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Explicit extension maps in intersections of non-quasi-analytic classes

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To our friend Klaus Dieter Bierstedt on the occasion of his 60th birthday

Abstract. We deal with projective limits of classes of functions and prove that: (a) the Chebyshev polynomials constitute an absolute Schauder basis of the nuclear Fréchet spaces $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$; (b) there is no continuous linear extension map from $\Lambda^{(r)}_{(\mathfrak{M})}$ into $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R}^r)$; (c) under some additional assumption on $\mathfrak{M}$, there is an explicit extension map from $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$ into $\mathcal{D}_{(\mathfrak{M})}([-2,2]^r)$ by use of a modification of the Chebyshev polynomials. These results extend the corresponding ones obtained by Beaugendre in [1] and [2].

1. Introduction. It is well known that there is no continuous linear extension map from ω into $\mathcal{C}^\infty(\mathbb{R})$. In 1961, Mityagin [4] proved that (a) $\mathcal{C}^\infty([-1,1])$ is a Fréchet nuclear space in which the Chebyshev polynomials constitute an absolute Schauder basis, and (b) there is a continuous linear extension map from $\mathcal{C}^\infty([-1,1])$ into $\mathcal{C}^\infty(\mathbb{R})$ by means of a modification of the Chebyshev polynomials.

Since then, (non-) existence of continuous linear extension maps has been extensively studied in the case of non-quasi-analytic classes. For instance, we mention the results by Petzsche [5] about the Borel case, i.e. the case $n = 1$ and $K = \{0\}$.

Recently Beaugendre [1], [2] has obtained similar results in the case of some projective limits of such classes.

In this paper we deal with these questions in a more general setting, i.e. in the case of countable intersections of classes of functions defined by means of matrices of positive elements.

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After having obtained some inequalities, we introduce in Section 4 the matrices $\mathfrak{m}$ and $\mathfrak{M}$ that allow us to define for instance the spaces $\mathcal{B}_{(\boldsymbol{M}_j)}(\Omega)$, $\mathcal{E}_{(\boldsymbol{M}_j)}([-1,1]^r)$ and $\Lambda^{(r)}_{(\boldsymbol{M}_j)}$ by use of the rows of $\mathfrak{M}$, and next their projective limits $\mathcal{B}_{(\mathfrak{M})}(\Omega)$, $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$ and $\Lambda^{(r)}_{(\mathfrak{M})}$.

We then prove that for every $r \in \mathbb{N}$,

(a) $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$ is a Fréchet nuclear space in which the Chebyshev polynomials constitute an absolute Schauder basis;
(b) there is no continuous linear extension map from $\Lambda^{(r)}_{(\mathfrak{M})}$ into $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R}^r)$.

Under some additional assumption on $\mathfrak{M}$, we also obtain the existence of a continuous linear extension map from $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$ into $\mathcal{D}_{(\mathfrak{M})}([-2,2]^r)$, hence into $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R}^r)$.

Finally, we prove that these results extend the corresponding ones obtained by Beaugendre in [1] and [2].

2. An auxiliary inequality

PROPOSITION 2.1. *Let $r, m \in \mathbb{N}$, Ω be an open subset of $\mathbb{R}^r$ and $f \in \mathcal{C}^m(\Omega)$. For every $s \in \{1,\dots,r\}$, let moreover ω_s be an open subset of $\mathbb{R}$ and $g_s \in \mathcal{C}^m(\omega_s)$ be a real function such that*

$$(g_1(t_1),\dots,g_r(t_r)) \in \Omega, \qquad \forall (t_1,\dots,t_r) \in \omega := \omega_1 \times \dots \times \omega_r,$$

and

$$M := \sup_{1\le s\le r} \sup_{0\le l\le m} \|g_s^{(l)}\|_{\omega_s} < \infty.$$

Then for every $\alpha \in \mathbb{N}_0^r$ such that $0 < |\alpha| \le m$, there are explicit functions A_β on ω such that

$$\frac{1}{\alpha!}\,(f(g_1,\dots,g_r))^{(\alpha)}(t) = \sum_{0\ne\beta\le\alpha} A_\beta(t)\,\frac{1}{\beta!}\,f^{(\beta)}(g_1(t_1),\dots,g_r(t_r))$$

and $\sum_{0\ne\beta\le\alpha} |A_\beta(t)| \le (1+M)^{|\alpha|}$ for every $t \in \omega$. Therefore for every $\alpha \in \mathbb{N}_0^r$ such that $0 \le |\alpha| \le m$ and $t \in \omega$,

$$\frac{1}{\alpha!}\,|(f(g_1,\dots,g_r))^{(\alpha)}(t)| \le (1+M)^{|\alpha|} \sup_{0\le\beta\le\alpha} \frac{1}{\beta!}\,|f^{(\beta)}(g_1(t_1),\dots,g_r(t_r))|.$$

Proof. Once the first assertion is established, the consequence is immediate since the case $\alpha = 0$ is trivial.

The case $r = 1$ is a direct consequence of the Faà di Bruno formula (cf. [7]) stating that, for every $s \in \{1,\dots,m\}$,

$$\frac{(f(g_1))^{(s)}(t)}{s!} = \sum_{k=1}^{s} \frac{f^{(k)}(g_1(t))}{k!}\,A_k(t), \qquad \forall t \in \omega_1,$$

where

$$A_k(t) = \sum_{\substack{k_1+\cdots+k_s=k \\ k_1+2k_2+\cdots+sk_s=s}} \frac{k!}{k_1!\cdots k_s!}\left(\frac{g_1^{(1)}(t)}{1!}\right)^{k_1}\cdots\left(\frac{g_1^{(s)}(t)}{s!}\right)^{k_s}$$

with

$$\sum_{\substack{k_1+\cdots+k_s=k \\ k_1+2k_2+\cdots+sk_s=s}} \frac{k!}{k_1!\dots k_s!} = \binom{s-1}{k-1}.$$

It is indeed sufficient to note that

$$\sum_{k=1}^{s} |A_k(t)| \le \sum_{k=1}^{s} M^k \binom{s-1}{k-1} = M(1+M)^{s-1} \le (1+M)^s.$$

To conclude by induction, we just have to prove that if the property is true for $r-1$ with $r \ge 2$, then it is also true for r.

Of course we may suppose $\alpha_1 \neq 0$. For every $t_1 \in \omega_1$ and $x_2,\dots,x_r \in \mathbb{R}$ such that $(g_1(t_1), x_2, \dots, x_r) \in \Omega$, the case $r=1$ provides

$$\frac{1}{\alpha_1!}\,(f(g_1,x_2,\dots,x_r))^{(\alpha_1)}(t_1) = \sum_{\beta_1=1}^{\alpha_1} A_{\beta_1}(t_1)\,\frac{1}{\beta_1!}\,f^{(\beta_1')}(g_1(t_1),x_2,\dots,x_r)$$

with $\beta_1' = (\beta_1, 0, \dots, 0)$ and $\sum_{\beta_1=1}^{\alpha_1} |A_{\beta_1}(t_1)| \le (1+M)^{\alpha_1}$ for every $t_1 \in \omega_1$.

Now we set $\gamma = (\alpha_2, \dots, \alpha_r)$. In the case $\gamma = 0$, we have

$$\frac{1}{\alpha!}\,(f(g_1,\dots,g_r))^{(\alpha)}(t) = \frac{1}{\alpha_1!}\,(f(g_1, g_2(t_2), \dots, g_r(t_r)))^{(\alpha_1)}(t_1)$$

and it suffices to set $A_\beta(t) = A_{\beta_1}(t_1)$ for every $\beta \in \mathbb{N}_0^r$ such that $0 \neq \beta \le \alpha$.

If $\gamma \neq 0$, the commutativity of the derivatives yields

$$\frac{(f(g_1,\dots,g_r))^{(\alpha)}(t)}{\alpha!} = \sum_{\beta_1=1}^{\alpha_1} \frac{A_{\beta_1}(t_1)}{\beta_1!}\,\frac{(f^{(\beta_1')}(g_1(t_1),g_2,\dots,g_r))^{(\gamma)}(t_2,\dots,t_r)}{\gamma!}.$$

Then we apply the case $r-1$ to get

$$\frac{(f^{(\beta_1')}(g_1(t_1),g_2,\dots,g_r))^{(\gamma)}(t_2,\dots,t_r)}{\gamma!} = \sum_{0\neq\eta\le\gamma} A_\eta(t_2,\dots,t_r)\,\frac{f^{(\beta_1,\eta)}(g_1(t_1),\dots,g_r(t_r))}{\eta!}$$

with $\sum_{0\neq\eta\le\gamma} |A_\eta(t_2,\dots,t_r)| \le (1+M)^{|\gamma|}$. So the formula is correct if we set

$$A_\beta(t_1,\dots,t_r) = \begin{cases} 0 & \text{if } \beta_1 = 0 \text{ or } (\beta_2,\dots,\beta_r) = 0, \\ A_{\beta_1}(t_1)A_{(\beta_2,\dots,\beta_r)}(t_2,\dots,t_r) & \text{otherwise,} \end{cases}$$

since then

$$\sum_{0\neq\beta\leq\alpha}|A_\beta(t)|=\sum_{\beta_1=1}^{\alpha_1}|A_{\beta_1}(t_1)|\sum_{0\neq\eta\leq(\beta_2,\dots,\beta_r)}|A_\eta(t_2,\dots,t_r)|\leq(1+M)^{|\alpha|}. \ \blacksquare$$

3. The Chebyshev polynomials. For every $n\in\mathbb{N}_0$, the *Chebyshev polynomial* T_n is the polynomial of degree n on $\mathbb{R}$ which coincides with the function $\cos(n\arccos(x))$ on the interval $[-1,1]$.

The following information about $|T_n^{(p)}(x)|$ will be used. For $T_0=\chi_{\mathbb{R}}$, the situation is clear. For every $n\in\mathbb{N}$, the *V. A. Markov inequality* ([6, 1.3.35 and 1.5.6]) states that

$$|T_n^{(p)}(x)|\leq T_n^{(p)}(1)=\frac{n^2}{1}\,\frac{n^2-1^2}{3}\cdots\frac{n^2-(p-1)^2}{2p-1}\leq\frac{n^{2p}}{p!}$$

for every $p\in\mathbb{N}_0$ and $x\in[-1,1]$. We need the following slight extension of this inequality.

Proposition 3.1. *For every $n\in\mathbb{N}$ and $p\in\mathbb{N}_0$, one has*

$$|T_n^{(p)}(x)|\leq e\,\frac{n^{2p}}{p!},\qquad\forall x\in\left[-1-\frac{1}{1+n^2},1+\frac{1}{1+n^2}\right].$$

Proof. As T_n is a polynomial of degree n, we only need to consider the case $p\leq n$. Since every $x\in[0,1+1/(1+n^2)]$ can be written as $x=y+z$ with $y\in[0,1]$ and $z\in[0,1/(1+n^2)]$, the Taylor formula provides the equality $T_n^{(p)}(x)=\sum_{k=0}^{n-p}T_n^{(p+k)}(y)z^k/k!$, which leads to

$$|T_n^{(p)}(x)|\leq\sum_{k=0}^{n-p}\frac{n^{2p}}{(p+k)!}\,\frac{n^{2k}}{(1+n^2)^k}\,\frac{1}{k!}\leq\frac{n^{2p}}{p!}\sum_{k=0}^{\infty}\frac{1}{k!}.$$

Hence the conclusion follows, since a similar argument works for the elements of the interval $[-1-1/(1+n^2),0]$. $\blacksquare$

As we will deal with functions defined on subsets of $\mathbb{R}^r$, the following considerations will be very useful.

Notation. For every $\gamma\in\mathbb{N}_0^r$, T_γ designates the polynomial defined on $\mathbb{R}^r$ by $T_\gamma(x)=T_{\gamma_1}(x_1)\cdots T_{\gamma_r}(x_r)$ for every $x\in\mathbb{R}^r$.

Then for every $f\in\mathcal{C}^\infty([-1,1]^r)$, $f(\cos(t_1),\dots,\cos(t_r))$ is a periodic $\mathcal{C}^\infty$-function on $\mathbb{R}^r$. So its Fourier development is

$$\sum_{\gamma\in\mathbb{N}_0^r}a_\gamma(f)\cos(\gamma_1t_1)\cdots\cos(\gamma_rt_r)$$

with

$$(1)\quad a_\gamma(f)=\frac{1}{2^{r(\gamma)}}\frac{1}{\pi^r}\int\limits_{[-\pi,\pi]^r} f(\cos(t_1),\dots,\cos(t_r))\cos(\gamma_1 t_1)\cdots\cos(\gamma_r t_r)\,dt,$$

where $r(\gamma)$ is the number of components of γ which are equal to 0. Moreover the derivatives of this series converge absolutely and uniformly on $\mathbb{R}^r$ to the corresponding derivatives of the function $f(\cos(t_1),\dots,\cos(t_r))$. Therefore we get

$$f(x)=f(x_1,\dots,x_r)=\sum_{\gamma\in\mathbb{N}_0^r} a_\gamma(f)T_\gamma(x)$$

and the derivatives term by term of this series converge absolutely and uniformly on $[-1,1]^r$ to the corresponding derivatives of f.

An estimate of the numbers $|a_\gamma(f)|$ can be obtained as follows.

Notation. Given $\gamma\in\mathbb{N}_0^r$, we set $\gamma^*=(\gamma_1^*,\dots,\gamma_r^*)$ with $\gamma_s^*=\gamma_s$ if $\gamma_s\neq 0$ and $\gamma_s^*=1$ if $\gamma_s=0$.

Moreover we set $\gamma^\alpha=(\gamma_1^*)^{\alpha_1}\dots(\gamma_r^*)^{\alpha_r}$ for every $\alpha\in\mathbb{N}_0^r$.

Then, by integration by parts, for every $\alpha\in\mathbb{N}_0^r$, formula (1) leads to

$$|a_\gamma(f)|=\frac{1}{2^{r(\gamma)}}\frac{1}{\pi^r}\frac{1}{\gamma^\alpha}\Big|\int\limits_{[-\pi,\pi]^r}(f(\cos(t_1),\dots,\cos(t_r)))^{(\alpha)}*(t)\,dt\Big|$$

with

$$*(t)=\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(\gamma_1 t_1)\dots\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(\gamma_r t_r),$$

hence

$$(2)\qquad |a_\gamma(f)|\le\alpha!\,\frac{2^{r+|\alpha|-r(\gamma)}}{\gamma^\alpha}\sup_{0\le\delta\le\alpha}\sup_{x\in[-1,1]^r}\frac{|f^{(\delta)}(x)|}{\delta!}$$

by Proposition 2.1.

4. The matrix $\mathfrak{m}$. To any sequence $\boldsymbol{m}=(m_p)_{p\in\mathbb{N}_0}$ of positive numbers, we associate as usual the sequence $\boldsymbol{M}=(M_p)_{p\in\mathbb{N}_0}$ defined by $M_p=m_0\cdots m_p$ for every $p\in\mathbb{N}_0$.

Throughout the paper, $\mathfrak{m}$ designates a matrix

$$\mathfrak{m}=(m_{j,p})_{\substack{j\in\mathbb{N}\\ p\in\mathbb{N}_0}}$$

such that the sequences $\boldsymbol{m}_j=(m_{j,p})_{p\in\mathbb{N}_0}$ satisfy the following conditions: for every $j\in\mathbb{N}$,

($\mathfrak{m}_1$) $m_{j,0}=1$ and $m_{j,p}\ge 1$ for every $p\in\mathbb{N}$;

($\mathfrak{m}_2$) $m_{j,p}/p\le m_{j,p+1}/(p+1)$ for every $p\in\mathbb{N}$; in particular the sequence $(M_{j,p}/p!)_{p\in\mathbb{N}_0}$ is increasing and $M_{j,p}\ge p!$ for every $p\in\mathbb{N}_0$;

($\mathfrak{m}_3$) $m_{j,p}\ge m_{j+1,p}$ for every $p\in\mathbb{N}_0$;

($\mathfrak{m}_4$) for every $A>0$, there is $A_j>1$ such that

$$M_{j+1,2(p+1)} \le A_j A^{-p} p! M_{j,p}, \qquad \forall p \in \mathbb{N}_0.$$

Conditions ($\mathfrak{m}_1$) and ($\mathfrak{m}_2$) are standard. Condition ($\mathfrak{m}_3$) is necessary to introduce the projective limits defined in the next section. Condition ($\mathfrak{m}_4$) is not standard: it provides links between the spaces defining these projective limits. Let us note at once that it leads to the following two consequences.

PROPOSITION 4.1. *For every $j \in \mathbb{N}$ and $B>0$, there is $c>0$ such that $M_{j,p}/p! \ge cB^{p+1}$ for every $p \in \mathbb{N}_0$.*

Proof. Indeed, by condition ($\mathfrak{m}_4$) with $A = B/4$, we get

$$\frac{M_{j,p}}{p!} \ge \frac{A^p}{A_j} \frac{M_{j+1,2(p+1)}}{(p!)^2} \ge \frac{A^p}{A_j}(2p+1)\frac{(2p)!}{(p!)^2} \ge \frac{(4A)^p}{A_j}. \blacksquare$$

NOTATION. For every $j \in \mathbb{N}$ and $\gamma \in \mathbb{N}_0^r$, set $M_{j,\gamma} = M_{j,\gamma_1} \cdots M_{j,\gamma_r}$.

PROPOSITION 4.2. *For every $j, r \in \mathbb{N}$ such that $r \ge 2$, there is a constant $B_{j,r} > 1$ such that $M_{j+r-1,|\delta|} \le B_{j,r}^{|\delta|} M_{j,\delta}$ for every $\delta \in \mathbb{N}_0^r$.*

Proof. When $r=2$, this is equivalent to establishing that, for every $j \in \mathbb{N}$, there is $B_j > 1$ such that $M_{j+1,p+q} \le B_j^{p+q} M_{j,p} M_{j,q}$ for every $p, q \in \mathbb{N}_0$ such that $p \ge q$. For $A=1$, condition ($\mathfrak{m}_4$) provides a constant $C_j \in \,]0,1[$ such that $C_j M_{j+1,2(p+1)} \le p! M_{j,p}$ for every $p \in \mathbb{N}_0$, hence

$$\begin{aligned} M_{j,p} M_{j,q} &\ge \frac{C_j}{p!} \frac{M_{j+1,2(p+1)}}{(2(p+1))!}(2(p+1))!q! \ge \frac{C_j}{p!}\frac{M_{j+1,p+q}}{(p+q)!}(2p)!q! \\ &\ge C_j \frac{M_{j+1,p+q}}{2^{p+q}} \frac{(2p)!}{p!^2} \ge \left(\frac{C_j}{2}\right)^{p+q} M_{j+1,p+q}. \end{aligned}$$

The case $r \ge 3$ is then immediate:

$$\begin{aligned} M_{j+r-1,|\delta|} &\le B_{j+r-2,2}^{|\delta|} M_{j+r-2,\delta_1+\cdots+\delta_{r-1}} M_{j,\delta_r} \\ &\le \cdots \le (B_{j+r-2,2} \ldots B_{j,2})^{|\delta|} M_{j,\delta_1} \cdots M_{j,\delta_r}. \blacksquare \end{aligned}$$

5. Some spaces

The Fréchet spaces $\Lambda^{(r)}_{(\boldsymbol{M}_j)}$ and $\Lambda^{(r)}_{(\mathfrak{M})}$. For every $j, r \in \mathbb{N}$, the Fréchet space $\Lambda^{(r)}_{(\boldsymbol{M}_j)}$ is the vector space of indexed families $\boldsymbol{a} = (a_\gamma)_{\gamma \in \mathbb{N}_0^r}$ of elements of $\mathbb{K}$ such that

$$|\boldsymbol{a}|_{j,h} := \sup_{\gamma \in \mathbb{N}_0^r} \frac{|a_\gamma|}{h^{|\gamma|} M_{j,|\gamma|}} < \infty, \qquad \forall h > 0,$$

endowed with the system of norms $\{|\cdot|_{j,1/m} : m \in \mathbb{N}\}$.

For every $r \in \mathbb{N}$, the space $\Lambda^{(r)}_{(\mathfrak{M})}$ is then defined as the projective limit of the sequence $(\Lambda^{(r)}_{(\boldsymbol{M}_j)})_{j\in\mathbb{N}}$; $\{|\cdot|_{j,1/j} : j \in \mathbb{N}\}$ is of course a fundamental system of norms for this space.

The Fréchet spaces $\mathcal{B}_{(\boldsymbol{M}_j)}(\Omega)$ and $\mathcal{B}_{(\mathfrak{M})}(\Omega)$. Given an open subset Ω of $\mathbb{R}^r$, the Fréchet space $\mathcal{B}_{(\boldsymbol{M}_j)}(\Omega)$ is the vector space of functions $f \in \mathcal{C}^\infty(\Omega)$ such that

$$\|f\|_{\Omega,j,h} := \sup_{\alpha\in\mathbb{N}_0^r} \frac{\|\mathrm{D}^\alpha f\|_\Omega}{h^{|\alpha|}M_{j,|\alpha|}} < \infty, \qquad \forall h > 0,$$

endowed with the countable system of norms $\{\|\cdot\|_{\Omega,j,1/m} : m \in \mathbb{N}\}$.

The Fréchet space $\mathcal{B}_{(\mathfrak{M})}(\Omega)$ is then defined as the projective limit of the sequence $(\mathcal{B}_{(\boldsymbol{M}_j)}(\Omega))_{j\in\mathbb{N}}$.

The space $\mathcal{D}_{(\mathfrak{M})}(K)$. Given a compact subset K of $\mathbb{R}^r$, the Fréchet space $\mathcal{D}_{(\mathfrak{M})}(K)$ is the topological subspace of $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R}^r)$ consisting of the elements with support contained in K.

The Banach spaces $\mathcal{B}_{j,r}$ and the Fréchet spaces $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$. The Banach space $\mathcal{B}_{j,r}$ is the vector space of functions $f \in \mathcal{C}^\infty([-1,1]^r)$ such that

$$\|f\|_{j,r} := \sup_{\alpha\in\mathbb{N}_0^r} \frac{j^{|\alpha|}\|\mathrm{D}^\alpha f\|_{[-1,1]^r}}{M_{j,|\alpha|}} < \infty,$$

endowed with the norm $\|\cdot\|_{j,r}$.

The Fréchet space $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$ is then defined as the projective limit of the sequence $(\mathcal{B}_{j,r})_{j\in\mathbb{N}}$.

6. Study of the space $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$

Proposition 6.1. *For every $j, r \in \mathbb{N}$, there is a constant $B_{j,r} > 1$ such that, for every $\alpha, \gamma \in \mathbb{N}_0^r$,*

$$|a_\gamma(f)| \le 2^r \,\frac{2^{|\alpha|}B_{j,r}^{|\alpha|}}{\gamma^\alpha}\, M_{j,\alpha}\|f\|_{j+r-1,r}, \qquad \forall f \in \mathcal{B}_{j+r-1,r}.$$

In particular, for every $j, r \in \mathbb{N}$, $(T_\gamma, a_\gamma(\cdot))_{\gamma\in\mathbb{N}_0^r}$ is a biorthogonal system in $\mathcal{B}_{j+r-1,r}$.

Proof. For every $f \in \mathcal{B}_{j+r-1,r}$ and $\alpha, \delta \in \mathbb{N}_0^r$ such that $\delta \le \alpha$, we clearly have

$$\begin{aligned}\frac{1}{\delta!}\,\|f^{(\delta)}\|_{[-1,1]^r} &\le \frac{1}{\delta!}\,\|f\|_{j+r-1,r}\,\frac{M_{j+r-1,|\delta|}}{(j+r-1)^{|\delta|}}\\ &\le B_{j,r}^{|\delta|}\,\frac{M_{j,\delta}}{\delta!}\,\|f\|_{j+r-1,r} \le B_{j,r}^{|\alpha|}\,\frac{M_{j,\alpha}}{\alpha!}\,\|f\|_{j+r-1,r}\end{aligned}$$

for some constant $B_{j,r} > 1$ independent of f (if $r = 1$, any number $B_{j,1} > 1$ is suitable; if $r \geq 2$, we use Proposition 4.2). So the announced inequality is an immediate consequence of the estimate (2) of $|a_\gamma(f)|$.

The particular case is clear: it is a fact that $a_\gamma(T_\delta)$ is equal to 1 if $\gamma = \delta$ and to 0 otherwise. Moreover the established inequality implies that the linear functionals $a_\gamma(\cdot)$ are continuous on $\mathcal{B}_{j+r-1,r}$. ∎

PROPOSITION 6.2. *For every $j, r \in \mathbb{N}$, the canonical injection from $\mathcal{B}_{j+r,r}$ into $\mathcal{B}_{j,r}$ is nuclear.*

Proof. For every $\beta, \gamma \in \mathbb{N}_0^r$, we use the V. A. Markov inequality to estimate the derivatives of T_γ, and Proposition 6.1 with the special value $\alpha = (2\beta_1 + 2, \ldots, 2\beta_r + 2)$ to estimate $|a_\gamma(f)|$. We thus get a constant $B_{j+1,r} > 1$ such that

$$\frac{j^{|\beta|}|a_\gamma(f)|\,\|\mathrm{D}^\beta T_\gamma\|_{[-1,1]^r}}{M_{j,|\beta|}} \leq \frac{j^{|\beta|}2^r 2^{2|\beta|+2r} B_{j+1,r}^{2|\beta|+2r}}{M_{j,|\beta|}} \, \frac{1}{\gamma_1^{*2} \cdots \gamma_r^{*2}} \, \frac{M_{j+1,\alpha}}{\beta!} \, \|f\|_{j+r,r}$$

for every $f \in \mathcal{B}_{j+r,r}$.

For $A = 4jB_{j+1,r}^2$, condition $(\mathfrak{m}_4)$ provides a constant $A_j > 1$ such that

$$M_{j+1,2(p+1)} \leq A_j A^{-p} p! M_{j,p}, \qquad \forall p \in \mathbb{N}_0.$$

This leads to the following estimate:

$$\frac{M_{j+1,\alpha}}{\beta!} = \frac{M_{j+1,2(\beta_1+1)}}{\beta_1!} \cdots \frac{M_{j+1,2(\beta_r+1)}}{\beta_r!} \leq A_j^r A^{-|\beta|} M_{j,\beta} \leq A_j^r A^{-|\beta|} M_{j,|\beta|}.$$

Therefore we get

$$(3) \qquad |a_\gamma(f)|\,\|T_\gamma\|_{j,r} \leq C_{j,r} \frac{1}{\gamma_1^{*2} \ldots \gamma_r^{*2}} \|f\|_{j+r,r}, \qquad \forall f \in \mathcal{B}_{j+r,r},$$

with $C_{j,r} = 2^{3r} A_j^r B_{j+1,r}^{2r} > 0$ and we conclude at once since the series $\sum_{\gamma \in \mathbb{N}_0^r} (\gamma_1^{*2} \cdots \gamma_r^{*2})^{-1}$ converges. ∎

Together with inequalities (3), Propositions 6.1 and 6.2 lead to the following result.

THEOREM 6.3. *For every $r \in \mathbb{N}$, the family $(T_\gamma, a_\gamma(\cdot))_{\gamma \in \mathbb{N}_0^r}$ is an absolute Schauder basis in the Fréchet nuclear space $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$.* ∎

NOTATION. As $\mathcal{E}_{(\mathfrak{M})}([-1,1])$ is a Fréchet nuclear space, the tensor products

$$\mathcal{E}_{(\mathfrak{M})}([-1,1]) \,\widehat{\otimes}_\varepsilon \overset{r}{\cdots} \widehat{\otimes}_\varepsilon\, \mathcal{E}_{(\mathfrak{M})}([-1,1])$$

and

$$\mathcal{E}_{(\mathfrak{M})}([-1,1]) \,\widehat{\otimes}_\pi \overset{r}{\cdots} \widehat{\otimes}_\pi\, \mathcal{E}_{(\mathfrak{M})}([-1,1])$$

coincide. To shorten notation, let us designate them by $\widehat{\bigotimes}^r \mathcal{E}_{(\mathfrak{M})}([-1,1])$.

PROPOSITION 6.4. *For every integer $r \geq 2$, the Fréchet nuclear spaces $\widehat{\bigotimes}^r \mathcal{E}_{(\mathfrak{M})}([-1,1])$ and $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$ coincide.*

Proof. Let $\mathcal{E}$ denote the tensor product

$$\mathcal{E}_{(\mathfrak{M})}([-1,1]) \otimes \overset{r}{\cdots} \otimes \mathcal{E}_{(\mathfrak{M})}([-1,1]).$$

By Theorem 6.3, we already know that $\mathcal{E}$ is a dense vector subspace of $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$.

We first prove that on $\mathcal{E}$ the π-topology is finer than the topology induced by $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$. For every $j \in \mathbb{N}$, it is well known that the set U_j of linear combinations

$$\sum_{s=1}^{l} b_s f_{s,1}(x_1) \cdots f_{s,r}(x_r) = \sum_{s=1}^{l} b_s f_{s,1} \otimes \cdots \otimes f_{s,r}(x)$$

with $l \in \mathbb{N}$, $b_1, \ldots, b_l \in \mathbb{K}$ such that $\sum_{s=1}^{l} |b_s| \leq 1$, and $f_{s,k} \in \mathcal{E}_{(\mathfrak{M})}([-1,1])$ such that $\|f_{s,k}\|_{j,1} \leq 1$ for all $s \in \{1, \ldots, l\}$ and $k \in \{1, \ldots, r\}$, is a neighbourhood in $\mathcal{E}$ endowed with the π-topology. To conclude it is then enough to verify that, for every $j \in \mathbb{N}$, U_j is a subset of the closed unit semi-ball of $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$ for the semi-norm $\|\cdot\|_{j,r}$. This is straightforward: for every element $f = \sum_{s=1}^{l} b_s f_{s,1} \otimes \cdots \otimes f_{s,r}$ of U_j, one just has to note that, for every $s \in \{1, \ldots, l\}$,

$$\|f_{s,1} \otimes \cdots \otimes f_{s,r}\|_{j,r} \leq \|f_{s,1}\|_{j,1} \cdots \|f_{s,r}\|_{j,1} \leq 1.$$

Now we prove that on $\mathcal{E}$, the ε-topology is weaker than the topology induced by $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$. For any elements $u_1, \ldots, u_r$ of the topological dual of $\mathcal{E}_{(\mathfrak{M})}([-1,1])$ and $f \in \mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$, set

$$\langle f, u_1 \otimes \cdots \otimes u_r \rangle := \sum_{\gamma \in \mathbb{N}_0^r} a_\gamma(f) \langle u_1, T_{\gamma_1} \rangle \cdots \langle u_r, T_{\gamma_r} \rangle.$$

For every $j \in \mathbb{N}$, let moreover V_j denote the polar set of

$$\{f \in \mathcal{E}_{(\mathfrak{M})}([-1,1]) \colon \|f\|_{[-1,1],j,1/j} \leq 1\}$$

in the topological dual of $\mathcal{E}_{(\mathfrak{M})}([-1,1])$. If $u_1, \ldots, u_r \in V_j$, we get

$$|\langle f, u_1 \otimes \cdots \otimes u_r \rangle| \leq \sum_{\gamma \in \mathbb{N}_0^r} |a_\gamma(f)| \, \|T_{\gamma_1}\|_{j,1} \cdots \|T_{\gamma_r}\|_{j,1}$$

with

$$\|T_{\gamma_1}\|_{j,1} \cdots \|T_{\gamma_r}\|_{j,1} = \sup_{\alpha \in \mathbb{N}_0^r} \frac{j^{|\alpha|} \|\mathrm{D}^\alpha T_\gamma\|_{[-1,1]^r}}{M_{j,\alpha}} \leq \sup_{\alpha \in \mathbb{N}_0^r} \frac{B_{j,r}^{|\alpha|} j^{|\alpha|} \|\mathrm{D}^\alpha T_\gamma\|_{[-1,1]^r}}{M_{j+r-1,|\alpha|}}$$

for some $B_{j,r} > 1$ independent of γ by Proposition 4.2. So if $m_{j,r}$ is an integer such that $m_{j,r} > j + r - 1$ and $m_{j,r} > jB_{j,r}$, we have

$$\|T_{\gamma_1}\|_{j,1} \cdots \|T_{\gamma_r}\|_{j,1} \leq \|T_\gamma\|_{m_{j,r},r}$$

for every $\gamma \in \mathbb{N}_0^r$. Now we apply inequality (3) to get

$$|\langle f, u_1 \otimes \cdots \otimes u_r\rangle| \le C_{m_{j,r},r}\|f\|_{m_{j,r},r} \sum_{\gamma\in\mathbb{N}_0^r} \frac{1}{\gamma_1^{*2}\cdots\gamma_r^{*2}}$$

for every $f \in \mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$, for some constant $C_{m_{j,r},r} > 0$ independent of f. Therefore $u_1 \otimes \cdots \otimes u_r$ is a continuous linear functional on $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$ and

$$\{u_1 \otimes \cdots \otimes u_r : u_1, \ldots, u_r \in V_j\}$$

is an equicontinuous subset of the topological dual of $\mathcal{E}_{(\mathfrak{M})}([-1,1]^r)$.

Hence the conclusion follows, since the topologies ε and π coincide on $\mathcal{E}$. ∎

7. Non-extension result. It is clear that the restriction map

$$R: \mathcal{B}_{(\mathfrak{M})}(\mathbb{R}^r) \to \Lambda^{(r)}_{(\mathfrak{M})}, \qquad f \mapsto (\mathrm{D}^\gamma f(0))_{\gamma\in\mathbb{N}_0^r},$$

is well defined, linear and continuous.

Proposition 7.1. *For every $r \in \mathbb{N}$, there is no continuous linear extension map from $\Lambda^{(r)}_{(\mathfrak{M})}$ into $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R}^r)$.*

Proof. We proceed by induction on r.

Case $r = 1$. Suppose that there is a continous linear extension map T from $\Lambda^{(1)}_{(\mathfrak{M})}$ into $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R})$. Then there are $r, s \in \mathbb{N}$, $h, k \in\,]0,1]$ and $H, K > 1$ such that

$$\|T\boldsymbol{a}\|_{\mathbb{R},1,1} \le H|\boldsymbol{a}|_{s,h} \quad \text{and} \quad \|T\boldsymbol{a}\|_{\mathbb{R},s+1,1} \le K|\boldsymbol{a}|_{r,k}$$

for every $\boldsymbol{a} \in \Lambda^{(1)}_{(\mathfrak{M})}$. For every $p \in \mathbb{N}_0$, denote by e_p the pth unit vector of $\Lambda^{(1)}_{(\mathfrak{M})}$ and set $\chi_p := Te_p$. We then have

$$\|\chi_p\|_{\mathbb{R},1,1} \le \frac{H}{h^p M_{s,p}} \quad \text{and} \quad \|\chi_p\|_{\mathbb{R},s+1,1} \le \frac{K}{k^p M_{r,p}}.$$

Moreover for every $x > 0$, the Taylor development provides some $y \in\,]0,x]$ such that

$$\chi_p^{(p)}(x) = \chi_p^{(p)}(0) + \frac{x^p}{p!}\,\chi_p^{(2p)}(y) = 1 + \frac{x^p}{p!}\,\chi_p^{(2p)}(y),$$

hence

$$|\chi_p^{(p)}(x) - 1| \le \frac{x^p}{p!}\|\chi_p\|_{\mathbb{R},s+1,1} M_{s+1,2p} \le \frac{x^p}{p!}\,\frac{KM_{s+1,2p}}{k^p M_{r,p}}.$$

So if we define $\tau_p > 0$ by $\tau_p^p = p!k^p M_{r,p}/(2KM_{s+1,2p})$, we get

$$|\chi_p^{(p)}(x) - 1| \le \frac{\tau_p^p}{p!}\,\frac{KM_{s+1,2p}}{k^p M_{r,p}} = \frac{1}{2}, \qquad \forall x \in\,]0,\tau_p],$$

hence $\Re\chi_p^{(p)}(x) \geq 1/2$ and therefore $\Re\chi_p(x) \geq x^p/(2p!)$ for every $x \in]0, \tau_p]$. This leads to

$$\frac{k^p M_{r,p}}{4KM_{s+1,2p}} = \frac{\tau_p^p}{2p!} \leq \Re\chi_p(\tau_p) \leq \|\chi_p\|_{\mathbb{R},1,1} M_{1,0} \leq \frac{H}{h^p M_{s,p}}.$$

Now we note that condition $(\mathfrak{m}_4)$ applied with $A = 1$ provides a constant $B_s > 0$ such that $M_{s+1,2(p+1)} \leq B_s p! M_{s,p}$ for every $p \in \mathbb{N}$. So the preceding inequalities lead to

$$\frac{M_{r,p}}{p!} \leq \frac{4HKM_{s+1,2p}}{p! h^p k^p M_{s,p}} \leq \frac{4HK}{p! h^p k^p} \frac{M_{s+1,2(p+1)}}{M_{s,p}} \leq \frac{4HK}{h^p k^p} B_s$$

and we obtain the existence of some $B > 0$ such that $M_{r,p}/p! \leq B^{p+1}$ for every $p \in \mathbb{N}_0$. As by Proposition 4.1, there is $c > 0$ such that $M_{r,p}/p! \geq c(2B)^{p+1}$ for every $p \in \mathbb{N}_0$, we arrive at the contradiction $c(2B)^{p+1} \leq B^{p+1}$ for every $p \in \mathbb{N}_0$.

General case. Suppose that for some $r > 1$, there is a continuous linear extension map S from $\Lambda_{(\mathfrak{M})}^{(r)}$ into $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R}^r)$. Of course, the map $V \colon \Lambda_{(\mathfrak{M})}^{(1)} \to \Lambda_{(\mathfrak{M})}^{(r)}$ defined by

$$(V\boldsymbol{a})_\gamma = \begin{cases} 0 & \text{if } (\gamma_2, \ldots, \gamma_r) \neq 0, \\ (a_{\gamma_1}, 0, \ldots, 0) & \text{otherwise}, \end{cases}$$

is well defined, linear and continuous. Moreover the map R_1 from $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R}^r)$ into $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R})$ defined by $(R_1 f)(t) = f(t, 0, \ldots, 0)$ for every $t \in \mathbb{R}$ is also well defined, linear and continuous. Therefore the map $R_1 \circ S \circ V$ is continuous and linear from $\Lambda_{(\mathfrak{M})}^{(1)}$ into $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R})$. As it is clearly an extension map, we have arrived at a contradiction. ∎

8. Conditions $(\mathfrak{m}_5)$ and $(\mathfrak{m}'_5)$

DEFINITION. (a) The matrix $\mathfrak{m}$ satisfies condition $(\mathfrak{m}_5)$ if there is some $\alpha > 1$ such that $\sum_{p=1}^{\infty} p^\alpha / m_{j,p} < \infty$ for every $j \in \mathbb{N}$.

(b) The matrix $\mathfrak{m}$ satisfies condition $(\mathfrak{m}'_5)$ if there is some $\beta > 2$ such that $\lim_{p\to\infty} p!^\beta / M_{j,p} = 0$ for every $j \in \mathbb{N}$.

PROPOSITION 8.1. *The matrix* $\mathfrak{m}$ *satisfies condition* $(\mathfrak{m}_5)$ *if and only if it satisfies condition* $(\mathfrak{m}'_5)$.

Proof. We will prove that if $\mathfrak{m}$ satisfies $(\mathfrak{m}_5)$ with $\alpha > 1$, then it satisfies $(\mathfrak{m}'_5)$ with $\beta = 1 + \alpha$. For every $j \in \mathbb{N}$, the sequence $(m'_{j,p})_{p\in\mathbb{N}_0}$ defined by $m'_{j,0} = 1$ and $m'_{j,p} = p^{-\alpha} m_{j,p}$ for every $p \in \mathbb{N}$ is non-quasi-analytic. Therefore the Denjoy–Carleman theorem (cf. [3, Theorem 1.3.8]) gives $\sum_{p=1}^{\infty} 1/L_{j,p} < \infty$ if we set

$$L_{j,p} = \inf\{(M'_{j,k})^{1/k} \colon k \geq p\}, \quad \forall p \in \mathbb{N}.$$

As $(1/L_{j,p})_{p\in\mathbb{N}}$ is a decreasing sequence of positive numbers, this leads to $p/L_{j,p} \to 0$. The conclusion is then clear since

$$\frac{p!^{1+\alpha}}{M_{j,p}} \le \frac{p^p p!^\alpha}{M_{j,p}} = \frac{p^p}{M'_{j,p}} \le \left(\frac{p}{L_{j,p}}\right)^p, \qquad \forall p \in \mathbb{N}.$$

If $\mathfrak{m}$ satisfies $(\mathfrak{m}'_5)$ with $\beta > 2$, the following argument proves that it satisfies $(\mathfrak{m}_5)$ for any $\alpha \in \,]1, \beta - 1[$. Let $j \in \mathbb{N}$ and choose $\gamma \in \,]2, \beta[$. An immediate application of the Stirling formula gives $\lim_{p\to\infty} p^{\gamma p}/(p!)^\beta = 0$, hence $\lim_{p\to\infty} p^{\gamma p}/M_{j,p} = 0$. So there is $p_j \in \mathbb{N}$ such that $p^{\gamma p} < M_{j,p}$, hence $p^\gamma < m_{j,p}$ for every $p \ge p_j$. Choosing now $\alpha > 1$ such that $\gamma - \alpha > 1$, we get

$$\frac{p^\alpha}{m_{j,p}} = \frac{p^\gamma}{m_{j,p}} \frac{1}{p^{\gamma-\alpha}} < \frac{1}{p^{\gamma-\alpha}}, \qquad \forall p \ge p_j,$$

hence $\sum_{p=1}^\infty p^\alpha/m_{j,p} < \infty$. ∎

9. Extension results

Construction. If the matrix $\mathfrak{m}$ satisfies condition $(\mathfrak{m}_5)$ with $\alpha > 1$, we make the following choices.

a) *The integers p_j and the numbers n_p and N_p.* We first choose a strictly increasing sequence $(p_j)_{j\in\mathbb{N}}$ of positive integers such that

$$\sum_{p=p_j+1}^{\infty} \frac{p^\alpha}{m_{j+1,p}} < 2^{-j}, \qquad \forall j \in \mathbb{N}.$$

We then define the sequence $(n_p)_{p\in\mathbb{N}_0}$ as follows:

$$n_p = \begin{cases} 1 & \text{if } p = 0, \\ p^{-\alpha} m_{1,p} & \text{if } p \in \{1, \ldots, p_1\}, \\ p^{-\alpha} m_{j+1,p} & \text{if } p \in \{p_j + 1, \ldots, p_{j+1}\} \text{ and } j \in \mathbb{N}. \end{cases}$$

Of course we then set $N_p = n_0 \cdots n_p$ for every $p \in \mathbb{N}_0$.

Note that, for every $j \in \mathbb{N}$, there is clearly a constant $c_j > 1$ such that $n_p \le c_j p^{-\alpha} m_{j,p}$ for every $p \in \mathbb{N}_0$, hence

$$N_p \le c_j^p p!^{-\alpha} M_{j,p}, \qquad \forall p \in \mathbb{N}_0. \tag{4}$$

b) *The functions S_n.* For $a := \sum_{p=1}^\infty 1/n_p$, a slight enhancement of property 1.3.5 of [3] provides the existence of a function $\varphi \in \mathcal{D}([-a, a])$ such that $0 \le \varphi \le 1$ as well as $\varphi^{(p)}(0) = \delta_{p,0}$ and $\|\varphi^{(p)}\|_{\mathbb{R}} \le 2^p N_p$ for every $p \in \mathbb{N}_0$.

For every $n \in \mathbb{N}_0$, we then set $v_n(x) = \varphi(a(1+n^2)x)$ for every $x \in \mathbb{R}$ and define the function u_n on $\mathbb{R}$ by setting

$$u_n(x) := \begin{cases} 0 & \text{if } x < -1 - 1/(1+n^2), \\ v_n(x+1) & \text{if } -1 - 1/(1+n^2) \le x \le -1, \\ 1 & \text{if } -1 < x < 1, \\ v_n(x-1) & \text{if } 1 \le x \le 1 + 1/(1+n^2), \\ 0 & \text{if } 1 + 1/(1+n^2) < x. \end{cases}$$

Finally, for every $n \in \mathbb{N}_0$, we set $S_n := T_n u_n$. It is a direct matter to verify that S_n belongs to $\mathcal{C}^\infty(\mathbb{R})$, is an extension of $T_n|_{[-1,1]}$ and has its support contained in $[-1-1/(1+n^2), 1+1/(1+n^2)]$.

c) *The functions S_γ.* Now everything is set up to introduce the following functions: for every $\gamma \in \mathbb{N}_0^r$, we define the function S_γ on $\mathbb{R}^r$ by

$$S_\gamma(x) = S_{\gamma_1}(x_1) \cdots S_{\gamma_r}(x_r), \quad \forall x \in \mathbb{R}^r.$$

It is clear that S_γ belongs to $\mathcal{C}^\infty(\mathbb{R}^r)$, is an extension of $T_\gamma|_{[-1,1]^r}$ and has its support contained in $[-2,2]^r$. A lot more can be said. ∎

LEMMA 9.1. *If the matrix $\mathfrak{m}$ satisfies condition $(\mathfrak{m}_5)$, then there is a constant $b > 1$ such that $\|S_0^{(p)}\|_{\mathbb{R}} \le b^{p+1} N_p$ and $\|S_n^{(p)}\|_{\mathbb{R}} \le (bn^2)^{p+1} N_p$ for every $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$.*

Proof. For every $n \in \mathbb{N}$, $x \in \mathbb{R}$ such that $|x| \le 1 + 1/(1+n^2)$ and $p \in \mathbb{N}_0$, Proposition 3.1 and the preceding construction lead to

$$\begin{aligned} |S_n^{(p)}(x)| &\le \sum_{l=0}^{p} \binom{p}{l} |T_n^{(l)}(x)|\,|u_n^{(p-l)}(x)| \\ &\le \sum_{l=0}^{p} \binom{p}{l} e\,\frac{n^{2l}}{l!}\, 2^{p-l} N_{p-l} a^{p-l} (1+n^2)^{p-l}. \end{aligned}$$

As the series $\sum_{p=0}^{\infty} 1/n_p$ converges, we have $n_p \ge 1$ for p large; this implies the existence of a constant $c > 1$ such that $N_q \le c N_p$ for every $p, q \in \mathbb{N}$ such that $q \le p$. Therefore we get

$$\begin{aligned} \|S_n^{(p)}\|_{\mathbb{R}} &\le ceN_p \sum_{l=0}^{p} \binom{p}{l} n^{2l} (2a(1+n^2))^{p-l} \\ &\le ceN_p (n^2 + 2a(1+n^2))^p, \quad \forall n \in \mathbb{N},\ p \in \mathbb{N}_0. \end{aligned}$$

As it is clear that we also have

$$\|S_0^{(p)}\|_{\mathbb{R}} = \|u_0^{(p)}\|_{\mathbb{R}} \le (2a)^p N_p, \quad \forall p \in \mathbb{N}_0,$$

we conclude at once. ∎

PROPOSITION 9.2. *If the matrix* $\mathfrak{m}$ *satisfies condition* $(\mathfrak{m}_5)$, *then there is* $q \in \mathbb{N}$ *such that, for every* $j, r \in \mathbb{N}$, *there is a constant* $D_{j,r,q} > 0$ *such that*

$$\|S_\gamma\|_{\mathbb{R}^r,j,1/j} \le D_{j,r,q} \|T_\gamma\|_{j+q+2r,r}, \qquad \forall \gamma \in \mathbb{N}_0^r.$$

In particular, for every $\gamma \in \mathbb{N}_0^r$, *the extension* S_γ *of* $T_\gamma|_{[-1,1]^r}$ *belongs to* $\mathcal{D}_{(\mathfrak{M})}([-2,2]^r)$.

Proof. Let us first remark that, for every $j \in \mathbb{N}$, condition $(\mathfrak{m}_4)$ applied with $A = 1$ provides a constant $E_j > 1$ such that

$$M_{j+1,2(p+1)} \le E_j p! M_{j,p}, \qquad \forall p \in \mathbb{N}_0, \tag{5}$$

hence

$$M_{j+1,p}^2 \le M_{j+1,2(p+1)} \le E_j p! M_{j,p}, \qquad \forall p \in \mathbb{N}_0. \tag{6}$$

Now we fix q by the following observation: as the matrix $\mathfrak{m}$ satisfies $(\mathfrak{m}_5)$, there is $\lambda > 1$ such that $\sum_{p=1}^\infty p^\lambda / m_{j,p} < \infty$ for every $j \in \mathbb{N}$, so we can choose $q \in \mathbb{N}$ with $2^{-q} < \lambda - 1$.

Next, for every $j, r \in \mathbb{N}$, we proceed as follows. First of all, to shorten notation, we set $\|T_\gamma\|_* = \|T_\gamma\|_{j+q+2r,r}$ for every $\gamma \in \mathbb{N}_0^r$. Then we recall that Proposition 6.1 provides some $B = B_{j+q+r+1,r} > 1$ depending only on j, r and q such that

$$1 = |a_\gamma(T_\gamma)| \le 2^r \frac{(2B)^{|\beta|}}{\gamma^\beta} M_{j+q+r+1,\beta} \|T_\gamma\|_*, \qquad \forall \beta, \gamma \in \mathbb{N}_0^r. \tag{7}$$

So, for every $\alpha \in \mathbb{N}_0^r$, using successively Lemma 9.1, inequality (4) and (7) with $\beta = (2(\alpha_1 + 1), \dots, 2(\alpha_r + 1)$ and setting $A = 8bc_{j+q+r}B^2$, we obtain

$$\|S_\gamma^{(\alpha)}\|_{\mathbb{R}^r} = \prod_{k=1}^r \|S_{\gamma_k}^{(\alpha_k)}\|_{\mathbb{R}} \le A^r \|T_\gamma\|_* \prod_{k=1}^r \frac{A^{\alpha_k}}{\alpha_k!^\lambda} M_{j+q+r,\alpha_k} M_{j+q+r+1,2(\alpha_k+1)}.$$

Now for every $k \in \{1, \dots, r\}$, we use (5) and (6) to get

$$M_{j+q+r,\alpha_k} M_{j+q+r+1,2(\alpha_k+1)} \le \alpha_k!^2 E_{j+q+r} E_{j+q+r-1} M_{j+q+r-1,\alpha_k}$$

and then we note that a repeated use of inequalities of type (6) leads to

$$E_{j+q+r-1} \frac{M_{j+q+r-1,\alpha_k}}{\alpha_k!} \le E_{j+q+r-1} E_{j+q+r-2}^{2^{-1}} \left(\frac{M_{j+q+r-2,\alpha_k}}{\alpha_k!} \right)^{2^{-1}} \le \cdots$$
$$\le E_{j+q+r-1} E_{j+q+r-2}^{2^{-1}} \cdots E_{j+r-1}^{2^{-q}} \left(\frac{M_{j+r-1,\alpha_k}}{\alpha_k!} \right)^{2^{-q}}.$$

Setting $E = E_{j+q+r} \prod_{k=0}^q E_{j+q+r-1-k}^{2^{-k}}$ and putting all this information together yields

$$\|S_\gamma^{(\alpha)}\|_{\mathbb{R}^r} \le (AE)^r \|T_\gamma\|_* \prod_{k=1}^r \frac{A^{\alpha_k}}{\alpha_k!^{\lambda-3}} \left(\frac{M_{j+r-1,\alpha_k}}{\alpha_k!} \right)^{2^{-q}},$$

hence

$$\frac{j^{|\alpha|}\|S_\gamma^{(\alpha)}\|_{\mathbb{R}^r}}{M_{j,|\alpha|}} \le \|S_\gamma^{(\alpha)}\|_{\mathbb{R}^r} \prod_{k=1}^r \frac{j^{\alpha_k}}{M_{j,\alpha_k}} \le \|S_\gamma^{(\alpha)}\|_{\mathbb{R}^r} \prod_{k=1}^r \frac{j^{\alpha_k}}{M_{j+r-1,\alpha_k}}$$
$$\le (AE)^r \|T_\gamma\|_* \prod_{k=1}^r \frac{(jA)^{\alpha_k}}{\alpha_k!^{\lambda-1-2^q}} \left(\frac{\alpha_k!^2}{M_{j+r-1,\alpha_k}}\right)^{1-2^{-q}}.$$

Finally, on the one hand the sequence $(p/m_{j,p})_{p\in\mathbb{N}}$ of positive numbers is decreasing and satisfies $\sum_{p=1}^\infty p/m_{j,p} < \infty$; this implies $\lim p^2/m_{j,p} = 0$, hence $\lim p!^2/M_{j,p} = 0$. On the other hand, the sequence $((jA)^p/p!^{\lambda-1-2^{-q}})_{p\in\mathbb{N}}$ converges to 0. These two facts and the last inequality together provide the existence of a constant $D_{j,r,q} > 0$ such that

$$\frac{j^{|\alpha|}\|S_\gamma^{(\alpha)}\|_{\mathbb{R}^r}}{M_{j,|\alpha|}} \le D_{j,r,q}\|T_\gamma\|_*, \qquad \forall \alpha \in \mathbb{N}_0^r,$$

hence the conclusion. ∎

THEOREM 9.3. *If the matrix* $\mathfrak{m}$ *satisfies condition* $(\mathfrak{m}_5)$, *then for every* $r \in \mathbb{N}$,

$$T\colon \mathcal{E}_{(\mathfrak{M})}([-1,1]^r) \to \mathcal{D}_{(\mathfrak{M})}([-2,2]^r), \qquad f \mapsto \sum_{\gamma\in\mathbb{N}_0^r} a_\gamma(f) S_\gamma,$$

is a continuous linear extension map.

Proof. This is a direct consequence of Theorem 6.3 and Proposition 9.2. ∎

10. Application

PROPOSITION 10.1. *If* $\Phi\colon [0,\infty[\, \to [0,\infty[$ *is an increasing and convex function such that* $\Phi(0) = 0$ *and* $\lim_{t\to\infty} \Phi(t)/t = \infty$, *then the matrix* $\mathfrak{m} = (m_{j,p})_{j\in\mathbb{N},\, p\in\mathbb{N}_0}$ *defined by* $m_{j,0} = 1$ *and*

$$m_{j,p} = pe^{\Phi(p/8^j)-\Phi((p-1)/8^j)}$$

for every $j,p \in \mathbb{N}$ *satisfies conditions* $(\mathfrak{m}_1)$ *to* $(\mathfrak{m}_4)$.

If moreover Φ *satisfies* $\lim_{p\to\infty} \Phi(t)/(t\log(t)) = \infty$, *then* $\mathfrak{m}$ *also satisfies* $(\mathfrak{m}_5)$.

Proof. By Proposition 7.1 of [8], we know that $\mathfrak{m}$ satisfies $(\mathfrak{m}_1)$ to $(\mathfrak{m}_3)$.

Let us prove that it also satisfies $(\mathfrak{m}_4)$. For every $p \in \mathbb{N}$, as we have $\binom{2p+2}{p} \le 2^{2p+2}$, $p+1 \le 2^p$ and $p+2 \le 2^{p+1}$, we get $(2p+2)! \le 2^{3+4p}p!^2$. So for every $j,p \in \mathbb{N}$, we have

$$M_{j+1,2(p+1)} = (2p+2)!e^{\Phi(2(p+1)/8^{j+1})} \le 2^{3+4p}p!^2 e^{\Phi(p/(2\cdot 8^j))}$$

since Φ is an increasing function. As also $2\Phi(t) \le \Phi(2t)$ for every $t \ge 0$, this leads to

$$M_{j+1,2(p+1)} \leq 2^{7p}p!^2 e^{\Phi(p/8^j)} e^{-\Phi(p/(2\cdot 8^j))} = 2^{7p}p! M_{j,p} e^{-\phi(p/(2\cdot 8^j))}$$

for every $j,p \in \mathbb{N}$. Now the condition $\lim \Phi(t)/t = \infty$ implies for every $A > 0$ the existence of some $p_A \in \mathbb{N}$ such that $\Phi(p/(2\cdot 8^j)) \geq p(\log(A) + \log(2^7))$, hence $e^{-\Phi(p/(2\cdot 8^j))} \leq A^{-p}2^{-7p}$ for every $p \geq p_A$. So altogether we have

$$M_{j+1,2(p+1)} \leq A^{-p}p! M_{j,p}, \quad \forall p \geq p_A;$$

hence the conclusion.

Finally, we prove that if Φ satisfies $\lim \Phi(t)/(t\log(t)) = \infty$, then $\mathfrak{m}$ satisfies $(\mathfrak{m}_5')$ with $\beta = 3$. Indeed, for every $j \in \mathbb{N}$, the supplementary condition provides $\lim \Phi(8^{-j}p)/(p\log(p)) = \infty$, hence there is $p_j \in \mathbb{N}$ such that $\Phi(8^{-j}p) \geq 3p\log(p)$ for every $p \geq p_j$. So for every $p \geq p_j$ we certainly have

$$\frac{p!^3}{M_{j,p}} = \frac{p!^2}{e^{\Phi(p/8^j)}} \leq p^{2p}e^{-\Phi(p/8^j)} \leq e^{-p\log(p)}$$

and we conclude at once. ∎

Let Φ be a real, increasing and convex function on $[0,\infty[$ such that $\Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(t)/t = \infty$.

The notations have been modified several times: the spaces $\mathrm{I}_{4,\Phi}(\{0\})$, $\mathrm{I}_{4,\Phi}([-1,1])$ and $\mathrm{I}_{4,\Phi}(\mathbb{R})$ introduced in [1] become $\mathrm{I}_{\Phi}(\{0\})$, $\mathrm{I}_{\Phi}([-1,1])$ and $\mathrm{I}_{\Phi}(\mathbb{R})$ in [2]. If one considers the matrix $\mathfrak{m}$ associated to Φ in Proposition 10.1, they coincide respectively with the spaces $\widehat{\Lambda}_{(\mathfrak{M})}$, $\widehat{\mathcal{E}}_{(\mathfrak{M})}([-1,1])$ and $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$ of [8] and with the spaces $\Lambda^{(1)}_{(\mathfrak{M})}$, $\mathcal{E}_{(\mathfrak{M})}([-1,1])$ and $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R})$ in this paper.

Let us also recall that in [8], it is proven that there are spaces $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R})$ for which there is no function Φ such that $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R}) = \mathrm{I}_{\Phi}(\mathbb{R})$. So Proposition 7.1 and Theorems 6.3 and 9.3 provide enhancements of the following results of Beaugendre.

THEOREM 10.2. *Let Φ be a real, increasing and convex function on the interval $[0,\infty[$ such that $\Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(t)/t = \infty$.*

(a) ([1, Proposition 3.1.1]) *There is no continuous linear extension map from $\Lambda^{(1)}_{(\mathfrak{M})}$ into $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R})$.*
(b) ([2, Theorem 3.6]) *The family $(T_n, a_n(\cdot))_{n\in\mathbb{N}_0}$ is an absolute Schauder basis in $\mathcal{E}_{(\mathfrak{M})}([-1,1])$.*
(c) ([2, Theorem 4.3]) *If Φ also satisfies $\lim_{t\to\infty}\Phi(t)/(t\log(t)) = \infty$, then there is a continuous linear extension map from $\mathcal{E}_{(\mathfrak{M})}([-1,1])$ into $\mathcal{B}_{(\mathfrak{M})}(\mathbb{R})$ by means of a modification of the Chebyshev polynomials.* ∎

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