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Continuity of the relative extremal function on analytic varieties in $\mathbb{C}^n$

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Abstract. Let V be an analytic variety in a domain $\Omega \subset \mathbb{C}^n$ and let $K \subset\subset V$ be a closed subset. By studying Jensen measures for certain classes of plurisubharmonic functions on V, we prove that the relative extremal function ω_K is continuous on V if Ω is hyperconvex and K is regular.

1. Introduction. Let $P(D)$ be a linear partial differential operator with constant coefficients. Hörmander [9] gave a characterization of when $P(D)$ is surjective on the space $\mathcal{A}(\Omega)$ of real-analytic functions when Ω is a convex domain in $\mathbb{R}^n$. This characterization is stated in terms of Phragmén–Lindelöf type estimates for plurisubharmonic functions on the zero variety of the symbol of $P(D)$. Hörmander's result has been the main inspiration for trying to find various kinds of geometric or algebraic criteria for recognizing varieties satisfying such Phragmén–Lindelöf estimates, and there are a large number of papers attacking this problem. One of the tools that have been used is the relative extremal function ω_K. (See for example [1, 2, 3].) The main goal of this paper is to prove that ω_K is continuous if K is regular. To state the result more precisely, we will require some preliminary definitions.

Definition 1.1. Let Ω be an open subset in $\mathbb{C}^n$. An *analytic variety* V in Ω is defined as a closed analytic subset of Ω. (See Chirka [5].) If $x \in V$ and there is an open neighborhood U of x such that $V \cap U$ is irreducible, we say that V is *locally irreducible at* x. We denote by V_{irr} the set of locally irreducible points in V. If $x \notin V_{\mathrm{irr}}$, we say that V is *reducible at* x; we denote the set of reducible points by V_{red}. Note that V_{irr} is an open dense subset of V containing all regular points of V.

Definition 1.2. Let V be an analytic variety in some open set $\Omega \subset \mathbb{C}^n$ and let U be an open subset of V. A function $u : U \to [-\infty, \infty)$ is *plurisub-*

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harmonic if it is upper semicontinuous and $u \circ f$ is subharmonic on $\mathbb{D}$ for every holomorphic $f : \mathbb{D} \to U$. (Here, $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$.) We let $\mathcal{PSH}(U)$ denote the set of all plurisubharmonic functions on U.

Remark. Functions in $\mathcal{PSH}(U)$ are sometimes called *weakly plurisubharmonic*. In the series of papers mentioned above, when studying Phragmén–Lindelöf estimates, a slightly larger class of plurisubharmonic functions is used. Let us say that u is *almost plurisubharmonic* on U, denoted $u \in \widetilde{\mathcal{PSH}}(U)$, if u is plurisubharmonic (in the usual sense) on U_{reg}, the regular points of U, and satisfies

$$u(z) = \varlimsup_{U_{\mathrm{reg}} \ni \zeta \to z} u(\zeta)$$

for all $z \in U \setminus U_{\mathrm{reg}}$. It turns out that in the study of the relative extremal function, it does not matter much whether we consider $\mathcal{PSH}(U)$ or $\widetilde{\mathcal{PSH}}(U)$. We will come back to these questions in Section 3.

A function on U is called *strongly plurisubharmonic* if it extends to a plurisubharmonic function on an open neighborhood of U in $\mathbb{C}^n$. By a deep result of Fornæss and Narasimhan [8], any weakly plurisubharmonic function is in fact strongly plurisubharmonic (this is also true in the more general setting of complex spaces), and we will make good use of this fact later on. On the other hand, almost plurisubharmonic functions do not necessarily extend to any neighborhood of the variety, as shown by the following example:

Example 1.3. Let $V = \{(z,w) \in \mathbb{B}^2 : zw = 0\}$, where $\mathbb{B}^2$ is the unit ball in $\mathbb{C}^2$, and define a function u on V by $u(z,0) = 0$ when $z \neq 0$, and $u(0,w) = 1$. Then $u \in \widetilde{\mathcal{PSH}}(V)$ (but $u \notin \mathcal{PSH}(V)$), and it is clear that u does not extend to a plurisubharmonic function in any neighborhood in $\mathbb{C}^2$ of the origin.

Most of the usual properties of plurisubharmonic functions in $\mathbb{C}^n$ carry over to the setting of plurisubharmonic functions on a variety, but when the variety is not locally irreducible, there are some subtle differences. Most notably, (the upper semicontinuous regularization of) the supremum of an upper bounded family of plurisubharmonic functions is not necessarily plurisubharmonic.

Example 1.4. Let $V = \{(z,w) \in \mathbb{B}^2 : zw = 0\}$ and for $\varepsilon > 0$, define a function u_ε on V by

$$u_\varepsilon(z,w) = \begin{cases} 1 + \varepsilon \log|w|, & z = 0, \\ \varepsilon \log|z|, & w = 0. \end{cases}$$

Then each u_ε is in $\mathcal{PSH}(V)$ but $(\sup u_\varepsilon)^* = u$, with u as in Example 1.3.

However, this is the only thing that can go wrong when forming suprema of locally upper bounded families of plurisubharmonic functions. More precisely, we have the following theorem:

THEOREM 1.5. *Let V be an analytic variety in $\Omega \subset \mathbb{C}^n$ and let $\mathcal{F} \subset \mathcal{PSH}(V)$ be a locally uniformly upper bounded family. Define $U(z) = \sup_{u\in\mathcal{F}} u(z)$. If V is locally irreducible, then $U^* \in \mathcal{PSH}(V)$. More generally, $U^* \in \mathcal{PSH}(V_{\mathrm{irr}})$. Furthermore, $U^* \in \widetilde{\mathcal{PSH}}(V)$ even if V is not locally irreducible.*

Similarly, if $\widetilde{\mathcal{F}} \subset \widetilde{\mathcal{PSH}}(V)$ is a uniformly locally upper bounded family, and $\widetilde{U}(z) = \sup_{u\in\widetilde{\mathcal{F}}} u(z)$, then $\widetilde{U}^* \in \widetilde{\mathcal{PSH}}(V)$.

Proof. The plurisubharmonicity of U^* (and $\widetilde{U}^*$) at every regular point in V follows readily from the theory of plurisubharmonic functions on $\mathbb{C}^n$. Also, upper semicontinuity of $\widetilde{U}^*$ and $\widetilde{U}^*$ implies that they are both almost plurisubharmonic on V.

To show that U^* is plurisubharmonic on V_{irr}, we can as well assume that V is locally irreducible (since the result is local). It follows from a removable singularity theorem for plurisubharmonic functions on locally irreducible varieties (if V is locally irreducible and $u \in \mathcal{PSH}(V \setminus X)$ where X is a subvariety of V and u is locally upper bounded near X, then $u^* \in \mathcal{PSH}(V)$) that U^* is in fact plurisubharmonic everywhere. See Demailly [6, Théorème 1.7] for a proof of the removable singularity theorem. The proof is not immediate, and relies on Hironaka's desingularization theorem. ∎

Let us move on to the definition of the relative extremal function.

DEFINITION 1.6. Let V be an analytic variety in $\Omega \subset \mathbb{C}^n$ and let K be a closed subset of V. We define the relative extremal function for K (and V, Ω) by

$$\omega_K(z) = \omega_{K,V,\Omega}(z) = \sup\{u(z) : u \in \mathcal{PSH}(V),\ u \le 0,\ u|_K \le -1\}.$$

In general, ω_K is not plurisubharmonic even on V_{irr} (it need not be upper semicontinuous), but the upper semicontinuous regularization ω_K^* is. If $\omega_K = \omega_K^*$ on V_{irr}, or equivalently if ω_K is continuous at K, we say that K is *regular*.

We can now state the main result of the paper:

THEOREM 1.7. *Let V be a locally irreducible analytic variety in $\Omega \subset \mathbb{C}^n$ and let K be a closed subset of V. If Ω is hyperconvex and K is regular, then $\omega_K \in C(\overline{V} \cap \overline{\Omega})$. In general, if K is regular and Ω is hyperconvex, but V is not necessarily locally irreducible, then ω_K is continuous on V_{irr}.*

REMARK. Here and in the following, $\overline{V}$ denotes the closure of V in $\overline{\Omega}$.

Recall also that a domain $\Omega \subset \mathbb{C}^n$ is said to be *hyperconvex* if there is a function $v \in \mathcal{PSH}(\Omega) \cap C(\overline{\Omega})$, with $v < 0$ on Ω and $v = 0$ on $\partial\Omega$.

To prove Theorem 1.7, we will introduce two different convex subcones of $\mathcal{PSH}(V)$ and their associated Jensen measures. By adapting an approximation result by Cegrell for plurisubharmonic functions on hyperconvex domains [4] (see also [11]), we will prove that the two classes of Jensen measures coincide, and from this fact, Theorem 1.7 will follow.

Remark. The analogous result for the relative extremal function for a regular compact subset of a hyperconvex domain is well known. (See [10] for the original proof.)

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2. Jensen measures and approximations of plurisubharmonic functions on analytic varieties. Let X be a compact metric space, and let $\mathcal{F}$ be a cone of real-valued, upper bounded and upper semicontinuous functions on X containing all the constants. If g is a real-valued function on X, then we define

$$Sg(z) = \sup\{u(z) : u \in \mathcal{F},\, u \le g\}.$$

Let $z \in X$ and define a class of positive measures by $\mathcal{J}_z^{\mathcal{F}} = \{\mu : u(z) \le \int u\, d\mu$ for all $u \in \mathcal{F}\}$. One can verify that $\mathcal{J}_z^{\mathcal{F}}$ is a convex, weak-* compact set. If g is a Borel function on X, we define $Ig(z) = \inf\{\int g\, d\mu : \mu \in \mathcal{J}_z^{\mathcal{F}}\}$. Note that every measure in $\mathcal{J}_z^{\mathcal{F}}$ is a probability measure.

The measures in $\mathcal{J}_z^{\mathcal{F}}$ are called *Jensen measures* for the cone $\mathcal{F}$, and the main reason for introducing these measures is the following duality theorem by Edwards [7]. A more accessible proof can be found in [11].

Theorem 2.1 (Edwards' Theorem). *Let $\mathcal{F}$ be as above. If g is lower semicontinuous on X, then $Sg = Ig$.*

For our purposes, $\mathcal{F}$ will be a convex subcone of $\mathcal{PSH}(V)$. In particular, let us introduce the following definitions:

Definition 2.2. Let V be an analytic variety in $\Omega \subset \mathbb{C}^n$. Define

$$\mathcal{PSH}^0(V) = \{u \in \mathcal{PSH}(V) : u^*|_{\overline{V}\cap\partial\Omega} = u_*|_{\overline{V}\cap\partial\Omega} = \text{const}\},$$

the set of plurisubharmonic functions with constant boundary values. (Here u^* and u_* denote the upper and lower semicontinuous regularizations of u, respectively.) Furthermore, define

$$\mathcal{PSH}^0_{\mathrm{c}}(V) = \mathcal{PSH}^0(V) \cap C(\overline{V} \cap \overline{\Omega}).$$

It is clear that $\mathcal{PSH}^0_{\mathrm{c}}(V) \subset \mathcal{PSH}^0(V)$ and that both classes are convex subcones of $\mathcal{PSH}(V)$. For $z \in \overline{V} \cap \overline{\Omega}$, we let $\mathcal{J}_z^0$ and $\mathcal{J}_{\mathrm{c},z}^0$ denote the corresponding Jensen measures.

THEOREM 2.3. *Let V be an analytic variety in $\Omega \subset \mathbb{C}^n$ and assume that Ω is hyperconvex. If $u \in \mathcal{PSH}^0(V)$, then there exists a sequence (u_j) with $u_j \in \mathcal{PSH}^0_{\mathrm{c}}(V)$ such that $u_j \searrow u^*$ on $\overline{V} \cap \overline{\Omega}$.*

Proof. By adding a constant, we may as well assume that $u^* = u_* = 0$ on $\overline{V} \cap \partial\Omega$. Let $h \in \mathcal{PSH}(\Omega) \cap C(\overline{\Omega})$ be a bounded plurisubharmonic exhaustion function for Ω with $h|_{\partial\Omega} = 0$. (Such a function exists, since Ω is assumed to be hyperconvex.)

By Fornæss and Narasimhan [8], we can extend u to be plurisubharmonic on some open neighborhood $U \supset V$ in Ω. This extension will still be denoted by u.

For $\varepsilon > 0$, define $\Omega_\varepsilon = \{z \in \Omega : h(z) < -\varepsilon\}$. For each positive integer j, let $\varepsilon_j = 1/2j^2$ and choose $r_j > 0$ such that $r_j < d_j := \operatorname{dist}(\Omega_{\varepsilon_j}, \partial\Omega)$ and $r_j < \operatorname{dist}(V \cap \Omega_{\varepsilon_j}, \partial U)$. By decreasing r_j further, we may assume that $r_j \searrow 0$. Define $U_j = \{z \in U \cap \Omega_{d_j} : \operatorname{dist}(z, \partial U) > r_j\}$.

Let $\psi \in C_0^\infty(\mathbb{C}^n)$ be a non-negative radial function with support in the unit ball and $\int \psi\, dV = 1$, and let $\psi_\delta(z) = \delta^{-n}\psi(z/\delta)$. Define $u_j = u * \psi_{r_j}$ and let

$$(2.1) \qquad \widetilde{u}_m(z) = \begin{cases} \max\{u_m(z) - 1/m, mh(z)\}, & z \in V \cap U_m, \\ mh(z), & z \in V \setminus U_m. \end{cases}$$

Note that on $V \cap \partial U_m$, $u_m \le -1/m$ and $mh = -1/2m$, so $\widetilde{u}_m$ is well defined, plurisubharmonic on V and continuous on $\overline{V}$. For simplicity of notation, we will use the shorthand notation $\widetilde{u}_m = \max\{u_m(z) - 1/m, mh(z)\}$ for (2.1).

Define $v_j(z) = \sup_{m \ge j}\{\widetilde{u}_m(z)\}$. We claim that each v_j is plurisubharmonic on V and continuous on $\overline{V}$. Clearly, v_j is lower semicontinuous on $\overline{V}$, being the supremum of continuous functions, and $v_j^*|_{\overline{V} \cap \partial\Omega} = (v_j)_*|_{\overline{V} \cap \partial\Omega} = 0$. Furthermore

$$\begin{aligned} v_j &= \sup_{m \ge j}\{\max\{u_m - 1/m, mh\}\} = \sup_{m \ge j}\{\max\{u_m, mh + 1/m\} - 1/m\} \\ &\le \max\{\max_{K \ge m \ge j}\{\max\{u_m, mh + 1/m\} - 1/m\},\ \max\{u_K, Kh + 1/K\}\} \end{aligned}$$

for any $K \ge j$, since

$$(2.2) \qquad \max\{u_m, mh + 1/m\} - 1/m \le \max\{u_K, Kh + 1/K\}$$

if $m \ge K$ because the right hand side of (2.2) is decreasing in K.

Finally, note that

$$\widetilde{v}_K^j := \max\{\max_{K \ge m \ge j}\{\max\{u_m, mh + 1/m\} - 1/m\}, \max\{u_K, Kh + 1/K\}\}$$

decreases to v_j as $K \to \infty$. Each $\widetilde{v}_K^j$ is continuous on $\overline{V}$ and hence v_j is upper semicontinuous. It is clear that $v_j \searrow u^*$ on V, and the proof is finished. ∎

As an immediate corollary, we deduce that the Jensen measures for $\mathcal{PSH}^0$ and $\mathcal{PSH}^0_{\mathrm{c}}$ coincide.

COROLLARY 2.4. *Let V be an analytic variety in $\Omega \subset \mathbb{C}^n$ and assume that Ω is hyperconvex. If $z \in V$, then $\mathcal{J}_z^0 = \mathcal{J}_{c,z}^0$.*

Proof. Clearly $\mathcal{J}_z^0 \subset \mathcal{J}_{c,z}^0$. Let $\mu \in \mathcal{J}_{c,z}^0$ and let $u \in \mathcal{PSH}^0$ be an arbitrary plurisubharmonic function. By Theorem 2.3, we can find a sequence $u_j \in \mathcal{PSH}_c^0$ with $u_j \searrow u^*$ on $\overline{V}$. Hence, by the monotone convergence theorem,

$$\int u^* \, d\mu = \lim_{j\to\infty} \int u_j \, d\mu \geq \overline{\lim_{j\to\infty}} \, u_j(z) = u^*(z).$$

Since u was arbitrary, $\mu \in \mathcal{J}_z^0$. ∎

3. Continuity of ω_K. If we use the results from Section 2, the proof of our main result is straightforward.

Proof of Theorem 1.7. Let $h \in \mathcal{PSH}(\Omega) \cap C(\overline{\Omega})$ be a bounded exhaustion function with $h|_{\partial\Omega} = 0$. If u is a member of the defining family for ω_K, then so is $\widetilde{u} = \max\{u, Mh\}$ for $M > 0$ large enough and $\widetilde{u} \in \mathcal{PSH}^0(V)$. Hence it is enough to consider functions in $\mathcal{PSH}^0$ when defining ω_K, i.e.

$$\omega_K(z) = \sup\{u(z) : u \in \mathcal{PSH}^0(V), u \leq 0, u|_K \leq -1\}.$$

Consequently, since $-\chi_K$ is lower semicontinuous, Edwards' Theorem and Corollary 2.4 imply that

$$\begin{aligned}\omega_K(z) &= \inf\Big\{\int -\chi_K \, d\mu : \mu \in \mathcal{J}_z^0\Big\} = \inf\Big\{\int -\chi_K \, d\mu : \mu \in \mathcal{J}_{c,z}^0\Big\} \\ &= \sup\{u(z) : u \in \mathcal{PSH}_c^0(V),\ u \leq 0,\ u|_K \leq -1\} =: \omega_K^c(z),\end{aligned}$$

for all $z \in V$. Clearly, ω_K^c is lower semicontinuous, being the supremum of continuous functions, and since we assume that K is regular, $\omega_K = \omega_K^*$ is plurisubharmonic and in particular upper semicontinuous on V_{irr}. Hence, $\omega_K \in \mathcal{PSH}(V_{\mathrm{irr}}) \cap C((\overline{V} \setminus V_{\mathrm{red}}) \cap \overline{\Omega})$. ∎

Note that the relative extremal function studied in [1, 2, 3] is defined in terms of almost plurisubharmonic functions. Let us conclude the paper by showing that, at least for a reasonable class of varieties, there is no difference between these two extremal functions.

DEFINITION 3.1. Let V be an analytic variety in $\Omega \subset \mathbb{C}^n$ and let K be a closed subset of V. Define

$$\widetilde{\omega}_K(z) = \widetilde{\omega}_{K,V,\Omega}(z) = \sup\{u(z) : u \in \widetilde{\mathcal{PSH}}(V),\ u \leq 0,\ u|_K \leq -1\}.$$

If $\widetilde{\omega}_K$ is continuous at K, we say that K is *$\widetilde{\mathcal{PSH}}$-regular.*

Since $\mathcal{PSH}(V) \subset \widetilde{\mathcal{PSH}}(V)$, it is clear that $\omega_K \leq \widetilde{\omega}_K$. However, we can prove the following result:

THEOREM 3.2. *Let V be an analytic variety in $\Omega \subset \mathbb{C}^n$, let K be a closed subset of V and assume that there is a negative function $\phi \in \widetilde{\mathcal{PSH}}(V)$*

with $\phi^{-1}(-\infty) = V_{\rm sing}$. *If* Ω *is hyperconvex and* K *is* $\widetilde{\mathcal{PSH}}$*-regular, then* $\widetilde{\omega}_K = \omega_K$ *on* $V_{\rm irr}$, *and in particular*, $\widetilde{\omega}_K \in C((\overline{V} \setminus V_{\rm red}) \cap \overline{\Omega})$.

Proof. First note that if $u \in \widetilde{\mathcal{PSH}}(V)$, then $u + \varepsilon\phi \in \mathcal{PSH}(V)$ for every $\varepsilon > 0$. Hence $\widetilde{\omega}_K \le \omega_K + \varepsilon\phi$ for all $\varepsilon > 0$. By letting $\varepsilon \to 0$, it follows that $\widetilde{\omega}_K = \omega_K$ on $V_{\rm reg}$, so by Theorem 1.7, $\widetilde{\omega}_K^* = \omega_K^* = \omega_K$ on $V_{\rm irr}$. ∎

REMARK. Note that any algebraic variety and more generally any variety in Ω whose defining functions extend to a common open neighborhood of $\overline{\Omega}$ satisfies the assumption in Theorem 3.2. Also note that if a compact set K is $\widetilde{\mathcal{PSH}}$-regular, then it is regular (since $-1 \le \omega_K \le \widetilde{\omega}_K$).

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