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On some elliptic boundary-value problems with discontinuous nonlinearities

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Abstract. We establish two existence results for elliptic boundary-value problems with discontinuous nonlinearities. One of them concerns implicit elliptic equations of the form $\psi(-\Delta u) = f(x,u)$. We emphasize that our assumptions permit the nonlinear term f to be discontinuous with respect to the second variable at each point.

1. Introduction. Throughout, Ω is a nonempty open bounded set in $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary $\partial\Omega$, a is a real positive number, p is a real number strictly greater than $n/2$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $\psi : [a, +\infty[\to \mathbb{R}$ are given functions.

This paper is motivated by the results of [7] and [8] where some elliptic boundary-value problems with discontinuous nonlinearities are studied. Specifically, in these papers, the following two problems are considered:

$$(P) \qquad \begin{cases} -\Delta u(x) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(P_1) \qquad \begin{cases} \psi(-\Delta u(x)) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplacian operator; for both, existence of strong solutions is established. As is well known, a strong solution for problem (P) or (P_1) is a function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying the corresponding equation for almost all $x \in \Omega$. There is wide literature on the existence of solutions for problem (P), and when f is a Carathéodory function, variational methods are usually employed.

Here we are interested in the case where f can be discontinuous with respect to the second variable. In this connection we refer to [7] and [8] (and the references therein). In particular, Theorems 3.1 and 4.2 of [7] give

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the existence of a strong solution for problem (P) and (P_1) respectively, assuming that $f(x,\cdot)$ is a Riemann-measurable function for almost all $x \in \Omega$ and, in the case of problem (P_1), that f is also independent of $x \in \Omega$. We recall that a Riemann-measurable function is a function whose set of discontinuity points has Lebesgue measure 0. In the same paper it is pointed out that the assumption on f cannot be weakened. Indeed, in Remark 3.2 of [7] it is shown that if $f(x,\cdot)$ is almost everywhere equal to a Riemann-measurable function for almost all $x \in \Omega$ only, then problem (P) may not have any strong solution. On the other hand, Theorem 3.1 of [8] gives the existence of a strong solution for problem (P_1) this time allowing f to depend on $x \in \Omega$ and assuming hypotheses substantially different from those of Theorem 4.2 of [7].

The purpose of the present paper is twofold: we give versions of Theorems 3.1 and 4.2 of [7] in which $f(x,\cdot)$ is supposed to be almost everywhere equal to a Riemann-measurable function for almost all $x \in \Omega$ extending, at the same time, the second one to the nonautonomous case. In the latter case, our result and Theorem 3.1 of [8] will turn out to be mutually independent. To prove the existence result relating to problem (P_1) we will use a recent selection theorem established in [1].

2. Basic definitions and notations. Let X, Y be two nonempty sets. A multifunction F from X into Y is a function from X into the family of all subsets of Y and we briefly denote it by $F : X \to 2^Y$. The set $\{(x,y) \in X \times Y : y \in F(x)\}$ is called the *graph* of F. For each $A \subseteq Y$, we denote by $F^-(A)$ the set $\{x \in X : F(x) \cap A \neq \emptyset\}$. We say that a function $f : X \to Y$ is a *selection* of F if $f(x) \in F(x)$ for all $x \in X$. If X, Y are topological spaces, a multifunction $F : X \to 2^Y$ is called *lower semicontinuous* (briefly l.s.c.) at $x \in X$ if for any $y \in F(x)$ and any neighborhood V of y there exists a neighborhood U of x such that

$$F(z) \cap V \neq \emptyset \quad \text{for all } z \in U.$$

If $(X, \Im)$ is a measurable space and Y is a topological space, a multifunction $F : X \to 2^Y$ is called *measurable* when $F^-(A) \in \Im$ for any open set $A \subseteq Y$.

We denote by $\mathcal{L}(\Omega)$ the Lebesgue σ-algebra of Ω and by m_n the Lebesgue measure in $\mathbb{R}^n$. Also, the symbol $\mathcal{B}(\mathbb{R})$ stands for the Borel σ-algebra of $\mathbb{R}$. For a subset A of $\mathbb{R}^n$, $\overline{\mathrm{co}}(A)$ and $\mathrm{int}(A)$ will denote the closed convex hull and the interior of A respectively. If d is the Euclidean distance in $\mathbb{R}^n$ and A is a nonempty set in $\mathbb{R}^n$, we put $d(x,A) = \inf_{y \in A} d(x,y)$ for all $x \in \mathbb{R}^n$. Finally, we denote by $\|\cdot\|_p$ the usual norm in $L^p(\Omega)$.

To close this section, recall that, thanks to Theorem 2 of [11], one has

$$\operatorname*{ess\,sup}_{x \in \Omega} |u(x)| \leq B \|\Delta u\|_p \quad \text{for every } u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \tag{1}$$

where

$$B = [m(\Omega)]^{2/n-1/p} \frac{\Gamma\left(1+\frac{n}{2}\right)}{\pi n(n-2)} \left[\frac{\Gamma\left(1+\frac{p}{p-1}\right)\Gamma\left(\frac{n}{n-2}\right) - \frac{p}{p-1}}{\Gamma\left(\frac{n}{n-2}\right)}\right]^{1-1/p}$$

and Γ denotes the Gamma function.

3. Main results. In this section we state and prove the main results. Throughout, we briefly write a.a. for "almost all".

THEOREM 1. *Let $\beta \in L^p(\Omega)$ with $\beta \neq 0$ and $A = [-B\|\beta\|_p, B\|\beta\|_p]$. Assume that there exist $E \subseteq A$ with $m_1(E) = 0$ and a function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfying the following properties*:

(α_1) *$\{t \in A : g(x,\cdot)$ is discontinuous at t or $g(x,t) \neq f(x,t)\} \subseteq E$ for a.a. $x \in \Omega$*;
(α_2) *$g(\cdot,t)$ is measurable for a.a. $t \in A$*;
(α_3) *$\sup_{t\in A} |g(x,t)| \le \beta(x)$ for a.a. $x \in \Omega$*;
(α_4) *there exists an open set $D \supseteq E$ such that*

$$\operatorname*{ess\,inf}_{x\in\Omega} \inf_{t\in D} g(x,t) > 0 \quad \text{or} \quad \operatorname*{ess\,sup}_{x\in\Omega} \sup_{t\in D} g(x,t) < 0.$$

Then there exists a strong solution u of problem (P) satisfying

$$|\Delta u(x)| \le \sup_{t\in A} |g(x,t)| \quad \text{for a.a. } x \in \Omega.$$

Proof. Without loss of generality, we can suppose that conditions (α_1) and (α_3) hold for all $x \in \Omega$. For every $x \in \Omega$, we define

$$\widehat{g}(x,t) = \begin{cases} g(x,-B\|\beta\|_p) & \text{if } t < -B\|\beta\|_p,\\ g(x,t) & \text{if } |t| \le B\|\beta\|_p,\\ g(x,B\|\beta\|_p) & \text{if } t > B\|\beta\|_p. \end{cases}$$

Clearly (α_1) holds with $\widehat{g}$ in place of g. Observe that by (α_2) we can find a countable set $P \subseteq \mathbb{R}\setminus E$ dense in $\mathbb{R}$ such that $\widehat{g}(\cdot,t)$ is measurable for all $t \in P$. Moreover, by (α_3), we can suppose that the function $\widehat{g}(x,\cdot)$ is bounded for all $x \in \Omega$. Thus, thanks to Proposition 2 of [3] (which is proved assuming that Ω is a real interval, but it is easy to see that the same holds if Ω is a bounded open subset of $\mathbb{R}^n$), the multifunction $F : \Omega\times\mathbb{R} \to 2^{\mathbb{R}}$ defined by

$$F(x,t) = \bigcap_{m\in\mathbb{N}} \overline{\mathrm{co}}\Big(\overline{\bigcup_{y\in P,\,|y-t|\le 1/m} \{\widehat{g}(x,y)\}}\Big)$$

has the following properties:

(a) $F(x,t)$ is nonempty and convex for all $(x,t) \in \Omega\times\mathbb{R}$;
(b) $F(\cdot,t)$ is measurable for all $t \in \mathbb{R}$;

(c) $F(x,\cdot)$ has closed graph for all $x \in \Omega$;
(d) if $x \in \Omega$ and $\widehat{g}(x,\cdot)$ is continuous at $t \in \mathbb{R}$, then $F(x,t) = \{\widehat{g}(x,t)\}$.

Now, thanks to Proposition 1 of [9] it is easy to deduce that

$$\sup_{t\in A} d(0, F(x,t)) = \sup_{t\in A\cap P} d(0, F(x,t))$$

for all $x \in \Omega$ (see, for instance, the proof of Theorem 3.1 of [7]). Consequently, the function $x \in \Omega \mapsto \sup_{t\in A} d(0, F(x,t))$ is measurable. Moreover, from condition (α_3), this function belongs to $L^p(\Omega)$ and

$$\|\sup_{t\in A} d(0, F(\cdot,t))\|_{L^p(\Omega)} \le \|\beta\|_p. \tag{2}$$

At this point, we can apply Theorem 2.2 of [7]. Hence, there exists $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$-\Delta u(x) \in F(x, u(x)) \tag{3}$$

and

$$|\Delta u(x)| \le \sup_{t\in A} d(0, F(x,t)) \tag{4}$$

for a.a. $x \in \Omega$. Observe that from (3) and (4) and in view of (1) we have

$$\operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \le B\|\Delta u(x)\|_{L^p(\Omega)} \le B\|\sup_{t\in A} d(0, F(x,t))\|_{L^p(\Omega)} \le B\|\beta\|_p,$$

that is,

$$u(x) \in A \quad \text{for a.a. } x \in \Omega. \tag{5}$$

Now, put

$$\Omega_0 = \{x \in \Omega : u(x) \in E\}.$$

We claim that $m_n(\Omega_0) = 0$. Indeed, by Proposition 2.1 of [7] one has

$$\Delta u(x) = 0 \in F(x, u(x)) \tag{6}$$

for almost all $x \in \Omega_0$. On the other hand, by condition (α_4), we have

$$\operatorname*{ess\,inf}_{x\in\Omega} \inf_{t\in D} \inf F(x,t) \ge \operatorname*{ess\,inf}_{x\in\Omega} \inf_{t\in D} g(x,t) > 0 \tag{7}$$

or

$$\operatorname*{ess\,sup}_{x\in\Omega} \sup_{t\in D} \sup F(x,t) \le \operatorname*{ess\,sup}_{x\in\Omega} \sup_{t\in D} g(x,t) < 0. \tag{8}$$

Clearly, (6) together with (7) or (8) imply $m_n(\Omega_0) = 0$. This latter fact, the definition of $\widehat{g}$, condition (α_1), properties (d) and (3)–(5) imply

$$-\Delta u(x) = f(x, u(x))$$

and

$$|\Delta u(x)| \le \sup_{t\in A} |g(x,t)|$$

for a.a. $x \in \Omega$. This completes the proof.

Remark 1. As observed in the introduction, an example in Remark 3.2 of [7] shows that if f is almost everywhere equal to a function fulfilling all the assumptions of Theorem 3.1 of [7], then problem (P) may not have any strong solution. Precisely, for f defined by

$$f(x,t)=\begin{cases}1 & \text{if } (x,t)\in\Omega\times\mathbb{Q},\\ 0 & \text{otherwise,}\end{cases}$$

problem (P) cannot have any strong solution. Observe that this f does not satisfy the hypotheses of Theorem 1. Indeed, it is easy to note that there is no function g fulfilling both (α_1) and (α_4). On the other hand, if we define

$$f(x,t)=\begin{cases}0 & \text{if } (x,t)\in\Omega\times\mathbb{Q},\\ 1 & \text{otherwise,}\end{cases}$$

then for $g(x,t)=1$ for all $(x,t)\in\Omega\times\mathbb{R}$ all the hypotheses of Theorem 1 are satisfied, and so problem (P) admits a strong solution. We emphasize that in this latter case f is discontinuous at each point of $\mathbb{R}$.

Remark 2. We observe that condition (α_3) of Theorem 1 is weaker than condition (α_3) of Theorem 3.1 of [7]. Indeed, the measurability of the function $x\in\Omega\mapsto\sup_{t\in A}|g(x,t)|$ is not required.

To prove the next result we need the following selection theorem for multifunctions of two variables:

Theorem A (Theorem 2 of [1]). *Let T,X be Polish spaces and let μ,ψ be positive regular Borel measures on T and X, respectively, with μ finite and ψ σ-finite. Let S be a separable metric space, $F:T\times X\to 2^S$ a multifunction with nonempty complete values, and let $E\subseteq X$ be a given set. Finally, let $\mathcal{B}(X)$ be the Borel σ-algebra of X and $\mathcal{T}_\mu$ the completion of the Borel σ-algebra of T with respect to μ. Assume that*:

(i) *F is $\mathcal{T}_\mu\otimes\mathcal{B}(X)$-measurable*;
(ii) *$\{x\in X : F(t,\cdot)$ is not lower semicontinuous at $x\}\subseteq E$ for a.a. $t\in T$.*

Then there exists a selection $\phi:T\times X\to S$ of F and a negligible set $R\subseteq X$ such that:

(i)′ *$\phi(\cdot,x)$ is $\mathcal{T}_\mu$-measurable for each $x\in X\setminus(E\cup R)$*;
(ii)′ *$\{x\in X:\phi(t,\cdot)$ is not continuous at $x\}\subseteq E\cup R$ for a.a. $t\in T$.*

Now, we are able to prove our second main result. From now on, if $C\subseteq\mathbb{R}$, $\mathcal{B}(C)$ will denote the Borel σ-algebra of C.

Theorem 2. *Let $\beta\in L^p(\Omega)$ with $\beta\neq 0$ and put $A=[-B\|\beta\|_p, B\|\beta\|_p]$. Assume that there exist $E,E_1\subset A$ with $m_1(E\cup E_1)=0$ and E_1 closed, and a function $g:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfying the following properties*:

(β_1) *$\{t \in A : g(x,\cdot)$ is discontinuous at $t\} \subseteq E_1$ and $\{t \in A : g(x,t) \neq f(x,t)\} \subseteq E$ for a.a. $x \in \Omega$;*

(β_2) *ψ is continuous in $[a,+\infty[$ and $\psi^{-1}(\sigma)$ has empty interior for every $\sigma \in \operatorname{int}\psi([a,+\infty])$;*

(β_3) *$g(\cdot,t)$ is measurable for a.a. $t \in A$;*

(β_4) *if one puts*

$$v(x) = \operatorname*{ess\,inf}_{t\in A} g(x,t) \quad \textit{and} \quad z(x) = \operatorname*{ess\,sup}_{t\in A} g(x,t)$$

for all $x \in \Omega$, then $[v(x),z(x)] \subseteq \psi([a,+\infty[)$ and $\psi^{-1}([v(x),z(x)]) \subseteq [a,\beta(x)]$ for a.a. $x \in \Omega$.

Then there exists a strong positive solution of problem (P_1).

Proof. Without loss of generality, we can suppose that the conditions (β_1) and (β_4) hold for all $x \in \Omega$. By (β_2) we can find a countable set $P \subseteq A\setminus E_1$ dense in A such that $g(\cdot,t)$ is measurable for all $t \in P$. Moreover, it is easy to see that

$$v(x) = \inf_{t\in A\setminus E_1} g(x,t) \quad \text{and} \quad z(x) = \sup_{t\in A\setminus E_1} g(x,t)$$

for all $x \in \Omega$. Hence, $g_{|\Omega\times(A\setminus E_1)}$ is $\mathcal{L}(\Omega)\otimes\mathcal{B}(A\setminus E_1)$-measurable by the Lemma on p. 198 of [6]. So, by Lemma III.39 of [2], the functions v and z are measurable. Now, define

$$\phi(x,t) = \begin{cases} g(x,t) & \text{if } (x,t) \in \Omega\times(A\setminus E_1),\\ z(x) & \text{if } (x,t) \in \Omega\times E_1. \end{cases}$$

Since E_1 is closed, condition (β_1) implies that

$$\{t \in A : \phi(x,\cdot) \text{ is discontinuous at } t\} \subseteq E_1. \tag{9}$$

Moreover, ϕ turns out to be $\mathcal{L}(\Omega)\otimes\mathcal{B}(A)$-measurable and satisfies

$$v(x) \le \phi(x,t) \le z(x) \tag{10}$$

for all $(x,t) \in \Omega\times A$. At this point, observe that the function ψ fulfils all the hypotheses of Theorem 2.4 of [10]. Hence, there exists a set $Y \subseteq [a,+\infty[$ such that $\psi^{-1}(\sigma)\cap Y$ is nonempty and closed in $\mathbb{R}$ for each $\sigma \in \psi([a,+\infty[)$ and the multifunction $\psi^{-1}(\cdot)\cap Y$ is l.s.c. in $\psi([a,+\infty[)$. Now, put

$$\Gamma(x,t) = \begin{cases} \psi^{-1}(\phi(x,t))\cap Y & \text{if } (x,t) \in \Omega\times A,\\ \mathbb{R} & \text{otherwise.} \end{cases}$$

Then Γ is an $\mathcal{L}(\Omega)\otimes\mathcal{B}(\mathbb{R})$-measurable multifunction and, further, by (9) one has

$$\{t \in \mathbb{R} : \Gamma(x,\cdot) \text{ is not l.s.c. at } t\} \subseteq E_1. \tag{11}$$

Consequently, by Theorem A, there exist $R \subseteq \mathbb{R}$ with $m_1(R) = 0$ and a selection γ of Γ such that

$$\{t \in \mathbb{R} : \gamma(x,\cdot) \text{ is discontinuous at } t\} \subseteq E_1 \cup R \tag{12}$$

for a.a. $x \in \Omega$ and $\gamma(\cdot,t)$ measurable for all $t \in \mathbb{R} \setminus (E_1 \cup R)$. From this latter property we can find a countable set $P_1 \subseteq \mathbb{R} \setminus (E_1 \cup R)$, dense in $\mathbb{R}$, such that $\gamma(\cdot,t)$ is measurable for all $t \in P_1$. Also, by conditions (β_4), (10) and since

$$\gamma(x,t) \in \psi^{-1}(\phi(x,t)) \quad \text{for all } (x,t) \in \Omega \times A \tag{13}$$

we deduce that

$$\begin{aligned} a \le \gamma(x,t) &\le \sup_{t\in A} \sup \psi^{-1}(\phi(x,t)) \\ &\le \sup \psi^{-1}([v(x), z(x)]) \le \beta(x) \end{aligned} \tag{14}$$

for all $(x,t) \in \Omega \times A$. At this point, we can apply Theorem 1. Hence, there exists $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$-\Delta u(x) = \gamma(x, u(x)) \tag{15}$$

and

$$|\Delta u(x)| \le \sup_{t\in A} \gamma(x,t) \le \beta(x) \tag{16}$$

for a.a. $x \in \Omega$. In particular, by (16) and (1), one has $\operatorname{ess\,sup}_{x\in\Omega} |u(x)| \le B\|\beta\|_p$, that is,

$$u(x) \in A \quad \text{for a.a. } x \in \Omega. \tag{17}$$

Taking into account (14) and (15), we can argue as in the proof of Theorem 1 to deduce that

$$m_n(\{x \in \Omega : u(x) \in E \cup E_1\}) = 0. \tag{18}$$

Consequently, by the definition of ϕ and conditions (β_1), (13), (15), (17) and (18) one has

$$\psi(-\Delta u(x)) = f(x, u(x))$$

for a.a. $x \in \Omega$. Moreover, the Maximum Principle and (14), (15) and (17) imply that u is positive. This completes the proof.

Remark 3. Notice that Theorem 2 is a nonautonomous version of Theorem 4.2 of [7].

Remark 4. When the function g of Theorem 2 does not depend on $x \in \Omega$, then we can take the function β equal to a constant ϱ. So we have $A = [-Bm_n(\Omega)^{1/p}\varrho, Bm_n(\Omega)^{1/p}\varrho]$ and condition (β_4) becomes

(β_4') $g([-Bm_n(\Omega)^{1/p}\varrho, Bm_n(\Omega)^{1/p}\varrho]) \subseteq \psi([a,+\infty])$ and $\psi^{-1}(g([-Bm_n(\Omega)^{1/p}\varrho, Bm_n(\Omega)^{1/p}\varrho])) \subseteq [a,\varrho]$.

We observe that condition (β_4') is stronger than condition (c_3) of Theorem 4.2 of [7]. Hence, the former does not generalize the latter to the nonautonomous case. Nevertheless, we point out that condition (c_3) of Theorem 4.2 of [7] must be replaced with (β_4'), otherwise inequality (14) and the subsequent inclusion in the proof of that result may not be true, as is easily checked.

REMARK 5. We observe that Theorem 3.1 of [8] deals with the vectorial case of problem (P_1), namely $\mathbb{R}$ is replaced by $\mathbb{R}^h$ where h is an integer greater than or equal to 1. When $h = 1$, it is immediate to check that this result and Theorem 2 are mutually independent.

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