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Quasianalytic perturbation of multi-parameter hyperbolic polynomials and symmetric matrices

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Abstract. This paper investigates hyperbolic polynomials with quasianalytic coefficients. Our main purpose is to prove factorization theorems for such polynomials, and next to generalize the results of K. Kurdyka and L. Paunescu about perturbation of analytic families of symmetric matrices to the quasianalytic setting.

Generally, the *perturbation problem* is whether, given a family of monic polynomials with coefficients from a certain class of functions, one can represent their roots as a function of this class. Hyperbolic polynomials with analytic coefficients in one variable were studied by Rellich [24, 25], which was connected with his investigation of the behavior of eigenvalues of symmetric matrices under one-parameter analytic perturbation. This one-parameter theory, initiated by Rellich, culminated in the work of Kato [8]. One-parameter families of hyperbolic polynomials were recently studied in [2, 10] and Kurdyka–Paunescu [12] developed a multi-parameter analytic perturbation theory. Our purpose is to carry over this multi-parameter theory to the quasianalytic setting. This is also the subject of a recent paper [23].

The main purpose of this paper is to establish certain splitting theorems for quasiordinary hyperbolic polynomials with quasianalytic coefficients. Our main results are Theorems 1 and 1* to the effect that every such polynomial splits into linear factors with coefficients which are arc-Q-analytic functions or smooth, definable in the real field with restricted Q-analytic functions, respectively. In order to prove Theorem 1, we introduce the concepts of arc-symmetric Q-set and noetherian arc-symmetric topology, which correspond to those of Kurdyka [11] for semialgebraic sets. Also, we use global (canonical) desingularization results by Bierstone–Milman [4, 5]. Theorem 1*

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is derived from Theorem 1 by means of a description of arc-Q-analytic functions due to Bierstone–Milman–Valette [7] and a quasianalytic version of Glaeser's composite function theorem from our paper [21] (where we demonstrate how to carry over the results of Bierstone–Milman–Pawłucki [6] to the quasianalytic setting). Finally, we derive some applications to perturbation of symmetric matrices with quasianalytic entries.

In our earlier approach to quasianalytic families of hyperbolic polynomials and symmetric matrices (cf. [17]), we applied a generalization of the Abhyankar–Jung theorem for henselian $k[x]$-algebras of formal power series, which are closed under reciprocal, power substitution and division by a coordinate, given in our paper [16]. This allowed us to carry over that theorem to the local rings of quasianalytic function germs in several variables in polynomially bounded o-minimal structures. Our proof of that theorem made use, however, of Luengo's statement that every quasiordinary Weierstrass polynomial in Tschirnhausen form is ν-quasiordinary in the sense of Hironaka. Therefore, those results of ours bear a relative character, because it turned out, as indicated in [9], that Luengo's proof seems to have an essential gap.

As in our previous papers [14, 15], we fix a family $\mathcal{Q} = (\mathcal{Q}_m)_{m\in\mathbb{N}}$ of sheaves of local $\mathbb{R}$-algebras of smooth functions on $\mathbb{R}^m$. For each open subset $U \subset \mathbb{R}^m$, $\mathcal{Q}(U) = \mathcal{Q}_m(U)$ is thus a subalgebra of the algebra $\mathcal{C}^\infty_m(U)$ of smooth real functions on U. By a *Q-analytic function* (or a *Q-function*, for abbreviation) we mean any function $f \in \mathcal{Q}(U)$. Similarly,

$$f = (f_1, \dots, f_k) : U \to \mathbb{R}^k$$

is called a *Q-analytic mapping* (or a *Q-mapping*) if so are its components $f_1, \dots, f_k$. We impose on this family of sheaves the following six conditions:

1. each algebra $\mathcal{Q}(U)$ contains the restrictions of all polynomials;
2. $\mathcal{Q}$ is closed under composition, i.e. the composition of Q-mappings is a Q-mapping (whenever it is well defined);
3. $\mathcal{Q}$ is closed under inverse, i.e. if $\varphi : U \to V$ is a Q-mapping between open subsets $U, V \subset \mathbb{R}^m$, $a \in U$, $b \in V$ and $(\partial\varphi/\partial x)(a) \neq 0$, then there are neighborhoods U_a and V_b of a and b, respectively, and a Q-diffeomorphism $\psi : V_b \to U_a$ such that $\varphi \circ \psi$ is the identity mapping on V_b;
4. $\mathcal{Q}$ is closed under differentiation;
5. $\mathcal{Q}$ is *closed under division by a coordinate*, i.e. if $f \in \mathcal{Q}(U)$ and $f(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m) = 0$ as a function in the variables x_j, $j \neq i$, then $f(x) = (x_i - a_i)g(x)$ with some $g \in \mathcal{Q}(U)$;
6. $\mathcal{Q}$ is *quasianalytic*, i.e. if $f \in \mathcal{Q}(U)$ and the Taylor series $\widehat{f_a}$ of f at a point $a \in U$ vanishes, then f vanishes in the vicinity of a.

We have not attempted to avoid redundancies among these conditions; for instance, it is easy to check, via the Taylor formula, that condition 5 implies condition 4.

By means of Q-mappings, one can build, in the ordinary manner, the category Q of Q-manifolds and Q-mappings, which is a subcategory of that of smooth manifolds and smooth mappings. Also, the category Q gives rise to the geometry of Q-semianalytic and Q-subanalytic sets. What is unavailable in the Q-analytic geometry are the Weierstrass preparation and division theorems, on which most of the classical analytic geometry relies. Instead, it is largely based on transformation to normal crossings by blowing up along smooth centers and desingularization algorithms, which are ensured by the above conditions (as proven in [4, 5, 26]).

Often it is convenient to use the terminology of o-minimal structures. Denote by $\mathcal{R}_Q$ the expansion of $\mathbb{R}$ by restricted Q-functions, which is an o-minimal polynomially bounded structure with exponent field $\mathbb{Q}$ and admits smooth quasianalytic cell decomposition (cf. [26, 14]. From now on, the word "definable" means "definable (with parameters) in the structure $\mathcal{R}_Q$".

Let $\Omega \subset \mathbb{R}^m$ be an open subset. We call a monic polynomial

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathcal{Q}(\Omega)[t]$$

hyperbolic if, for each value of the parameter $x \in \Omega$, all the roots of f are real. This is an abbreviated name for a "quasianalytic family of hyperbolic polynomials". The following result refers to one-parameter hyperbolic polynomials.

PROPOSITION 1. *Every hyperbolic polynomial*

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathcal{Q}(c,d)[t]$$

with coefficients quasianalytic on an interval $(c,d) \subset \mathbb{R}$ *splits into linear factors over* $\mathcal{Q}(c,d)$.

The problem being local, we can regard the coefficients of the polynomial as Q-analytic germs at $0 \in \mathbb{R}$. The counterpart of the Newton–Puiseux theorem holds for the local ring $\mathcal{Q}_1$ of Q-analytic germs at $0 \in \mathbb{R}$, because it is a discrete valuation ring (cf. [18, Remark 3]). Therefore all roots of $f(x;t)$ are of the form

$$\sum_{k=0}^{r-1} \alpha_k(x) \cdot x^{k/r}, \quad \text{where} \quad \alpha_k(x) \in \mathcal{Q}_1,\ r = n!$$

(*loc. cit.*, Remark 2). But all algebraic conjugates

$$\sum_{k=0}^{r-1} \alpha_k(x) \cdot (\epsilon^i x^{1/r})^k, \qquad i = 0, \ldots, r-1,$$

where ϵ is a primitive rth root of unity, are roots of $f(x;t)$ too. Since all the roots are real, it follows that

$$\alpha_k(x) \equiv 0 \quad \text{for all } k = 1, \dots, r-1,$$

as required.

Let M be a Q-manifold. We call a function $g : M \to \mathbb{R}$ *arc-quasianalytic* (or *arc-Q-analytic*) if, for every Q-analytic arc $\gamma : (-\delta, \delta) \to M$, the superposition $g \circ \gamma$ is Q-analytic. It is well known that every arc-Q-function is continuous (cf. [11, 3, 20]).

Now, consider a monic polynomial

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \dots + a_0(x) \in \mathcal{Q}(\Omega)[t]$$

with quasianalytic coefficients on an open subset $\Omega \subset \mathbb{R}^m$. We call $f(x;t)$ a *quasiordinary polynomial* if its discriminant $D(x)$ is a normal crossing on Ω. Our main purpose is to establish a multi-parameter counterpart of the foregoing result, namely, the following theorem on splitting a hyperbolic quasiordinary polynomial into linear factors over the ring of arc-Q-analytic functions:

Theorem 1 (Splitting of hyperbolic quasiordinary polynomials). *Let Ω be an open, simply connected subset of $\mathbb{R}^m$. Then every hyperbolic quasiordinary polynomial*

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \dots + a_0(x) \in \mathcal{Q}(\Omega)[t]$$

splits into linear factors of the form

$$f(x;t) = \prod_{i=1}^{n} (t - \psi_i(x)), \quad x \in \Omega,$$

where $\psi_i(x)$ are arc-Q-functions on Ω. Furthermore, the functions $\psi_i(x)$, $i = 1, \dots, n$, enjoy at every point $P \in \Omega$ the following property (T)*:*

(T) *there exist formal power series Ψ_i, $i = 1, \dots, n$, centered at P such that for each Q-analytic arc $\gamma(\tau)$ through P, $\gamma(0) = P$, not contained in the zero locus of the discriminant $D(x)$ of $f(x;t)$, the Taylor series of $\psi_i \circ \gamma$ at zero is $\Psi_i \circ \widehat{\gamma}$, $i = 1, \dots, n$; here $\widehat{\gamma}$ denotes the Taylor series of γ at zero.*

Before proceeding with the proof of this theorem, we shall introduce some terminology and results indispensable for our approach. Let M be a definable, locally closed, smooth submanifold of an affine space $\mathbb{R}^N$. By a *Q-leaf* in M we mean a definable, connected, locally closed subset that is a smooth submanifold of M. We say that a definable subset $E \subset M$ is an *arc-symmetric Q-subset* when one of the two equivalent conditions holds:

(i) if $\gamma : (-1,1) \to M$ is a Q-analytic arc such that $\gamma(-1,0) \subset E$, then $\gamma(-1,1) \subset E$;
(ii) if $\gamma : (-1,1) \to M$ is a Q-analytic arc such that $\mathrm{Int}(\gamma^{-1}(E)) \neq \emptyset$, then $\gamma(-1,1) \subset E$.

This is an o-minimal generalization of the concept of an arc-symmetric semialgebraic subset, introduced by Kurdyka [11] and inspired by [13]. By the curve selection lemma, every arc-symmetric Q-subset E of M is closed.

Applying smooth definable stratifications (and even decompositions into Q-leaves instead) and the global (canonical) desingularization of quasianalytic hypersurfaces due to Bierstone–Milman [4, 5], we are able to carry over Kurdyka's results about arc-symmetric semialgebraic sets to the case of arc-symmetric Q-sets. Below, we state three quasianalytic counterparts of his Theorem 1.4, Theorem 2.6 and Corollary 2.8, whose proofs can be repeated almost verbatim.

PROPOSITION 2. *There exists a unique noetherian topology $\mathcal{AS}$ on M whose closed sets are precisely the arc-symmetric Q-subsets of M.*

This follows, by induction on the dimension of the ambient space M, from the decomposition of a definable set into finitely many Q-leaves and from

OBSERVATION. *Let $E \subset M$ be an arc-symmetric Q-subset and $\Gamma \subset M$ a Q-leaf. If $\dim(\Gamma \cap R) = \dim \Gamma$, then $\Gamma \subset E$.*

The topology described above will be called the *arc-symmetric topology.*

THEOREM 2. *Let Y be a reduced Q-analytic hypersurface in $\mathbb{R}^m$ of dimension $m-1$. Further, let U be an open, definable, relatively compact subset of $\mathbb{R}^m$, $X := Y \cap U$ and E an arc-symmetric Q-subset of X of dimension $m-1$. Consider a (canonical) desingularization $\sigma : \widetilde{X} \to X$ which is a finite sequence of global blowings-up with smooth centers. Then E is irreducible in the arc-symmetric topology iff there is a (unique) connected component C of $\widetilde{X}$ such that $\sigma(C)$ coincides with the closure (in the ordinary topology) of the set $\mathrm{Reg}_{m-1}(E)$ of those points at which E is smooth of dimension $m-1$.*

Kurdyka's proof applies directly to the above quasianalytic version of his Theorem 2.6, because the image $\sigma(E)$ of the exceptional divisor E of σ is an arc-symmetric Q-subset of dimension $< m-1$.

COROLLARY 1. *Under the foregoing assumptions, if E is an arc-symmetric Q-subset of X of dimension $m-1$, then the following two conditions are equivalent:*

(i) *E is irreducible in the arc-symmetric topology;*
(ii) *any two points of the closure $\overline{\mathrm{reg}_{m-1}(E)}$ of $\mathrm{reg}_{m-1}(E)$ can be joined by a Q-analytic arc lying in $\overline{\mathrm{reg}_{m-1}(E)}$.*

REMARK 1. In the proof of this corollary, Kurdyka's argument showing that any given points P, Q on a connected real analytic submanifold V in a real affine space $\mathbb{R}^N$ can be joined by an analytic arc (by means of geodesics) should be replaced by two observations. First, the problem reduces to the case of a connected open subset of $\mathbb{R}^N$ via a tubular neighborhood (in the category of Q-manifolds) (W, ϱ) of the Q-submanifold V under study, where $\varrho : W \to V$ is a retraction. Indeed, one must compose a Q-analytic arc in W with the retraction ϱ. Next, take a broken line $\gamma : [0,1] \to W$, $\gamma(0) = P$, $\gamma(1) = Q$, which lies in W and joins the two points $P = (P_1, \ldots, P_N)$, $Q = (Q_1, \ldots, Q_N) \in V \subset \mathbb{R}^N$. Then, applying a variant of the Weierstrass approximation theorem, stated below, it suffices to approximate the components $\gamma_i(t)$, $i = 1, \ldots, N$, of γ by real polynomials $p_i(t)$ in one variable t such that $p_i(0) = P_i$ and $p_i(1) = Q_i$ for $i = 1, \ldots, N$.

LEMMA 1. *Consider a continuous real function $f(t)$ on a compact interval $[c, d]$ and a finite number of points $\xi_1, \ldots, \xi_s \in [c, d]$. Then for each real number $\varepsilon > 0$, there exists a polynomial $p(t)$ such that*

$$|f(t) - p(t)| < \varepsilon \quad \text{for all } t \in [c,d] \quad \text{and} \quad f(\xi_1) = p(\xi_1), \ldots, f(\xi_s) = p(\xi_s).$$

This lemma is a combination of Weierstrass approximation and Lagrange interpolation with nodes $\xi_1, \ldots, \xi_s$.

At this stage we can turn to the proof of our main result.

Proof of Theorem 1. The zero locus

$$Y := \{(x,t) \in \Omega \times \mathbb{R}_t : f(x;t) = 0\} \subset \mathbb{R}^m_x \times \mathbb{R}_t$$

is a reduced hypersurface of codimension 1. Due to the identity principle for arc-Q-functions, the splitting problem is local. We may thus confine our proof to an open, relatively compact subset

$$U = (-\delta, \delta)^m \times (-M, M)$$

of $\Omega \times \mathbb{R}$ which contains all roots of $f(x;t)$ over $(-\delta,\delta)^m$, and such that the discriminant of $f(x,t)$ is of the form $D(x) = x^\alpha \cdot u(x)$, where $\alpha \in \mathbb{N}^m$ and $u(x)$ is a nowhere vanishing, Q-analytic function on $(-\delta,\delta)^m$. In what follows, we shall apply Theorem 2 to the trace $X := Y \cap U$.

The set $\{x \in (-\delta,\delta)^m : D(x) \neq 0\}$ contains the union of 2^m open cubes Q_j cut out from U by the cross $x_1 \cdot \ldots \cdot x_m = 0$. Obviously, the set X over each cube Q_j is the union of the graphs of n Q-analytic and Q-subanalytic functions $\psi_{j,1}(x), \ldots, \psi_{j,n}(x)$ on Q_j such that

$$\psi_{j,1}(x) < \cdots < \psi_{j,n}(x) \quad \text{for all } x \in Q_j.$$

By the classical Abhyankar–Jung theorem (see e.g. [1, 9, 18]), the formal roots $\Psi_1(x), \ldots, \Psi_n(x)$ of $f(x;t)$ at $0 \in \mathbb{R}^m$ satisfy

$$\Psi_1(x), \ldots, \Psi_n(x) \in \mathbb{C}[[x^{1/r}]], \qquad x^{1/r} = (x_1^{1/r}, \ldots, x_m^{1/r}),\ r = n!.$$

But, by Proposition 1, the roots of the restriction of $f(x;t)$ to each polynomial arc γ through zero can be arranged as n Q-analytic functions at zero. Hence, the restrictions of the fractional power series $\Psi_i(x^{1/r})$ to each polynomial arc γ through zero are real formal power series. Therefore $\Psi_1(x),\dots,\Psi_n(x) \in \mathbb{R}[[x]]$ by the lemma below.

LEMMA 2. *Let*

$$\Psi(x,y) = \sum_{i,\alpha} a_{i,\alpha} x^{i/r} y^{\alpha/r} \in \mathbb{C}[[x^{1/r}, y^{1/r}]], \quad i \in \mathbb{N},\ \alpha \in \mathbb{N}^m,\ r \in \mathbb{N},\ r > 1,$$

be a fractionary power series, where $x = x_1$ and $y = (y_1,\dots,y_m)$. Suppose that $a_{j,\beta} \neq 0$ for some $j \in \mathbb{N} \setminus r\mathbb{N}$, $\beta \in \mathbb{N}^m$. Then there is a polynomial arc γ such that $\Psi \circ \gamma$ is fractionary but not a power series.

Indeed, consider polynomial arcs γ parametrized by

$$x = \tau, \quad y = (c\tau)^r, \quad c \in \mathbb{C}^m.$$

Then

$$(\Psi \circ \gamma)(\tau) = \sum_p \Big(\sum_{i/r+|\alpha|=p} a_{i,\alpha} c^\alpha \Big) \tau^p.$$

Put $q := j/r + |\beta|$. Clearly, the polynomial

$$P(C) := \sum_{i/r+|\alpha|=q} a_{i,\alpha} C^\alpha, \quad C = (C_1,\dots,C_m),$$

does not vanish identically, because $a_{j,\beta} \neq 0$, and if $i/r + |\alpha| = j/r + |\beta|$ and $\alpha = \beta$, then $i = j$. Now it suffices to take $c \in \mathbb{C}^m$ for which $P(c) \neq 0$.

The ring $\mathbb{R}[[\tau]]$ of formal power series in one variable τ is an ordered ring by putting $\tau > 0$. Consider formal arcs $\gamma(\tau) = (\gamma_1(\tau),\dots,\gamma_m(\tau)) \in (\mathbb{R}[[\tau]])^m$ through zero. In the following lemma about the roots $\Psi_i(x)$, the open cubes Q_j are regarded, by abuse of notation, as the subsets of $(\mathbb{R}[[\tau]])^m$ given by the same inequalities as those of $\mathbb{R}^m$.

LEMMA 3. *Suppose a formal arc $\gamma(\tau)$ through zero lies in an open cube Q_j. If $\Psi_i(\gamma) < \Psi_k(\gamma)$, then this inequality remains true for every formal arc $\vartheta(\tau)$ through zero lying in that cube. In other words, one can order the formal roots $\Psi_i(x)$, $i = 1,\dots,n$, with respect to any open cube Q_j.*

We first observe that the formal power series $\Psi_i(x)$ can be approximated in the Krull topology by convergent power series $\Psi_{i,\nu}(x)$, $i = 1,\dots,n$, $\nu \in \mathbb{N}$, such that the discriminants of the polynomials

$$f_\nu(x) := \prod_{i=1}^n (t - \Psi_{i,\nu}(x)), \quad \nu \in \mathbb{N},$$

are of the form $D_\nu(x) = x^\alpha \cdot u_\nu(x)$ with $u_\nu(0) \neq 0$. Indeed, the discriminant $D(x) = x^\alpha \cdot u(x)$ with $u(0) \neq 0$ of the polynomial $f(x,t)$ can be expressed as follows:

$$D(x) = P(\Psi_1(x), \dots, \Psi_n(x)) \in \mathbb{R}[[x]],$$

where $P(T_1, \dots, T_n) \in \mathbb{Z}[T_1, \dots, T_n]$ is a polynomial with integer coefficients; here we identify Q-analytic function germs at zero with their Taylor series. Our assertion thus follows from the Artin approximation theorem applied to the system of two polynomial equations

$$P(A_1, \dots, A_n) = x^\alpha \cdot U, \qquad U \cdot V = 1,$$

with indeterminates $A_1, \dots, A_n, U, V$.

Therefore, $\Psi_{i,\nu}(\gamma) < \Psi_{k,\nu}(\gamma)$ for ν large enough. By approximation in the Krull topology, we may assume that γ is a convergent arc through zero. It is clear that this inequality remains true for every convergent arc through zero lying in the cube Q_j. Let ϑ be a formal arc through zero lying in Q_j. Then, again by approximation in the Krull topology, $\Psi_{i,\nu}(\vartheta) \leq \Psi_{k,\nu}(\vartheta)$, and hence $\Psi_i(\vartheta) \leq \Psi_k(\vartheta)$. But $\Psi_i(\vartheta) \neq \Psi_k(\vartheta)$ because the discriminant $D \circ \vartheta$ of the restriction of the polynomial $f(x;t)$ to the formal arc ϑ is not zero, and thus Lemma 3 follows.

Lemma 3 is valid, of course, not only at $0 \in U$, but at each point $a \in U$. Theorem 1 will be established once we prove that the arc-symmetric Q-set X has, in the $\mathcal{AS}$ topology, precisely n irreducible components C_i, $i = 1, \dots, n$, each of which is the graph of an arc-Q-function.

To this end, observe that in the desingularization from Theorem 2 one can ensure that the set

$$\widetilde{X} \setminus \bigcup_{ji} \sigma^{-1}(\mathrm{graph}(\psi_{j,i}))$$

is a normal crossing divisor (locally, the zero locus of the Jacobian determinant of σ or a submanifold of codimension 1; cf. [5, Theorem 5.10]). Since σ is a proper modification which is a Q-diffeomorphism over each $\mathrm{graph}(\psi_{j,i})$, we get

$$\sigma(\partial(\sigma^{-1}(\mathrm{graph}(\psi_{j,i}))) = \partial(\mathrm{graph}(\psi_{j,i}));$$

here ∂A denotes the frontier of a set A. Let Q_j, Q_l be two adjacent cubes with common face $Q_{j,l}$. We first consider our problem of irreducible components in the arc-symmetric topology over the open subset $\Omega_{j,l} := Q_j \cup Q_l \cup Q_{j,l}$. In a similar fashion, we can proceed with attaching successive cubes.

We introduce the following notation:

$$X_j := X \cap (Q_j \times \mathbb{R}), \quad X_l := X \cap (Q_l \times \mathbb{R}), \quad X_{j,l} := X \cap (\Omega_{j,l} \times \mathbb{R})$$

$$\widetilde{X}_j := \widetilde{X} \cap \sigma^{-1}(Q_j \times \mathbb{R}), \ \widetilde{X}_l := \widetilde{X} \cap \sigma^{-1}(Q_l \times \mathbb{R}), \ \widetilde{X}_{j,l} := \widetilde{X} \cap \sigma^{-1}(\Omega_{j,l} \times \mathbb{R}),$$

$$A_{j,i} = \mathrm{graph}(\psi_{j,i}) \quad \text{and} \quad \widetilde{A}_{j,i} = \sigma^{-1}(A_{j,i}).$$

It is not difficult to check that the closure (in the ordinary topology) of each $A_{j,i}$ is the graph of a continuous function on $Q_j \cup Q_{j,l}$. Since the roots of a monic polynomial depend continuously on the coefficients, we get $\partial X_j = \partial X_l$; here the frontier ∂ is with respect to the ambient space $\Omega_{j,l} \times \mathbb{R}$. Then

$$\widetilde{X}_j = \bigcup_i \widetilde{A}_{j,i}, \quad \widetilde{X}_l = \bigcup_k \widetilde{A}_{l,k}$$

and $\widetilde{X}_{j,l} \setminus (\widetilde{X}_j \cup \widetilde{X}_l)$ is a normal crossing divisor. Clearly,

$$\partial\widetilde{A}_{j,i} = \bigcup_{k\neq i}(\partial\widetilde{A}_{j,i} \cap \partial\widetilde{A}_{j,k}) \cup \bigcup_k(\partial\widetilde{A}_{j,i} \cap \partial\widetilde{A}_{l,k}),$$

whence

$$\partial A_{j,i} = \sigma(\partial\widetilde{A}_{j,i}) = \bigcup_{k\neq i} \sigma(\partial\widetilde{A}_{j,i} \cap \partial\widetilde{A}_{j,k}) \cup \bigcup_k \sigma(\partial\widetilde{A}_{j,i} \cap \partial\widetilde{A}_{l,k}).$$

We still need the following

Claim. *The sets $\partial\widetilde{A}_{j,i}$, $i = 1,\dots,n$, are pairwise disjoint, as also are the sets $\sigma(\partial\widetilde{A}_{j,i} \cap \partial\widetilde{A}_{l,k})$, $k = 1,\dots,n$.*

Suppose, on the contrary, that two sets $\partial\widetilde{A}_{j,i}$ and $\partial\widetilde{A}_{j,p}$, with $p \neq i$, have a common point $\widetilde{P}$. Take a Q-analytic arc $\eta : (-1,1) \to \widetilde{X}_{j,l}$ such that $\eta(-1,0) \subset \widetilde{A}_{j,p}$, $\eta(0,1) \subset \widetilde{A}_{j,i}$ and $\eta(0) = \widetilde{P}$. Then

$$(\sigma \circ \eta)(\tau) = (\gamma(\tau), \psi(\tau)), \quad \tau \in (-1,1),$$

for some Q-analytic γ and ψ. Let $\pi : \mathbb{R}^m_x \times \mathbb{R}_t \to \mathbb{R}^m_x$ be the canonical projection.

We may assume, by a linear change of variables, that $(\pi\circ\sigma)(\widetilde{P}) = 0 \in \mathbb{R}^m$. Keep the notation of Lemma 3 and order the formal roots $\Psi_i(x)$, $i = 1,\dots,n$, with respect to the open cube Q_j. The arcs $\gamma(\tau)$ and $\gamma(-\tau)$ lie in the cube Q_j, $\psi(\tau)$ occurs at the ith place among $\Psi_1(\gamma(\tau)),\dots,\Psi_n(\gamma(\tau))$, and $\psi(-\tau)$ occurs at the pth place among $\Psi_1(\gamma(-\tau)),\dots,\Psi_n(\gamma(-\tau))$, which contradicts the conclusion of Lemma 3.

To prove the second assertion, suppose that two sets

$$\sigma(\partial\widetilde{A}_{j,i} \cap \partial\widetilde{A}_{l,k}) \quad \text{and} \quad \sigma(\partial\widetilde{A}_{j,i} \cap \partial\widetilde{A}_{l,q}), \qquad \text{for some } q \neq k,$$

have a common point $P = \sigma(\widetilde{P}) = \sigma(\widetilde{S})$ with

$$\widetilde{P} \in \partial\widetilde{A}_{j,i} \cap \partial\widetilde{A}_{l,k} \quad \text{and} \quad \widetilde{S} \in \partial\widetilde{A}_{j,i} \cap \partial\widetilde{A}_{l,q}.$$

Take two Q-analytic arcs $\eta, \zeta : (-1,1) \to \widetilde{X}_{j,l}$ such that $\eta(-1,0) \subset \widetilde{A}_{l,k}$, $\eta(0,1) \subset \widetilde{A}_{j,i}$, $\eta(0) = \widetilde{P}$, $\zeta(-1,0) \subset \partial\widetilde{A}_{l,q}$, $\zeta(0,1) \subset \partial\widetilde{A}_{j,i}$ and $\zeta(0) = \widetilde{S}$.

Then

$$(\sigma \circ \eta)(\tau) = (\gamma(\tau), \psi(\tau)), \quad \tau \in (-1,1),$$
$$(\sigma \circ \zeta)(\tau) = (\vartheta(\tau), \varphi(\tau)), \quad \tau \in (-1,1),$$

for some Q-analytic γ, ϑ, ψ and φ.

As before, assume that $\pi(P) = 0 \in \mathbb{R}^m$. The arcs $\gamma(\tau)$, $\vartheta(\tau)$ lie in the cube Q_j, and the arcs $\gamma(-\tau)$, $\vartheta(-\tau)$ lie in the cube Q_l. Clearly,

$$\psi(\tau) = \Psi_i(\gamma(\tau)) \quad \text{and} \quad \varphi(\tau) = \Psi_i(\vartheta(\tau)),$$

and thus

$$\psi(-\tau) = \Psi_i(\gamma(-\tau)) \quad \text{and} \quad \varphi(-\tau) = \Psi_i(\vartheta(-\tau)).$$

Hence and by Lemma 3, the roots $\psi(-\tau)$ and $\varphi(-\tau)$ for $\tau > 0$ belong to a common set $A_{l,p}$. This contradiction completes the proof of the claim.

In view of the claim, the connected set $\partial A_{j,i}$ is the disjoint union of the closed subsets $\sigma(\partial \widetilde{A}_{j,i} \cap \partial \widetilde{A}_{l,k})$, $k = 1, \ldots, n$. Consequently,

$$\partial A_{j,i} = \sigma(\partial \widetilde{A}_{j,i} \cap \partial \widetilde{A}_{l,p})$$

for a unique $p = \omega(i) \in \{1, \ldots, n\}$, and

$$\partial \widetilde{A}_{j,i} \cap \partial \widetilde{A}_{l,k} = \emptyset \quad \text{for all } k = 1, \ldots, n,\ k \neq \omega(i).$$

Hence $\partial \widetilde{A}_{j,i} \subset \partial \widetilde{A}_{l,\omega(i)}$, and thus, by symmetry, we get $\partial \widetilde{A}_{j,i} = \partial \widetilde{A}_{l,\omega(i)}$. Consequently,

$$\widetilde{A}_{j,i} \cup \widetilde{A}_{l,\omega(i)} \cup \partial \widetilde{A}_{j,i}$$

is the connected component (in the ordinary topology) of $\widetilde{X}_{j,l}$. Hence and by Theorem 2,

$$A_{j,i} \cup A_{l,\omega(i)} \cap \partial A_{j,i}$$

is the irreducible component (in the arc-symmetric topology) of $X_{j,l}$ containing $A_{j,i}$. Therefore the arc-symmetric Q-subset $X_{j,l}$ of $\Omega_{j,l} \times \mathbb{R}$ has precisely n irreducible arc-symmetric components C_i, $i = 1, \ldots, n$, of the form

$$C_i = A_{j,i} \cup A_{l,\omega(i)} \cap \partial A_{j,i} = \operatorname{graph}(\chi_i),$$

where $i \mapsto \omega(i)$ is a permutation of $\{1, \ldots, n\}$ and each χ_i is a continuous function which extends the functions $\psi_{j,i}$ and $\psi_{l,\omega(i)}$.

It remains to show that each χ_i is an arc-Q-function with property (T). So let $\gamma : (-1,1) \to \Omega_{j,l}$ be a Q-analytic arc and put

$$X_\gamma := \{(\tau, t) \in (-1,1) \times \mathbb{R} : f(\gamma(\tau), t) = 0\} = (\gamma \times \mathrm{Id})^{-1}(X).$$

By Proposition 1, X_γ is the union of the graphs of n, not necessarily distinct, Q-analytic functions on $(-1,1)$, say, $\varphi_1(\tau), \ldots, \varphi_n(\tau)$. Then

$$C_i \subset \bigcup_{k=1}^{n} (\gamma, \varphi_k)(-1,1) \quad \text{and} \quad \bigcup_{k=1}^{n} (\gamma, \varphi_k)^{-1}(C_i) = (-1,1).$$

Hence $\mathrm{Int}(\gamma, \varphi_k)^{-1}(C_i) \neq \emptyset$ for some k, and thus $(\gamma, \varphi_k)(-1, 1) \subset C_i$, because C_i is an arc-symmetric Q-subset. This demonstrates that

$$(\chi_i \circ \gamma)(\tau) = \varphi_k(\tau) \quad \text{ for all } \tau \in (-1, 1),$$

and thus $\chi_i \circ \gamma$ is a Q-analytic function, as required. Property (T) follows directly from Lemma 3.

Finally, we can repeat the above arguments when attaching successive cubes Q_p. In this manner, we see that X has precisely n irreducible components in the $\mathcal{AS}$ topology, each of which is the graph of an arc-Q-function. This completes the proof of Theorem 1.

Now, let A be a normal domain, K its quotient field and $f(t) \in A[t]$ a monic polynomial. It is well known from commutative algebra that if $f(t)$ is irreducible over A, then it is so over K. Consequently, if the field K is of characteristic zero, then every irreducible monic polynomial $f(t) \in A[t]$ is square-free.

We cannot apply this algebraic assertion directly to the domain $\mathcal{Q}(\Omega)$. However, each Q-meromorphic fraction

$$a(x)/b(x), \quad a(x), b(x) \in \mathcal{Q}(\Omega),$$

which is integral over $\mathcal{Q}(\Omega)$, can be transformed to a Q-analytic function over any open, relatively compact subset $U \subset \Omega$ by a finite sequence of global blowings-up with smooth centers. Indeed, one can transform the functions $a(x)$ and $b(x)$ by global blowing up to normal crossings, which are locally of the form $y^\alpha \cdot u(y)$ and $y^\beta \cdot v(y)$, respectively, where $u(y)$, $v(y)$ are units and either $\alpha \leq \beta$ or $\alpha \geq \beta$. In our case, since the fraction $y^{\alpha-\beta} \cdot u(y)v^{-1}(y)$ is integral over the ring of Q-analytic functions, we must have $\alpha \geq \beta$, which is the desired result. Consequently, one can adapt the foregoing algebraic assertion to the quasianalytic setting as follows.

PROPOSITION 3 (On square-free factorization). *If $f(x; t)$ is a monic polynomial with coefficients in $\mathcal{Q}(\Omega)$ and $U \subset \Omega$ is an open, relatively compact subset, then there exists a Q-modification $\sigma : \widetilde{U} \to U$ such that*

- (i) *σ is a composite of finitely many global blowings-up with smooth centers;*
- (ii) *the pull-back polynomial $f^\sigma(y; t)$ factorizes globally into a product of square-free monic polynomials with coefficients in $\mathcal{Q}(\widetilde{U})$.*

Hence and via transformation to normal crossings by blowing up, we obtain the following

COROLLARY 2. *Every monic polynomial $f(x; t) \in \mathcal{Q}(\Omega)[t]$ with quasianalytic coefficients factorizes, after a suitable transformation of its coefficients by a finite sequence of global blowings-up with smooth centers over any open,* *relatively compact subset $U \subset \Omega$, into a product of quasiordinary polynomials with quasianalytic coefficients. Moreover, if the polynomial $f(x;t)$ is hyperbolic, so are its modified quasiordinary factors.*

Corollary 3 and Theorem 1 yield immediately the following theorem on the splitting of hyperbolic polynomials:

Corollary 3. *If $f(x;t)$ is a hyperbolic polynomial with coefficients in $\mathcal{Q}(\Omega)$ and $U \subset \Omega$ is an open, relatively compact subset, then there exists a Q-modification $\sigma : \widetilde{U} \to U$ such that*

(i) *σ is a composite of finitely many blowings-up with smooth centers;*
(ii) *the pull-back polynomial $f^{\sigma}(y;t)$ has, locally in the vicinity of each point $b \in \widetilde{U}$, a factorization of the form*

$$f(y;t) = \prod_{i=1}^{n} (t - \psi_i(y)),$$

where $\psi_i(y)$ are arc-Q-functions near b.

Before drawing subsequent corollaries, we state two theorems on rectilinearization of quasianalytic functions and arc-quasianalytic functions. The latter will play a key role in our approach to hyperbolic polynomials. They are a counterpart of two theorems by Bierstone–Milman [3] from real analytic geometry, and are proven in a yet unpublished manuscript by Bierstone–Milman–Valette [7]. It should be emphasized, however, that the quasianalytic versions are much weaker than the analytic ones, because they refer only to quasianalytic functions which satisfy a non-trivial Q-analytic equation. The passage from functions with subanalytic graphs to functions with semianalytic graphs needs much stronger algebro-analytic methods unavailable in the quasianalytic setting such as, for instance, complexification and the techniques of flattening or equidimensionality (see also [19, 20]).

Theorem 3 (Rectilinearization of Q-subanalytic functions). *Let M be a (connected) Q-manifold, $U \subset M$ a relatively compact, open, definable subset, and $g : M \to \mathbb{R}$ a continuous Q-subanalytic function which satisfies a Q-analytic equation of the form $\Phi(x, g(x)) = 0$, $\Phi \not\equiv 0$. Then there exists a surjective Q-mapping $\sigma : \widetilde{U} \to U$ such that*

(i) *σ is a composite of finitely many Q-mappings, each of which is either a blowing-up with smooth center or a surjection of the form $\coprod U_j \to \bigcup U_j$, where $(U_j)_j$ is a finite covering of the target space by coordinate charts and $\coprod$ denotes disjoint union, or a local power substitution of the form*

$$U_j \to U_j, \quad (u_1, \dots, u_m) \mapsto (\varepsilon_1 u_1^{r_1}, \dots, \varepsilon_m u_m^{r_m}),$$

where $(u_1, \dots, u_m)$ *are the coordinates of* U_j, $r_1, \dots, r_m > 0$ *are integers and* $\varepsilon_1 \dots, \varepsilon_m = \pm 1$;

(ii) $g \circ \sigma$ *is a Q-analytic function.*

THEOREM 4 (Description of arc-Q-analytic functions). *Let* M *be a (connected) Q-manifold,* $U \subset M$ *a relatively compact, open, definable subset, and* $g : M \to \mathbb{R}$ *an arc-Q-analytic function which satisfies a Q-analytic equation of the form* $\Phi(x, g(x)) = 0$, $\Phi \not\equiv 0$. *Then there exists a surjective Q-mapping* $\sigma : \widetilde{U} \to U$ *such that*

(i) σ *is a composite of finitely many Q-mappings, each of which is either a blowing-up with smooth center or a surjection of the form* $\coprod U_j \to \bigcup U_j$, *where* $(U_j)_j$ *is a finite covering of the target space by coordinate charts;*

(ii) $g \circ \sigma$ *is a Q-analytic function.*

From Corollary 4 and Theorem 4 we can immediately derive the following two results, which carry over those from [12] to the quasianalytic setting.

COROLLARY 4. *If* $f(x;t)$ *is a hyperbolic polynomial with coefficients in* $\mathcal{Q}(\Omega)$ *and* $U \subset \Omega$ *is an open, relatively compact subset, then there exists a Q-modification* $\sigma : \widetilde{U} \to U$ *such that*

(i) σ *is a composite of finitely many Q-mappings, each of which is either a blowing-up with smooth center or a surjection of the form* $\coprod U_j \to \bigcup U_j$, *where* $(U_j)_j$ *is a locally finite covering of the target space by coordinate charts;*

(ii) *the pull-back polynomial* $f^\sigma(y;t)$ *has locally, in the vicinity of each* $b \in \widetilde{U}$, *a Q-analytic factorization of the form*

$$f^\sigma(y;t) = \prod_{i=1}^{n} (t - \psi_i(y)).$$

COROLLARY 5. *Under the foregoing assumptions, there exists a Q-subanalytic subset* $\Sigma \subset \Omega$ *of codimension at least* 2 *such that, in the vicinity of each* $a \in \Omega \setminus \Sigma$, *the hyperbolic polynomial* $f(x;t)$ *has a Q-analytic factorization*

$$f(x;t) = \prod_{i=1}^{n} (t - \psi_i(x)).$$

We shall now apply the foregoing description of arc-Q-analytic functions and the quasianalytic version of Glaeser's composite function theorem from [21] to achieve the following strengthening of Theorem 1.

THEOREM 1*. *Let* Ω *be an open, simply connected subset of* $\mathbb{R}^m$. *Then every hyperbolic quasiordinary polynomial*

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \dots + a_0(x) \in \mathcal{Q}(\Omega)[t]$$

splits into linear factors of the form

$$f(x;t) = \prod_{i=1}^{n} (t - \psi_i(x)), \quad x \in \Omega,$$

where $\psi_i(x)$ are (C^∞) smooth functions on Ω definable in the structure $\mathcal{R}_Q$.

As in the case of Theorem 1, we may confine our proof to an open, relatively compact subset

$$U = (-\delta, \delta)^m \times (-M, M)$$

of $\Omega \times \mathbb{R}$ which contains all roots of $f(x;t)$ over $(-\delta,\delta)^m$. Consider a surjective Q-mapping $\sigma : \widetilde{U} \to U$ which is a finite composite of local blowings-up with smooth centers such that the superpositions

$$\widetilde{\psi}_i := \psi_i(x) \circ \sigma : \widetilde{U} \to \mathbb{R}, \quad i = 1, \ldots, n,$$

are Q-analytic functions. We show that each function $\widetilde{\psi}_i$ is formally composite with σ, i.e. for each $P \in U$, there is a formal power series Φ at P such that, for every point $\widetilde{P} \in \sigma^{-1}(P)$ on the fiber over P, the Taylor series $\mathrm{T}_{\widetilde{P}} \widetilde{\psi}_i$ of $\widetilde{\psi}_i$ at $\widetilde{P}$ coincides with $\Phi \circ \mathrm{T}_{\widetilde{P}}\, \sigma$. For Φ, we should take the formal power series Ψ_i from property (T) in the conclusion of Theorem 1. Indeed, for a generic straight line $l(\tau)$ through $\widetilde{P}$, $l(0) = \widetilde{P}$, we have

$$\Psi_i \circ \mathrm{T}_{\widetilde{P}}\, \sigma \circ l = \Psi_i \circ \mathrm{T}_{\widetilde{P}}\, (\sigma \circ l) = \mathrm{T}_{\widetilde{P}}\, (\psi_i \circ \sigma \circ l) = \mathrm{T}_{\widetilde{P}}\, (\widetilde{\psi}_i \circ l) = \mathrm{T}_{\widetilde{P}}\, \widetilde{\psi}_i \circ l.$$

Hence $\Psi_i \circ \mathrm{T}_{\widetilde{P}}\, \sigma = \mathrm{T}_{\widetilde{P}}\, (\widetilde{\psi}_i)$, as required.

Since the mapping σ is generically a submersion, it follows from the quasianalytic version of the composite function theorem from [21] that $\widetilde{\psi}_i$ is a composite function with σ, i.e. there is a smooth mapping $\varphi_i : U \to \mathbb{R}$ such that $\widetilde{\varphi}_i = \varphi_i \circ \sigma$. Therefore, the mappings φ_i and ψ_i coincide over an open and dense subset of U, hence over the whole of U. This completes the proof.

Before turning to perturbation of symmetric matrices, we introduce some terminology. We shall state some results about perturbation of matrices with quasianalytic entries in terms of quasianalytic function germs. Denote by $\mathcal{M}_{m,a}$ the field of quasi-meromorphic germs at $a \in \mathbb{R}^m$, i.e. the quotient field of the local ring $\mathcal{Q}_{m,a}$ of Q-analytic germs at a point $a \in \mathbb{R}^m$. In the vector spaces $\mathbb{R}^n$ and $(\mathcal{M}_{m,a})^n$ over the fields $\mathbb{R}$ and $\mathcal{M}_{m,a}$, respectively, the standard inner products are given by

$$v \bullet w = v_1 w_1 + \cdots + v_n w_n, \quad f(x) \bullet g(x) = f_1(x) g_1(x) + \cdots + f_n(x) g_n(x).$$

The spectral theorem for symmetric matrices is valid for any real closed field. The assumption of real closedness is necessary to ensure that the characteristic polynomial of a given symmetric matrix, which is always hyperbolic, factorizes into linear factors. We are able to dispense with it, but we must apply Corollary 5 instead, i.e. split the characteristic polynomial into linear factors via blowing up. Therefore, repeating mutatis mutandis the proof of the spectral theorem from linear algebra, we obtain the following counterpart over the field of quasi-meromorphic function germs.

THEOREM 5 (Spectral theorem with quasianalytic parameters). *Let N be a symmetric $n \times n$ matrix with quasianalytic entries from $\mathcal{Q}_{m,0}$. Then we can find a modification $\sigma : W \to U$ of a neighborhood U of $0 \in \mathbb{R}^m$, which is a finite composite of local blowings-up with smooth centers, such that for each point $b \in \sigma^{-1}(0) \subset W$ the vector space $(\mathcal{M}_{W,b})^n$ over the field $\mathcal{M}_{W,b}$ has an orthogonal basis*

$$w^1(y), \ldots, w^n(y) \in (\mathcal{Q}_{W,b})^n$$

that consists of eigenvectors of the pull-back matrix N^σ.

A matrix with entries from the $\mathbb{R}$-algebra $\mathcal{Q}_{m,a}$ may be regarded as a quasianalytic family of real matrices parametrized by x. Our next objective is to achieve a simultaneous quasianalytic diagonalization of the pull-back matrix N^σ after performing a suitable modification σ which is a finite composite of local blowings-up with smooth centers. Let

$$\lambda_1(y), \ldots, \lambda_n(y) \in \mathcal{Q}_{W,b}$$

be the eigenvalues of N^σ, which may not be pairwise distinct. The above theorem yields a quasianalytic family

$$w^1(y), \ldots, w^n(y) \in \mathbb{R}^n$$

of orthogonal eigenvectors which form a basis of $\mathbb{R}^n$ generically near $b \in W$, say over $W_0 = W \setminus \Sigma$ where $\Sigma \subset W$ is a closed Q-analytic subset of codimension at least one. Fix a vector $w(b) = w^j(b)$, $j = 1, \ldots, n$, and its eigenvalue $\lambda(b) = \lambda_j(b)$. Take any sequence $(b_k) \subset W_0$ with $b_k \to b$ and such that the limit

$$v(b) = \lim_{k\to\infty} v(b_k), \quad \text{where} \quad v(b_k) = \frac{w(b_k)}{\|w(b_k)\|},$$

exists. We obviously have

$$(N^\sigma(b) - \lambda(b)) \cdot v(b) = \lim_{k\to\infty} (N^\sigma(b_k) - \lambda(b_k)) \cdot v(b_k) = 0.$$

Since the vectors $w^j(y)$ are pairwise orthogonal, we obtain in this manner an orthonormal basis

$$v^1(b), \ldots, v^n(b) \in \mathbb{R}^n$$

that consists of eigenvectors of the matrix $N^\sigma(b)$.

We wish to construct a quasianalytic family

$$v^1(y), \ldots, v^n(y) \in \mathbb{R}^n$$

of orthonormal bases that consist of eigenvectors of $N^\sigma(y)$. Clearly, this will be possible once we know that all the components of each vector $w^j(y)$, $j = 1, \ldots, n$, are divisible in $\mathcal{Q}_{W,b}$ by one of them. It is well known that the last condition can be ensured via a successive transformation to normal crossings by blowing up. Consequently, we have obtained the theorem below, which generalizes to the quasianalytic setting the result of Kurdyka–Paunescu [12] about real analytic perturbation of symmetric matrices.

THEOREM 6 (On quasianalytic diagonalization of symmetric matrices). *Consider a symmetric $n \times n$ matrix N with entries from $\mathcal{Q}_{m,0}$. Then there exists a modification $\sigma : W \to U$ of a neighborhood U of zero, which is a finite composite of local blowings-up with smooth centers, such that the pull-back matrix N^σ admits a simultaneous quasianalytic diagonalization near each point $b \in \sigma^{-1}(0) \subset W$. This diagonalization can be performed by a Q-analytic choice of orthonormal bases that consist of eigenvectors of N^σ.*

REMARK 2. Both Theorems 5 and 6 with quasianalytic parameters remain valid, with the same proof, in the case of quasianalytic families of hermitian matrices with entries from $\mathcal{Q}_{m,0} \otimes_{\mathbb{R}} \mathbb{C}$.

REMARK 3. As shown by Kurdyka–Paunescu, all the above results can be carried over to the case of polynomials with purely imaginary roots, and hence to antisymmetric matrices. Indeed, a polynomial

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathcal{Q}_{m,0}[t]$$

has purely imaginary roots iff the polynomial $i^{-n} f(x;it)$ is hyperbolic (cf. [12] for details).

REMARK 4. It is well known that, in general, one cannot find bases of eigenvectors even in a continuous way. This is caused by the fact that the angle between linearly independent eigenvectors, which correspond to distinct eigenvalues, may tend to zero when approaching a given point b.

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