J^{1,1,1}Y$ on an (m_1,m_2,n_1,n_2)-dimensional fibred-fibred manifold $Y \to M$ and a torsion-free projectable classical linear connection ∇ on the fibred manifold M form the one-parameter family $\mathcal{J}^{1,1,1}(\Gamma,\nabla) + t\mathcal{R}_\Gamma$, $t \in \mathbb{R}$. A similar classification of all general connections $D(\Gamma,\nabla) : J^1Y \to J^1J^1Y$ on $J^1Y \to M$ canonically depending on $\Gamma : Y \to J^{1,1,1}Y$ and ∇ is also presented.

All manifolds and maps in the present paper are assumed to be of class C^∞.

2. On constructions on connections on fibred manifolds. Let $\Gamma : Y \to J^1Y$ be a general connection on a fibred manifold $p : Y \to M$ and ∇ be a torsion-free classical linear connection on M. Let $\mathcal{J}^1(\Gamma,\nabla) : J^1Y \to J^1J^1Y$ be the induced general connection on $J^1Y \to M$ (see Introduction). Using $\mathcal{J}^1(\Gamma,\nabla)$ one can produce the following family of general connections on $J^1Y \to M$.

Example 2.1. It is well-known that $p^1 : J^1Y \to Y$ (the target jet projection) is (canonically) an affine bundle with the corresponding vector bundle $T^*M \otimes VY$. Then $p^1 : J^1J^1Y \to J^1Y$ is (canonically) an affine bundle with the corresponding vector bundle $T^*M \otimes VJ^1Y$. The curvature tensor $\mathcal{R}_\Gamma : Y \to \bigwedge^2 T^*M \otimes VY$ of $\Gamma : Y \to J^1Y$ can be (in an obvious way) considered as a fibred map $\mathcal{R}_\Gamma : J^1Y \to T^*M \otimes T^*M \otimes VY \subset T^*M \otimes VJ^1Y$ covering the identity map id_{J^1Y}, where the inclusion is induced by the injection $T^*M \otimes VY \to VJ^1Y$ from the known exact sentence $0 \to T^*M \otimes VY \to VJ^1Y \to VY \to 0$ of vector bundles over J^1Y (the obvious pull-backs are not indicated). So, for any $t \in \mathbb{R}$ we have the general connection $D_t(\Gamma, \nabla) := \mathcal{J}^1(\Gamma, \nabla) + t\mathcal{R}_\Gamma : J^1Y \to J^1J^1Y$ on $J^1Y \to M$.

Remark 2.2. The most general concept of natural operators can be found in [KMS]. In particular, an $\mathcal{FM}_{m,n}$*-natural operator* $D : J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(J^1 \to \mathcal{B})$ transforming general connections Γ on fibred manifolds $Y \to M$ and torsion-free classical linear connections ∇ on M into general connections $D(\Gamma, \nabla) : J^1Y \to J^1J^1Y$ on $J^1Y \to M$ is a family of $\mathcal{FM}_{m,n}$-invariant regular operators

$$D : \mathrm{Con}(Y \to M) \times Q_\tau(M) \to \mathrm{Con}(J^1Y \to M)$$

for all $\mathcal{FM}_{m,n}$-objects $Y \to M$, where $\mathrm{Con}(Y \to M)$ is the set of all general connections on $Y \to M$ and $Q_\tau(M)$ is the set of all torsion-free classical linear connections on M. The $\mathcal{FM}_{m,n}$-invariance means that $D(\Gamma, \Lambda)$ is J^1f-related to $D(\Gamma_1, \Lambda_1)$ for any $\Gamma \in \mathrm{Con}(Y \to M)$, $\Gamma_1 \in \mathrm{Con}(Y_1 \to M_1)$, $\nabla \in Q_\tau(M)$ and $\nabla_1 \in Q_\tau(M_1)$ such that Γ is f-related to Γ_1 by an $\mathcal{FM}_{m,n}$-map $f : Y \to Y_1$ covering $\underline{f} : M \to M_1$ (i.e. $J^1f \circ \Gamma = \Gamma_1 \circ f$) and ∇ is $\underline{f}$-related (or more precisely $T\underline{f}$-related) to ∇_1 (i.e. $J^1T\underline{f} \circ \nabla = \nabla_1 \circ T\underline{f}$). The regularity means that D transforms smoothly parametrized families of connections into smoothly parametrized ones.

Thus (because of the canonical character of the construction of $D_t(\Gamma, \nabla)$ in Example 2.1) we have the corresponding $\mathcal{FM}_{m,n}$-natural operator $D_t : J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(J^1 \to \mathcal{B})$ for any $t \in \mathbb{R}$.

We see that the classification result [KMS, Proposition 45.8] mentioned in Introduction can be immediately reformulated as follows.

Proposition 2.3. *All* $\mathcal{FM}_{m,n}$*-natural operators* $D : J^1 \times Q_\tau(\mathcal{B}) \rightsquigarrow J^1(J^1 \to \mathcal{B})$ *form the one-parameter family* $D_t := \mathcal{J}^1 + t\mathcal{R}$, $t \in \mathbb{R}$.

3. On constructions on connections on fibred-fibred manifolds. In this section we extend the results presented in the previous section to fibred-fibred manifolds instead of fibred manifolds.

A *fibred-fibred manifold* is a fibred surjective submersion $p = (p, \underline{p}) : (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ between fibred manifolds $p_Y : Y \to \underline{Y}$ and $p_M : M \to \underline{M}$ covering $\underline{p} : \underline{Y} \to \underline{M}$ such that the restrictions of p to the fibres are submersions, or (equivalently) it is a fibred square $p = (p, p_Y, p_M, \underline{p})$, i.e. a commutative square diagram with arrows being surjective submersions $p : Y \to M$, $p_Y : Y \to \underline{Y}$, $p_M : M \to \underline{M}$ and $\underline{p} : \underline{Y} \to \underline{M}$ such that the system $(p, p_Y) : Y \to M \times_{\underline{M}} \underline{Y}$ of maps p and p_Y is a submersion. If $p^1 = (p^1, \underline{p}^1) : (p^1_{Y^1} : Y^1 \to \underline{Y}^1) \to (p^1_{M^1} : M^1 \to \underline{M}^1)$ is another fibred-fibred manifold then a *fibred-fibred map* $f : Y \to Y^1$ is a system $f = (f, f_1, f_2, \underline{f})$ of maps $f : Y \to Y^1$, $f_1 : \underline{Y} \to \underline{Y}^1$, $f_2 : M \to M^1$ and $\underline{f} : \underline{M} \to \underline{M}^1$ such that the obvious cubic diagram is commutative.

A fibred-fibred manifold $p = (p, \underline{p}) : (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ is of *dimension* (m_1, m_2, n_1, n_2) if $\dim(Y) = m_1 + m_2 + n_1 + n_2$, $\dim(M) = m_1 + m_2$, $\dim(\underline{Y}) = m_1 + n_1$ and $\dim(\underline{M}) = m_1$. The fibred-fibred manifolds of dimension (m_1, m_2, n_1, n_2) and their local fibred-fibred diffeomorphisms form a local admissible category over manifolds (in the sense of [KMS, Section 18]), which will be denoted by $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$. Any $\mathcal{F}^2\mathcal{M}_{m_1.m_2,n_1,n_2}$-object is locally isomorphic to the trivial fibred square (denoted by $\mathbb{R}^{m_1,m_2,n_1,n_2}$) with vertices $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, $\mathbb{R}^{m_1} \times \mathbb{R}^{n_1}$ and $\mathbb{R}^{m_1}$ and arrows being obvious projections.

Let r, s, q be non-negative integers with $s \geq r \leq q$. Let $p = (p, \underline{p}) : (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ be a fibred-fibred manifold of dimension (m_1, m_2, n_1, n_2). According to [KMS, Section 12.19], two fibred sections $\sigma_1, \sigma_2 : (p_M : M \to \underline{M}) \to (p_Y : Y \to \underline{Y})$ of $p : Y \to M$ (i.e. fibred maps with $p \circ \sigma_i = \mathrm{id}_M$) covering sections $\underline{\sigma}_1, \underline{\sigma}_2 : \underline{M} \to \underline{Y}$ of $\underline{p} : \underline{Y} \to \underline{M}$ have the same (r, s, q)-jet $j^{r,s,q}_x \sigma_1 = j^{r,s,q}_x \sigma_2$ at $x \in M$ iff $j^r_x \sigma_1 = j^r_x \sigma_2$, $j^q_{\underline{x}} \underline{\sigma}_1 = j^q_{\underline{x}} \underline{\sigma}_2$, $j^s_x(\sigma_{1|M_{\underline{x}}}) = j^s_x(\sigma_{2|M_{\underline{x}}})$, where $M_{\underline{x}}$ is the fibre of M over $\underline{x} = p_M(x) \in \underline{M}$. The space $J^{r,s,q}Y$ of (r, s, q)-jets of fibred sections $M \to Y$ of $p : Y \to M$ is a fibred manifold over Y with respect to the target projection $p^{r,s,q} : J^{r,s,q}Y \to Y$. If $f = (f, f_1, f_2, f_3) : Y \to Y^1$ is an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-morphism then we have the fibred map $J^{r,s,q}f : J^{r,s,q}Y \to J^{r,s,q}Y^1$ covering f given by $J^{r,s,q}f(j^{r,s,q}_x \sigma) = j^{r,s,q}_{f_2(x)}(f \circ \sigma \circ f_2^{-1})$, $j^{r,s,q}_x \sigma \in J^{r,s,q}Y$. The correspondence $J^{r,s,q} : \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \to \mathcal{FM}$ is a (regular) bundle functor in the sense of [KMS], which is called the *(r, s, q)-jet prolongation functor*. This functor $J^{r,s,q}$ was first introduced by the second author in [M1].

The space $J^{r,s,q}Y$ is also a fibred manifold over $J^q\underline{Y}$ with respect to the projection $p^{r,s,q}_q : J^{r,s,q}Y \to J^q\underline{Y}$ given by $p^{r,s,q}_q(j^{r,s,q}_x \sigma) = j^q_{\underline{x}}\underline{\sigma}$. Consequently, $J^{r,s,q}Y$ can be considered as a fibred-fibred manifold $p^{r,s,q} = (p^{r,s,q}, \underline{p}^q) : (p^{r,s,q}_q : J^{r,s,q}Y \to J^q\underline{Y}) \to (p_Y : Y \to \underline{Y})$, where $\underline{p}^q : J^q\underline{Y} \to \underline{Y}$ is the target projection of the q-jet prolongation of the fibred manifold $\underline{p} : \underline{Y} \to \underline{M}$.

The fibred-fibred manifold $p^{r,s,q}$ is called the (r,s,q)-*jet prolongation* of the fibred-fibred manifold p.

The concept of higher order connections on fibred manifolds can be extended to the one of higher order projectable connections on fibred-fibred manifolds as follows.

Definition 3.1. Let r,s,q be non-negative integers with $s \geq r \leq q$. An (r,s,q)-*order projectable connection* on a fibred-fibred manifold $p = (p, \underline{p}) : (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ is a fibred section $\Theta : (p_Y : Y \to \underline{Y}) \to (p_q^{r,s,q} : J^{r,s,q}Y \to J^q\underline{Y})$ (or briefly a fibred section $\Theta : Y \to J^{r,s,q}Y$) of $p^{r,s,q} = (p^{r,s,q}, \underline{p}^q) : (p_q^{r,s,q} : J^{r,s,q}Y \to J^q\underline{Y}) \to (p_Y : Y \to \underline{Y})$ (or briefly of $p^{r,s,q} : J^{r,s,q}Y \to Y$) covering a section $\underline{\Theta} : \underline{Y} \to J^q\underline{Y}$ of $\underline{p}^q : J^q\underline{Y} \to \underline{Y}$, where $p^{r,s,q}$ is the (r,s,q)-jet prolongation of p.

A *projectable general connection* on a fibred-fibred manifold p is a $(1,1,1)$-order projectable connection $\Gamma : Y \to J^{1,1,1}Y$ on p, or (equivalently) it is a square connection in the sense of [K2] on the fibred square p (i.e. a pair of general connections $\Gamma : Y \times_M TM \to TY$ and $\underline{\Gamma} : \underline{Y} \times_{\underline{M}} T\underline{M} \to T\underline{Y}$ on the fibred manifolds $p : Y \to M$ and $\underline{p} : \underline{Y} \to \underline{M}$ (respectively) such that $\underline{\Gamma} \circ (p_Y \times_{\mathrm{id}_M} Tp_M) = Tp_Y \circ \Gamma$). If $p = (p, \underline{p}) : (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ is a *fibred-fibred vector bundle* (i.e. a fibred-fibred manifold such that $p : Y \to M$ and $\underline{p} : \underline{Y} \to \underline{M}$ are vector bundles and $p_Y : Y \to \underline{Y}$ is a vector bundle map covering $p_M : M \to \underline{M}$), then an (r,s,q)-*order projectable linear connection* on p is by definition an (r,s,q)-order projectable connection $\Theta : Y \to J^{r,s,q}Y$ on the fibred-fibred manifold p such that $\Theta : (p : Y \to M) \to (p \circ p^{r,s,q} : J^{r,s,q}Y \to M)$ is a vector bundle map covering id_M (and consequently $\underline{\Theta} : (\underline{p} : \underline{Y} \to \underline{M}) \to (\underline{p} \circ \underline{p}^q : J^q\underline{Y} \to \underline{M})$ is a vector bundle map covering $\mathrm{id}_{\underline{M}}$). An (r,s,q)-*order projectable linear connection on a fibred manifold* $p_M : M \to \underline{M}$ is an (r,s,q)-order projectable linear connection $\Lambda : TM \to J^{r,s,q}TM$ on the fibred-fibred vector tangent bundle $p_M^T = (p_M^T, p_{\underline{M}}^T) : (Tp_M : TM \to T\underline{M}) \to (p_M : M \to \underline{M})$, or (equivalently) it is a linear square connection of order (r,s,q) in the sense of [K2] on p_M. A *projectable classical linear connection* on a fibred manifold $p_M : M \to \underline{M}$ is a $(1,1,1)$-order projectable linear connection ∇ on $p_M : M \to \underline{M}$, or (equivalently) a classical linear connection ∇ on the manifold M such that there is a (unique) p_M-related (to ∇) classical linear connection $\underline{\nabla}$ on $\underline{M}$.

Let $F : \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \to \mathcal{FM}$ be a (regular) bundle functor of order r in the sense of [KMS]. The construction $\mathcal{F}(\Gamma, \Lambda)$ from [K3] (mentioned in Introduction) can be adapted to the fibred-fibred manifold situation as follows.

EXAMPLE 3.2. Let $\Gamma : Y \to J^{1,1,1}Y$ be a projectable general connection on an (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifold $p = (p, \underline{p}) : (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ and let $\Lambda : TM \to J^{r,r,r}TM$ be an (r, r, r)-order projectable linear connection on the fibred manifold $p_M : M \to \underline{M}$. Let us recall that a *projectable-projectable vector field* on p is a vector field $X \in \mathcal{X}(Y)$ on Y such that there exist underlying vector fields $X_M \in \mathcal{X}(M)$, $X_{\underline{Y}} \in \mathcal{X}(\underline{Y})$ and $X_{\underline{M}} \in \mathcal{X}(\underline{M})$ such that X is p-related to X_M, X is p_Y-related to $X_{\underline{Y}}$, X_M is p_M-related to $X_{\underline{M}}$ and $X_{\underline{Y}}$ is $\underline{p}$-related to $X_{\underline{M}}$, or (equivalently) the flow $\mathrm{Exp}(tX)$ of X is formed by $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-morphisms. So, similarly to the fibred manifold case, the flow operator $\mathcal{F}$ of F lifting projectable-projectable vector fields X on p into vector fields $\mathcal{F}X := \frac{\partial}{\partial t}_{|t=0} F(\mathrm{Exp}(tX))$ on FY (we can apply F as $\mathrm{Exp}(tX)$ is an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map) is of order r, and then it can be interpreted as the flow morphism $\mathcal{F} : FY \times_Y J^r T_{\text{proj-proj}} Y \to TFY$, $\mathcal{F}(v, j^r_y X) = \mathcal{F}X(v)$, $v \in F_y Y$, $y \in Y$, $X \in \mathcal{X}_{\text{proj-proj}}(p : Y \to M)$. Since the general connection $\Gamma : Y \times_M TM \to TY$ on p is projectable, the Γ-horizontal lift $\underline{X}^\Gamma$ of a projectable vector field $\underline{X}$ on p_M (defined by $\underline{X}^\Gamma_{|z} = \Gamma(z, \underline{X}_{|p(z)})$, $z \in Y$) is a projectable-projectable vector field on p. Then (as in the fibred manifold case) we have $\widetilde{\mathcal{F}}\Gamma : FY \times_M J^r T_{\text{proj}} M \to TFY$, $\widetilde{\mathcal{F}}\Gamma(v, j^r_x \underline{X}) = \mathcal{F}(v, j^r_y(\underline{X}^\Gamma))$, $v \in F_y Y$, $y \in Y_x$, $x \in M$, $\underline{X} \in \mathcal{X}_{\text{proj}}(p_M : M \to \underline{M})$. So, applying $\Lambda : TM \to J^{r,r,r}TM = J^r T_{\text{proj}} M$, we get a general connection $\mathcal{F}(\Gamma, \Lambda) =: \widetilde{\mathcal{F}}\Gamma \circ (\mathrm{id}_{FY} \times \Lambda) : FY \times_M TM \to TFY$ on $FY \to M$.

In particular, if $F = J^{1,1,1}$ we have the general connection $\mathcal{J}^{1,1,1}(\Gamma, \nabla) : J^{1,1,1}Y \to J^1 J^{1,1,1}Y$ on $J^{1,1,1}Y \to M$ for any projectable general connection $\Gamma : Y \to J^{1,1,1}Y$ on the fibred-fibred manifold p and a (torsion-free) projectable classical linear connection ∇ on the fibred manifold p_M. Now, quite similarly to Section 2, using $\mathcal{J}^{1,1,1}(\Gamma, \nabla)$ one can produce the following family of general connections on $J^{1,1,1}Y \to M$ from a projectable general connection $\Gamma : Y \to J^{1,1,1}Y$ on p by means of a (torsion-free) projectable classical linear connection ∇ on p_M.

EXAMPLE 3.3. In Example 2.1, we observed that the curvature tensor $\mathcal{R}_\Gamma : Y \to \bigwedge^2 T^*M \otimes VY$ of Γ (treated as a general connection on the fibred manifold $p : Y \to M$) can be considered as the fibred map $\mathcal{R}_\Gamma : J^1Y \to T^*M \otimes VJ^1Y$. Now (see Remark 4.3 in the next section), using the "special" coordinates from Lemma 4.2 and the characterization (4.1) (see the next section) of $VJ^{1,1,1}Y$ (the vertical bundle of $J^{1,1,1}Y \to M$) and recalling what is the curvature of Γ (e.g. from [KMS]), one can rather easily verify that (in our situation of projectable Γ) $\mathcal{R}_\Gamma$ restricts to a fibred map $\mathcal{R}_\Gamma : J^{1,1,1}Y \to T^*M \otimes VJ^{1,1,1}Y$ covering $\mathrm{id}_{J^{1,1,1}Y}$. On the other hand,

$J^1(J^{1,1,1}Y \to M) \to J^{1,1,1}Y$ is (canonically) an affine bundle with the corresponding vector bundle $T^*M \otimes VJ^{1,1,1}Y$. So, given $t \in \mathbb{R}$ we have the general connection $D_t(\Gamma, \nabla) := \mathcal{J}^{1,1,1}(\Gamma, \nabla) + t\mathcal{R}_\Gamma : J^{1,1,1}Y \to J^1J^{1,1,1}Y$ on $J^{1,1,1}Y \to M$.

REMARK 3.4. Quite similarly to Remark 2.2, an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-*natural operator* $D : J^{1,1,1}_{\mathrm{proj}} \times Q_{\tau\text{-proj}}(\mathcal{B}) \rightsquigarrow J^1(J^{1,1,1} \to \mathcal{B})$ is a family of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariant regular operators

$$D : \mathrm{Con}_{\mathrm{proj}}(p : Y \to M) \times Q_{\tau\text{-proj}}(p_M : M \to \underline{M}) \to \mathrm{Con}(J^{1,1,1}Y \to M)$$

for all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-objects p (as above), where $\mathrm{Con}_{\mathrm{proj}}(p : Y \to M)$ is the set of all projectable general connections on the fibred-fibred manifold p, $Q_{\tau\text{-proj}}(p_M : M \to \underline{M})$ is the set of all torsion-free projectable classical linear connections on the fibred manifold p_M and $\mathrm{Con}(J^{1,1,1}Y \to M)$ is the set of all general connections on $J^{1,1,1}Y \to M$.

Thus (because of the canonical character of the construction $D_t(\Gamma, \nabla)$ from Example 3.3) we have the corresponding $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator $D_t : J^{1,1,1}_{\mathrm{proj}} \times Q_{\tau\text{-proj}}(\mathcal{B}) \rightsquigarrow J^1(J^{1,1,1} \to \mathcal{B})$ for any $t \in \mathbb{R}$.

4. The main result. The main result of the present paper is the following classification theorem extending Proposition 2.3.

THEOREM 4.1. *If $m_1 \geq 2$ and $n_2 \geq 1$, then all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D : J^{1,1,1}_{\mathrm{proj}} \times Q_{\tau\text{-proj}}(\mathcal{B}) \rightsquigarrow J^1(J^{1,1,1} \to \mathcal{B})$ form the one-parameter family $D_t := \mathcal{J}^{1,1,1} + t\mathcal{R}$, $t \in \mathbb{R}$.*

The proof of the above theorem will occupy the rest of this section.

For $j = 1, \ldots, m_2$ and $s = 1, \ldots, n_2$ we put $[j] := m_1 + j$ and $\langle s\rangle := n_1 + s$.

Let $x^i, x^{[j]}, y^q, y^{\langle s\rangle}$ be the usual fibred-fibred coordinates on the trivial fibred square $\mathbb{R}^{m_1,m_2,n_1,n_2}$, $y^q_i = \frac{\partial y^q}{\partial x^i}$, $y^q_{[j]} = \frac{\partial y^q}{\partial x^{[j]}}$, $y^{\langle s\rangle}_i = \frac{\partial y^{\langle s\rangle}}{\partial x^i}$, $y^{\langle s\rangle}_{[j]} = \frac{\partial y^{\langle s\rangle}}{\partial x^{[j]}}$ be the additional coordinates on the first jet prolongation $J^1\mathbb{R}^{m_1,m_2,n_1,n_2}$ of the fibred manifold $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, and $Y^q = dy^q$, $Y^{\langle s\rangle} = dy^{\langle s\rangle}$, $Y^q_i = dy^q_i$, $Y^q_{[j]} = dy^q_{[j]}$, $Y^{\langle s\rangle}_i = dy^{\langle s\rangle}_i$, $Y^{\langle s\rangle}_{[j]} = dy^{\langle s\rangle}_{[j]}$ be the essential coordinates on the vertical bundle $VJ^1\mathbb{R}^{m_1,m_2,n_1,n_2}$ of $J^1\mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1+m_2}$, $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$, $q = 1, \ldots, n_1$, $s = 1, \ldots, n_2$.

The $(1,1,1)$-jet prolongation of the fibred-fibred manifold $\mathbb{R}^{m_1,m_2,n_1,n_2}$ can be characterized as the subset $J^{1,1,1}\mathbb{R}^{m_1,m_2,n_1,n_2} \subset J^1\mathbb{R}^{m_1,m_2,n_1,n_2}$ satisfying the equalities $y^q_{[j]} = 0$, $q = 1, \ldots, n_1$, $j = 1, \ldots, m_2$. Similarly, the vertical bundle of $J^{1,1,1}\mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1+m_2}$ is the subset $VJ^{1,1,1}\mathbb{R}^{m_1,m_2,n_1,n_2} \subset VJ^1\mathbb{R}^{m_1,m_2,n_1,n_2}$ satisfying (on $J^{1,1,1}\mathbb{R}^{m_1,m_2,n_1,n_2}$) the equalities

$$Y^q_{[j]} = 0, \quad q = 1, \ldots, n_1,\ j = 1, \ldots, m_2. \tag{4.1}$$

Consequently, on $J^{1,1,1}\mathbb{R}^{m_1,m_2,n_1,n_2}$ we have the additional coordinates $y^q_i = \frac{\partial y^q}{\partial x^i}$, $y^{\langle s\rangle}_i = \frac{\partial y^{\langle s\rangle}}{\partial x^i}$, $y^{\langle s\rangle}_{[j]} = \frac{\partial y^{\langle s\rangle}}{\partial x^{[j]}}$, and on $VJ^{1,1,1}\mathbb{R}^{m_1,m_2,n_1,n_2}$ we have the essential coordinates $Y^q = dy^q$, $Y^{\langle s\rangle} = dy^{\langle s\rangle}$, $Y^q_i = dy^q_i$, $Y^{\langle s\rangle}_i = dy^{\langle s\rangle}_i$, $Y^{\langle s\rangle}_{[j]} = dy^{\langle s\rangle}_{[j]}$, $q = 1,\dots,n_1$, $s = 1,\dots,n_2$, $i = 1,\dots,m_1$, $j = 1,\dots,m_2$.

The following lemma can be treated as a fibred-fibred manifold version of [M2, Proposition 2.2(a) for $r = 1$].

LEMMA 4.2. *Let $\tilde{\Gamma} : Y \times_M TM \to TY$ be a projectable general connection on an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object $p = (p, \underline{p}) : (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ and ∇ be a torsion-free projectable classical linear connection on $p_M : M \to \underline{M}$. Let $y_o \in Y$ and $x_o = p(y_o) \in M$. Then there exists an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-chart ψ on Y covering a ∇-normal fibred coordinate system on M with centre x_o such that $\psi(y_o) = (0,0,0,0)$ and $j^1_{(0,0,0,0)}(\psi_*\tilde{\Gamma}) = j^1_{(0,0,0,0)}\Gamma$, where Γ is of the form*

$$
\begin{aligned}
(4.2)\qquad \Gamma = {} & \sum_{i=1}^{m_1} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{j=1}^{m_2} dx^{[j]} \otimes \frac{\partial}{\partial x^{[j]}} \\
& + \sum_{i_1,i_2=1}^{m_1}\sum_{q=1}^{n_1} A^q_{i_1i_2} x^{i_1} dx^{i_2} \otimes \frac{\partial}{\partial y^q} + \sum_{i_1,i_2=1}^{m_1}\sum_{s=1}^{n_2} B^s_{i_1i_2} x^{i_1} dx^{i_2} \otimes \frac{\partial}{\partial y^{\langle s\rangle}} \\
& + \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\sum_{s=1}^{n_2} C^s_{ij} x^i dx^{[j]} \otimes \frac{\partial}{\partial y^{\langle s\rangle}} + \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\sum_{s=1}^{n_2} D^s_{ji} x^{[j]} dx^i \otimes \frac{\partial}{\partial y^{\langle s\rangle}} \\
& + \sum_{j_1,j_2=1}^{m_2}\sum_{s=1}^{n_2} E^s_{j_1j_2} x^{[j_1]} dx^{[j_2]} \otimes \frac{\partial}{\partial y^{\langle s\rangle}}
\end{aligned}
$$

for some real numbers $A^q_{i_1i_2}$, $B^s_{i_1i_2}$, C^s_{ij}, D^s_{ji} and $E^s_{j_1j_2}$ satisfying

$$(4.3)\qquad A^q_{i_1i_2} = -A^q_{i_2i_1},\quad B^s_{i_1i_2} = -B^s_{i_2i_1},\quad C^s_{ij} = -D^s_{ji},\quad E^s_{j_1j_2} = -E^s_{j_2j_1}$$

for $i, i_1, i_2 = 1,\dots,m_1$, $j, j_1, j_2 = 1,\dots,m_2$, $q = 1,\dots,n_1$, $s = 1,\dots,n_2$.

If ψ is a chart having the above properties, then so is $(A \times B) \circ \psi$ for any $A \in \mathrm{GL}(m_1,m_2)$ (= the group of fibred linear isomorphisms $(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_1}) \to (\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_1})$) and $B \in \mathrm{GL}(n_1,n_2)$.

Proof. Choose an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-chart φ on Y covering a ∇-normal fibred coordinate system on M with centre $x_o \in M$ such that $\varphi(y_o) = (0,0,0,0)$. Replacing $(\tilde{\Gamma}, \nabla)$ by $\psi_*(\tilde{\Gamma}, \nabla)$, we can additionally assume that $Y = \mathbb{R}^{m_1,m_2,n_1,n_2}$, $y_o = (0,0,0,0)$ and that the identity map on $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ is a ∇-normal fibre coordinate system with centre $(0,0)$. So, one can write

$$j^1_{(0,0,0,0)}(\tilde{\Gamma}) = j^1_{(0,0,0,0)}\Bigg(\sum_{i=1}^{m_1} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{j=1}^{m_2} dx^{[j]} \otimes \frac{\partial}{\partial x^{[j]}} + \cdots\Bigg)$$

with the dots denoting

$$
\begin{aligned}
(4.4)\qquad & \sum_{i_1,i_2=1}^{m_1}\sum_{q=1}^{n_1} A^q_{i_1 i_2} x^{i_1} dx^{i_2} \otimes \frac{\partial}{\partial y^q} + \sum_{i_1,i_2=1}^{m_1}\sum_{s=1}^{n_2} B^s_{i_1 i_2} x^{i_1} dx^{i_2} \otimes \frac{\partial}{\partial y^{\langle s\rangle}} \\
& + \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\sum_{s=1}^{n_2} C^s_{ij} x^i dx^{[j]} \otimes \frac{\partial}{\partial y^{\langle s\rangle}} + \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\sum_{s=1}^{n_2} D^s_{ji} x^{[j]} dx^i \otimes \frac{\partial}{\partial y^{\langle s\rangle}} \\
& + \sum_{j_1,j_2=1}^{m_2}\sum_{s=1}^{n_2} E^s_{j_1 j_2} x^{[j_1]} dx^{[j_2]} \otimes \frac{\partial}{\partial y^{\langle s\rangle}} + \sum_{i=1}^{m_1}\sum_{q,q_1=1}^{n_1} a^q_{q_1 i} y^{q_1} dx^i \otimes \frac{\partial}{\partial y^q} \\
& + \sum_{i=1}^{m_1}\sum_{q=1}^{n_1}\sum_{s=1}^{n_2} b^s_{qi} y^q dx^i \otimes \frac{\partial}{\partial y^{\langle s\rangle}} + \sum_{i=1}^{m_1}\sum_{s,s_1=1}^{n_2} c^s_{s_1 i} y^{\langle s_1\rangle} dx^i \otimes \frac{\partial}{\partial y^{\langle s\rangle}} \\
& + \sum_{j=1}^{m_2}\sum_{q=1}^{n_1}\sum_{s=1}^{n_2} d^s_{qj} y^q dx^{[j]} \otimes \frac{\partial}{\partial y^{\langle s\rangle}} + \sum_{j=1}^{m_2}\sum_{s,s_1=1}^{n_2} e^s_{s_1 j} y^{\langle s_1\rangle} dx^{[j]} \otimes \frac{\partial}{\partial y^{\langle s\rangle}} \\
& + \sum_{i=1}^{m_1}\sum_{q=1}^{n_1} f^q_i dx^i \otimes \frac{\partial}{\partial y^q} + \sum_{i=1}^{m_1}\sum_{s=1}^{n_2} g^s_i dx^i \otimes \frac{\partial}{\partial y^{\langle s\rangle}} \\
& + \sum_{j=1}^{m_2}\sum_{s=1}^{n_2} h^s_j dx^{[j]} \otimes \frac{\partial}{\partial y^{\langle s\rangle}}
\end{aligned}
$$

for some real numbers $A^q_{i_1 i_2}, \ldots, h^s_j$ (because of the projectability of $\tilde{\Gamma}$).

Now, replacing $\tilde{\Gamma}$ by $(\psi_1)_*\tilde{\Gamma}$, where $\psi_1 : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ is an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map such that (defined by)

$$
\psi_1(v,w) = \Big(v, \Big(w^q - \sum_{i=1}^{m_1} f^q_i v^i\Big)_{q=1}^{n_1}, \Big(w^{\langle s\rangle} - \sum_{j=1}^{m_2} h^s_j v^{[j]} - \sum_{i=1}^{m_1} g^s_i v^i\Big)_{s=1}^{n_2}\Big)
$$

for any $v = (v^i, v^{[j]}) \in \mathbb{R}^{m_1+m_2}$ and $w = (w^q, w^{\langle s\rangle}) \in \mathbb{R}^{n_1+n_2}$, we can additionally assume that in (4.4) we have $f^q_i = 0$ and $g^s_i = 0$, $h^s_j = 0$.

Next, replacing $\tilde{\Gamma}$ by $(\psi_2)_*\tilde{\Gamma}$, where $\psi_2 : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ is a local $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map such that (defined by)

$$
\psi_2(v,w) = \Big(v, \Big(w^q - \sum_{i=1}^{m_1}\sum_{q_1=1}^{n_1} a^q_{q_1 i} v^i w^{q_1}\Big)_{q=1}^{n_1}, (w^{\langle s\rangle} - \cdots)_{s=1}^{n_2}\Big)
$$

with the dots denoting

$$
\sum_{j=1}^{m_2}\sum_{s_1=1}^{n_2} e^s_{s_1 j} w^{\langle s_1\rangle} v^{[j]} + \sum_{j=1}^{m_2}\sum_{q=1}^{n_1} d^s_{qj} v^{[j]} w^q + \sum_{i=1}^{m_1}\sum_{s_1=1}^{n_2} c^s_{s_1 i} v^i w^{\langle s_1\rangle} + \sum_{i=1}^{m_1}\sum_{q=1}^{n_1} b^s_{qi} v^i w^q
$$

for any $v = (v^i, v^{[j]}) \in \mathbb{R}^{m_1+m_2}$ and $w = (w^q, w^{\langle s\rangle}) \in \mathbb{R}^{n_1+n_2}$, we can additionally assume that in (4.4) we have $e^s_{s_1 j} = 0$, $d^s_{qj} = 0$, $c^s_{s_1 i} = 0$, $b^s_{qi} = 0$ and $a^q_{q_1 i} = 0$.

Finally, replacing $\tilde{\Gamma}$ by $(\psi_3)_*\tilde{\Gamma}$, where $\psi_3 : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ is a local $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map such that (defined by)

$$\psi_3(v,w) = \left(v, \left(w^q - \frac{1}{2}\sum_{i_1,i_2=1}^{m_1}(A^q_{i_1 i_2} + A^q_{i_2 i_1})v^{i_1}v^{i_2}\right)_{q=1}^{n_1}, \left(w^{[s]} - \frac{1}{2}(\cdots)\right)_{s=1}^{n_2}\right)$$

with the dots denoting

$$\sum_{i_1,i_2=1}^{m_1}(B^s_{i_1 i_2}+B^s_{i_2 i_1})v^{i_1}v^{i_2} + \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}(C^s_{ij}+D^s_{ji})v^i v^{[j]} + \sum_{j_1,j_2=1}^{m_2}(E^s_{j_1 j_2}+E^s_{j_2 j_1})v^{[j_1]}v^{[j_2]}$$

for any $v = (v^i, v^{[j]}) \in \mathbb{R}^{m_1+m_2}$ and $w = (w^q, w^{\langle s\rangle}) \in \mathbb{R}^{n_1+n_2}$, we can additionally assume that in (4.4) we have $A^q_{i_1 i_2} = -A^q_{i_2 i_1}$, $B^s_{i_1 i_2} = -B^s_{i_2 i_1}$, $C^s_{ij} = -D^s_{ji}$ and $E^s_{j_1 j_2} = -E^s_{j_2 j_1}$.

Thus the proof of the main part of the lemma is complete.

The last sentence of the lemma is a simple observation. ∎

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Put $\Delta(\Gamma,\nabla) := D(\Gamma,\nabla) - \mathcal{J}^{1,1,1}(\Gamma,\nabla) : J^{1,1,1}Y \to T^*M \otimes VJ^{1,1,1}Y$. As $D(\Gamma,\nabla)$ is determined by $\Delta(\Gamma,\nabla)$, it suffices to study the $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator Δ corresponding to the construction $\Delta(\Gamma,\nabla)$.

Using the invariance of Δ with respect to the homotheties $t\,\mathrm{id}_{\mathbb{R}^{m_1,m_2,n_1,n_2}}$ for $t > 0$, the non-linear Peetre theorem (see [KMS]) and the homogeneous function theorem one can easily observe that Δ is of order 1 in Γ and of order 0 in ∇. Then (using Lemma 4.2, the invariance of Δ with respect to $\mathcal{F}\mathcal{M}_{m_1,m_2,n_1,n_2}$-charts, the regularity of Δ and the density of respective $\mathrm{GL}(m_1,m_2) \times \mathrm{GL}(n_1,n_2)$th orbits) one can rather standardly deduce that Δ is determined by the values (contractions)

$$\left\langle Y^{\langle n_2\rangle}_{m_1\,|\rho}, \left\langle \Delta(\Gamma,\nabla^o)(\rho), \frac{\partial}{\partial x^{m_1-1}}_{|(0,0)}\right\rangle\right\rangle \in \mathbb{R} \tag{4.5}$$

and

$$\left\langle Y^{\langle n_2\rangle}_{|\rho}, \left\langle \Delta(\Gamma,\nabla^o)(\rho), \frac{\partial}{\partial x^{m_1-1}}_{|(0,0)}\right\rangle\right\rangle \in \mathbb{R} \tag{4.6}$$

for all $\rho \in (J^{1,1,1}\mathbb{R}^{m_1,m_2,n_1,n_2})_{(0,0,0,0)}$ and all projectable general connections Γ on $\mathbb{R}^{m_1,m_2,n_1,n_2}$ of the form (4.2) with coefficients satisfying (4.3) ($Y^{\langle n_2\rangle}_{m_1}$ and $\frac{\partial}{\partial x^{m_1-1}}$ exist as $m_1 \geq 2$ and $n_2 \geq 1$), where ∇^o is the flat projectable classical linear connection on the trivial bundle $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_1}$.

One can easily see that the (local) $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-map $\psi : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ given by

$$\psi^{-1}(v,w) = (v, (w^q)_{q=1}^{n_1}, w^{\langle 1\rangle}, \ldots, w^{\langle n_2-1\rangle}, w^{\langle n_2\rangle} + (w^{\langle n_2\rangle})^2),$$

where $v = (v^i, v^{[j]}) \in \mathbb{R}^{m_1+m_2}$, $w = (w^q, w^{\langle s\rangle}) \in \mathbb{R}^{n_1+n_2}$, preserves $\frac{\partial}{\partial x^{m_1-1}}_{|(0,0)}$, $j^1_{(0,0,0,0)}\Gamma$, ∇^o and sends $Y^{\langle n_2\rangle}_{m_1}$ into $Y^{\langle n_2\rangle}_{m_1} + 2y^{\langle n_2\rangle}_{m_1} Y^{\langle n_2\rangle}$ over $(0,0,0,0) \in \mathbb{R}^{m_1,m_2,n_1,n_2}$ (we have $y^{\langle n_2\rangle} = 0$ over $(0,0,0,0)$). Then (by the invariance of Δ with respect to ψ) the values (4.6) for all Γ satisfying (4.2) and (4.3) and all ρ as above are determined by the values (4.5) for all Γ satisfying (4.2) and (4.3) and all ρ as above.

Consequently, Δ is uniquely determined by the values (4.5) for all Γ satisfying (4.2) and (4.3) and all ρ as above.

On the other hand, by the invariance of Δ with respect to the "homotheties" $\psi_{t,\tau} : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ (for all $t = (t_i, t_{[j]}) \in \mathbb{R}^{m_1+m_2}_+$ and $\tau = (\tau_q, \tau_{\langle s\rangle}) \in \mathbb{R}^{n_1+n_2}_+$) given by

$$\psi_{t,\tau}(v,w) = \left(\left(\frac{1}{t_i}v^i\right), \left(\frac{1}{t_{[j]}}v^{[j]}\right), (\tau_q w^q), (\tau_{\langle s\rangle} w^{\langle s\rangle})\right), \tag{4.7}$$

$v = (v^i, v^{[j]}) \in \mathbb{R}^{m_1+m_2}$, $w = (w^q, w^{\langle s\rangle}) \in \mathbb{R}^{n_1+n_2}$, we deduce (using the homogeneous function theorem) that the value (4.5) for Γ satisfying (4.2) and (4.3) and ρ as above is a constant multiple of $B^{n_2}_{(m_1-1)m_1} = -B^{n_2}_{m_1(m_1-1)}$.

Therefore the vector space of all Δ (as above) is of dimension ≤ 1.

The proof of Theorem 4.1 is complete. ∎

Remark 4.3. In Example 3.3 we used the inclusion $\mathrm{im}(\mathcal{R}_\Gamma) \subset T^*M \otimes VJ^{1,1,1}Y$. We can prove this inclusion as follows. We see that $\mathcal{R}_\Gamma$ is of first order in Γ. Then (because of Lemma 4.2 and equalities (4.1)) it suffices to observe that

$$\left\langle Y^q_{[j]|\rho}, \left\langle \mathcal{R}_\Gamma(\rho), \frac{\partial}{\partial x^i}_{|(0,0)} \right\rangle \right\rangle = 0 \tag{4.8}$$

and

$$\left\langle Y^q_{[j]|\rho}, \left\langle \mathcal{R}_\Gamma(\rho), \frac{\partial}{\partial x^{[j_1]}}_{|(0,0)} \right\rangle \right\rangle = 0 \tag{4.9}$$

for any Γ of the form (4.2) with coefficients satisfying (4.3), $j, j_1 = 1, \ldots, m_2$, $q = 1, \ldots, n_1$, $i = 1, \ldots, m_1$ and any $\rho \in (J^{1,1,1}\mathbb{R}^{m_1,m_2,n_1,n_2})_{(0,0,0,0)}$. To show (4.8) and (4.9) we use the invariance of the operator $\mathcal{R}$ with respect to the homotheties (4.7) and then apply the homogeneous function theorem.

Example 4.4. Considering (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifolds $p = (p, \underline{p}) : (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ as (m_1+m_2, n_1+n_2)-dimensional fibred manifolds $p : Y \to M$ we have the "inclusion"

$\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \to \mathcal{F}\mathcal{M}_{m_1+m_2,n_1+n_2}$ (the "forgetting" functor being injective on morphisms). So, we have the "restriction" $J^1 : \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \to \mathcal{F}\mathcal{M}$ of $J^1 : \mathcal{F}\mathcal{M}_{m_1+m_2,n_1+n_2} \to \mathcal{F}\mathcal{M}$. Any projectable general connection on an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object p is also a general connection on the $\mathcal{F}\mathcal{M}_{m_1+m_2,n_1+n_2}$-object p. Any torsion-free projectable classical linear connection on the fibred manifold M is also a torsion-free classical linear connection on the manifold M. So (because of Example 2.1), for any $t \in \mathbb{R}$ we have the $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator $\mathcal{J}^1 + t\mathcal{R} : J^1_{\mathrm{proj}} \times Q_{\tau\text{-proj}}(\mathcal{B}) \rightsquigarrow J^1(J^1 \to \mathcal{B})$ producing general connections $\mathcal{J}^1(\Gamma, \nabla) + t\mathcal{R}_\Gamma : J^1Y \to J^1J^1Y$ on $J^1Y \to M$ from projectable general connections Γ on $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-objects $p = (p, \underline{p}) : (p_Y : Y \to \underline{Y}) \to (p_M : M \to \underline{M})$ by means of torsion-free projectable classical linear connections ∇ on $p_M : M \to \underline{M}$.

Quite similarly to Theorem 4.1 one can prove the following one.

THEOREM 4.5. *If $m_1 \geq 2$ and $n_2 \geq 1$ then all $\mathcal{F}\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D : J^1_{\mathrm{proj}} \times Q_{\tau\text{-proj}}(\mathcal{B}) \rightsquigarrow J^1(J^1 \to \mathcal{B})$ form the one-parameter family $\mathcal{J}^1 + t\mathcal{R}$, $t \in \mathbb{R}$.*

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