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# On the geometry of tangent bundles with a class of metrics

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**Abstract.** We introduce a class of metrics on the tangent bundle of a Riemannian manifold and find the Levi-Civita connections of these metrics. Then by using the Levi-Civita connection, we study the conformal vector fields on the tangent bundle of the Riemannian manifold. Finally, we obtain some relations between the flatness (resp. local symmetry) properties of the tangent bundle and the flatness (resp. local symmetry) on the base manifold.

**1. Introduction.** Tangent bundles of differentiable manifolds are of great importance in many areas of mathematics and physics. In the last decades, a large number of publications have been devoted to the study of their special differential geometric properties [CS1, D, FP, FIP, GTS, Mu1, Mu2, OP].

The geometry of tangent bundles goes back to the fundamental paper [S] of Sasaki published in 1958. He used a given Riemannian metric $g$ on a differentiable manifold $M$ to construct a metric $g^S$ on the tangent bundle $TM$. Today this metric is a standard notion in differential geometry, called the Sasaki metric and defined by

$$g^S = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)\delta y^i \otimes \delta y^j,$$

where $g_{ij}(x)$ are the components of the Riemannian metric $g$. The Sasaki metric has been extensively studied by several authors, including Yano and Davies [YD], Kowalski [Ko], Musso and Tricerri [MT], Aso [A], Cenzer and Salimov [CS2].

For a given Riemannian metric $g$ on a differentiable manifold $M$, there are certain other (pseudo-) Riemannian metrics on $TM$, constructed from $g$. One of them, introduced by Yano and Ishihara [YI], is defined by

$$\widetilde{g} = 2g_{ij}(x)dx^i \otimes \delta y^j + g_{ij}(x)\delta y^i \otimes \delta y^j. \tag{1.1}$$

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Hasegawa and Yamauchi investigated infinitesimal conformal and projective transformations on $(TM, \widetilde{g})$ [HY]. By replacing $g_{ij}(x)$ in $\widetilde{g}$ with the components $h_{ij}(x,y)$ of a generalized Lagrange metric [MA], one gets a class of pseudo-Riemannian metrics

$$G = 2h_{ij}(x)dx^i \otimes \delta y^j + h_{ij}(x)\delta y^i \otimes \delta y^j. \tag{1.2}$$

In particular, $h_{ij}(x,y)$ could be a deformation of $g_{ij}(x)$, a case studied by Anastasiei and Shimada [AS].

In this paper, we consider $G$ when $h_{ij}(x,y)$ is the following special deformation of $g_{ij}(x)$:

$$h_{ij}(x,y) = a(L^2)g_{ij}(x), \tag{1.3}$$

where $L^2 = g_{ij}(x)y^iy^j$, $y_i = g_{ij}(x)y^j$ and $a : \mathrm{Im}(L^2) \subseteq [0,\infty) \to \mathbb{R}^+$ with $a > 0$.

We calculate the Levi-Civita connection of $G$ and we show that the horizontal distribution $HTM$ (resp. vertical distribution $VTM$) is totally geodesic if and only if $(M,g)$ is locally flat ($K = 0$, respectively). We also study the conformal vector fields on $TM$ with respect to $G$. We prove that the complete lift of $X \in \mathcal{X}(M)$ is conformal on $TM$ if and only if $X$ is homothetic. Finally, we find the components of the Riemannian curvature tensor of $G$ and show that if $(TM,G)$ is flat (resp. locally symmetric), then $(M,g)$ is flat (locally symmetric, respectively).

**2. Preliminaries.** Let $(M,g)$ be a real $n$-dimensional Riemannian manifold and $(U,x)$ a local chart on $M$, where the coordinates of the point $p \in U$ are denoted by $x(p)$ or $(x^i)$. Using the coordinates $(x^i)$ on $M$, we have the local field of frames $\{\partial/\partial x^i\}$ on $T_pM$. Let $\nabla$ be a Riemannian connection on $M$ with coefficients $\Gamma_{ij}^k$ where the indices $a,b,c,h,i,j,k,m,\dots$ run over the range $1,\dots,n$. The Riemannian curvature tensor is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad \forall X,Y,Z \in \mathcal{X}(M). \tag{2.1}$$

Locally, we have

$$R_{ijk}{}^m = \partial_i\Gamma_{jk}^m - \partial_j\Gamma_{ik}^m + \Gamma_{ia}^m\Gamma_{jk}^a - \Gamma_{ja}^m\Gamma_{ik}^a,$$

where $\partial_i := \partial/\partial x^i$ and $R(\partial_i,\partial_j)\partial_k := R_{ijk}{}^m\partial_m$. Let $TM$ be the tangent bundle of $M$, and $\pi$ the natural projection from $TM$ to $M$. Consider $\pi_* : TTM \to TM$ and put

$$\ker \pi_*^v = \{z \in TTM \mid \pi_*^v(z) = 0\}, \qquad \forall v \in TM.$$

Then the vertical vector bundle on $M$ is defined by $VTM = \bigcup_{v\in TM} \ker \pi_*^v$. A *non-linear connection* or a *horizontal distribution* on $TM$ is a complementary distribution $HTM$ for $VTM$ on $TTM$. Here, the reason for using the term "non-linear connection" for $HTM$ is that $HTM$ is completely determined by the functions $N_j^i(x,y)$ which are non-linear with respect to $y$, in general. These functions are called *coefficients* of the non-linear connection. It is clear that $HTM$ is a horizontal vector bundle. By definition, we have the decomposition

$$TTM = VTM \oplus HTM. \tag{2.2}$$

Using the induced coordinates $(x^i, y^i)$ on $TM$, where $x^i$ and $y^i$ are called respectively *position* and *direction* of a point on $TM$, we have the local field of frames $\{\partial/\partial x^i, \partial/\partial y^i\}$ on $TTM$. Let $\{dx^i, dy^i\}$ be the dual basis of $\{\partial/\partial x^i, \partial/\partial y^i\}$. It is well known that we can choose a local field of frames $\{\delta/\delta x^i, \partial/\partial y^i\}$ adapted to the above decomposition, namely $\delta/\delta x^i \in \mathcal{X}(HTM)$ and $\partial/\partial y^i \in \mathcal{X}(VTM)$ are sections of the horizontal and vertical subbundles $HTM$ and $VTM$, defined by

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \tag{2.3}$$

where $N_i^j(x,y)$ are functions on $TM$ which have the following transformation rule in local coordinates $(x^i, y^i)$ and $(x^{i'}, y^{i'})$ on $TM$:

$$N_{i'}^{h'} = \frac{\partial x^{h'}}{\partial x^h}\left(\frac{\partial x^i}{\partial x^{i'}} N_i^h + \frac{\partial^2 x^h}{\partial x^{i'} \partial x^{a'}} y^{a'}\right).$$

To see a relation between linear and non-linear connections, let $\Gamma_{ji}^k$ be the coefficients of the Riemannian connection of $(M,g)$. Then it is easy to check that $y^a \Gamma_{ai}^k$ satisfy the above relation and thus can be regarded as coefficients of a non-linear connection on $TM$. Hence we can rewrite (2.3) as follows:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^a \Gamma_{ai}^j \frac{\partial}{\partial y^j}. \tag{2.4}$$

We put $\delta_h = \delta/\delta x^h$ and $\dot{\partial}_h = \partial/\partial y^h$. Then $\{\delta_h, \dot{\partial}_h\}$ is the adapted local field of frames of $TM$. Let $\{dx^h, \delta y^h\}$ be the dual basis of $\{\delta_h, \dot{\partial}_h\}$, where $\delta y^h = dy^h + y^a \Gamma_a{}^h{}_i dx^i$ and the indices $i, j, h, \dots$ run over the range $1, \dots, n$.

Let $\varphi$ be a transformation on $M$. Then $\varphi$ is called a *conformal* transformation on $M$ if it preserves angles. Let $X$ be a vector field on $M$ and $\{\varphi_t\}$ be the local one-parameter group of local transformations on $M$ generated by $X$. Then $X$ is called a *conformal vector field* on $M$ if each $\varphi_t$ is a local conformal transformation of $M$. It is well known that $X$ is a conformal vector field on $M$ if and only if there exists a scalar function $\rho$ on $M$ such that $\pounds_X g = 2\rho g$, where $\pounds_X$ denotes Lie derivation with respect to the vector field $X$. In particular, $X$ is called *homothetic* if $\rho$ is constant, and *Killing* when $\rho$ vanishes.

Let $TM$ be the tangent bundle of $M$, and $\phi$ be a transformation of $TM$. Then $\phi$ is called *fiber-preserving* if it preserves fibres. Let $\widetilde{X}$ be a vector field on $TM$, and consider the local one-parameter group $\{\phi_t\}$ of local transformations of $TM$ generated by $\widetilde{X}$. Then $\widetilde{X}$ is called fiber-preserving if each $\phi_t$ is a local fiber-preserving transformation of $TM$. Let $G$ be a (pseudo-) Riemannian metric of $TM$. A fiber-preserving vector field $\widetilde{X}$ on $TM$ is said to be *conformal* if there exists a scalar function $\widetilde{\rho}$ on $TM$ such that $\pounds_{\widetilde{X}} G = 2\widetilde{\rho} G$. In particular, $\widetilde{X}$ is called *inessential* if $\widetilde{\rho}$ is depends only on $(x^h)$, and *essential* when $\widetilde{\rho}$ depends only on $(y^h)$ [HB].

**3. The Levi-Civita connection of the metric $G$.** In this section, we calculate the Levi-Civita connection with respect to the lift metric $G$ and by using it, we find conditions for the horizontal (or vertical) distribution to be totally geodesic.

Let $(M, g)$ be a Riemannian space and $(TM, \pi, M)$ be its tangent bundle. On a domain $U \subset M$ of a local chart, $g$ has the components $g_{ij}(x)$ $(i, j, \ldots = 1, \ldots, n)$. On $\pi^{-1}(U) \subset TM$, we consider

$$\tau = L^2 = g_{ij}(x) y^i y^j. \tag{3.1}$$

Then $\tau$ is globally defined and differentiable on $TM$. With the above notation, we can rewrite the metric $G$ defined by (1.2) and (1.3) as follows:

$$G = 2a(\tau) g_{ij}(x) dx^i \otimes \delta y^j + a(\tau) g_{ij}(x) \delta y^i \otimes \delta y^j. \tag{3.2}$$

It is easy to check that $(TM, G)$ is a pseudo-Riemannian space, depending only on the metric $g$. Obviously, if $a = 1$, then we have (1.1).

Remark. In [AM], Abbassi and Calvaruso introduced the $g$-natural metric on $TM$ which is characterized by

$$\widetilde{G}_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(\tau) g_x(X, Y) + (\beta_1 + \beta_3)(\tau) g_x(X, u) g_x(Y, u),$$
$$\widetilde{G}_{(x,u)}(X^v, Y^v) = \alpha_1(\tau) g_x(X, Y) + \beta_1(\tau) g_x(X, u) g_x(Y, u),$$
$$\widetilde{G}_{(x,u)}(X^h, Y^v) = \widetilde{G}_{(x,u)}(X^v, Y^h) = \alpha_2(\tau) g_x(X, Y) + \beta_2(\tau) g_x(X, u) g_x(Y, u),$$

where $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, are smooth functions and $X^h$ and $X^v$ are the horizontal lift and the vertical lift of a vector $X \in T_x M$, respectively. In particular:

(i) The Sasaki metric $g^S$ is obtained for $\alpha_1(\tau) = 1$ and $\alpha_2(\tau) = \alpha_3(\tau) = \beta_1(\tau) = \beta_2(\tau) = \beta_3(\tau) = 0$.
(ii) The Cheeger–Gromoll metric $g^{CG}$ (see [CG]) is obtained for $\alpha_2(\tau) = \beta_2(\tau) = 0$, $\alpha_1(\tau) = \beta_1(\tau) = -\beta_3(\tau) = \frac{1}{1+\tau}$ and $\alpha_3(\tau) = \frac{\tau}{1+\tau}$.

Further, if we set $\alpha_1(\tau) = \alpha_2(\tau) = a(\tau)$, $\alpha_3(\tau) = -a(\tau)$ and $\beta_1(\tau) = \beta_2(\tau) = \beta_3(\tau) = 0$ then we obtain the metric $G$ defined by (3.2).

By direct calculation, we obtain the following lemma.

LEMMA 3.1. *Let $(M,g)$ be a Riemannian manifold. Then*

$$[\delta_i, \delta_j] = -y^a R_{ija}{}^k \dot{\partial}_k, \tag{3.3}$$

$$[\dot{\partial}_i, \dot{\partial}_j] = 0, \tag{3.4}$$

$$[\delta_i, \dot{\partial}_j] = \Gamma^k_{ij} \dot{\partial}_k. \tag{3.5}$$

PROPOSITION 3.2. *Let $(M,g)$ be a Riemannian manifold and $TM$ be its tangent bundle equipped with the metric $G$. Then the corresponding Levi-Civita connection $\widetilde{\nabla}$ satisfies the following relations:*

$$\widetilde{\nabla}_{\dot{\partial}_i} \dot{\partial}_j = -K g_{ij} y^k \delta_k + K(\delta^k_j y_i + \delta^k_i y_j)\dot{\partial}_k, \tag{3.6}$$

$$\widetilde{\nabla}_{\delta_i} \delta_j = (\Gamma^k_{ij} - \tfrac{1}{2} y^a R_{aij}{}^k - \tfrac{1}{2} y^a R_{aji}{}^k)\delta_k + y^a R_{aij}{}^k \dot{\partial}_k, \tag{3.7}$$

$$\widetilde{\nabla}_{\delta_i} \dot{\partial}_j = (K\delta^k_i y_j - K g_{ij} y^k - \tfrac{1}{2} y^a R_{aji}{}^k)\delta_k + (\Gamma^k_{ij} + \tfrac{1}{2} y^a R_{aji}{}^k)\dot{\partial}_k, \tag{3.8}$$

$$\widetilde{\nabla}_{\dot{\partial}_i} \delta_j = (K\delta^k_j y_i - K g_{ij} y^k - \tfrac{1}{2} y^a R_{aij}{}^k)\delta_k + \tfrac{1}{2} y^a R_{aij}{}^k \dot{\partial}_k, \tag{3.9}$$

*where $K = a'/a$.*

*Proof.* We only prove (3.6). Since the $g_{ij}$ only depend on $(x^h)$, we obtain

$$\dot{\partial}_k \tau = \dot{\partial}_k (g_{ij}(x) y^i y^j) = g_{ij}(x)\delta^i_k y^j + g_{ij}(x)\delta^j_k y^i = 2y_k. \tag{3.10}$$

If $\nabla$ is the Levi-Civita connection on $(M,g)$, then

$$0 = \nabla_k g_{ij} = \delta_k g_{ij} - g_{ir}\Gamma^r_{kj} - g_{jr}\Gamma^r_{ki}. \tag{3.11}$$

By using (3.1), we obtain

$$\delta_k (g_{ij} y^i y^j) = (\delta_k g_{ij}) y^i y^j + g_{ij}(\delta_k y^i) y^j + g_{ij} y^i (\delta_k y^j). \tag{3.12}$$

But (2.4) gives

$$\delta_k y^i = -y^a \Gamma^r_{ak} \delta^i_r = -y^a \Gamma^i_{ak}. \tag{3.13}$$

By inserting (3.13) in (3.12) and using (3.11), we infer that

$$\delta_k \tau = \delta_k (g_{ij} y^i y^j) = (\nabla_k g_{ij}) y^i y^j = 0. \tag{3.14}$$

Now let

$$\widetilde{\nabla}_{\dot{\partial}_i} \dot{\partial}_j = A^k_{ij} \delta_k + B^k_{ij} \dot{\partial}_k. \tag{3.15}$$

Writing the Koszul formula, we have

$$\begin{aligned} 2G(\widetilde{\nabla}_{\dot{\partial}_i} \dot{\partial}_j, \delta_k) &= \dot{\partial}_i G(\dot{\partial}_j, \delta_k) + \dot{\partial}_j G(\delta_k, \dot{\partial}_i) - \delta_k G(\dot{\partial}_i, \dot{\partial}_j) \\ &\quad + G([\dot{\partial}_i, \dot{\partial}_j], \delta_k) - (G[\dot{\partial}_j, \delta_k], \dot{\partial}_i) + G([\delta_k, \dot{\partial}_i], \dot{\partial}_j). \end{aligned}$$

Combining (3.4), (3.5) and (3.15) with the above equation and using (3.10), (3.11) and (3.14), we obtain

$$a g_{rk} B^r_{ij} = a'(y_i g_{jk} + y_j g_{ik}).$$

Contracting the above equation with $g^{kh}$ implies that

$$B^h_{ij} = K(y_i \delta^h_j + y_j \delta^h_i). \tag{3.16}$$

In the same way we obtain

$$A^h_{ij} = -K g_{ij} y^h. \tag{3.17}$$

From (3.16), (3.17) and (3.15), we deduce (3.6). ■

We recall that the vertical distribution $VTM$ is *totally geodesic* in $TTM$ if $\mathcal{H}\widetilde{\nabla}_{\dot{\partial}_i}\dot{\partial}_j = 0$, where $\mathcal{H}$ denotes the horizontal projection. Similarly, if we denote the vertical projection by $\mathcal{V}$, then we say that the horizontal distribution $HTM$ is totally geodesic in $TTM$ if $\mathcal{V}\widetilde{\nabla}_{\delta_i}\delta_j = 0$. In the following proposition, we find necessary and sufficient conditions for the vertical or horizontal distributions in $TTM$ to be totally geodesic.

Proposition 3.3. *Let $(M, g)$ be a Riemannian manifold and $TM$ be its tangent bundle with the metric $G$. Then*

1. *The vertical distribution $VTM$ is totally geodesic in $TTM$ if and only if $K = 0$.*
2. *The horizontal distribution $HTM$ is totally geodesic in $TTM$ if and only if $(M, g)$ is a locally flat manifold.*

*Proof.* By using (3.6) we infer that $\mathcal{H}\widetilde{\nabla}_{\dot{\partial}_i}\dot{\partial}_j = -K g_{ij} y^k \delta_k$. Hence $VTM$ is a totally geodesic distribution in $TTM$ if and only if $K = 0$. According to (3.7) we also have

$$\mathcal{V}\widetilde{\nabla}_{\delta_i}\delta_j = y^a R_{aij}{}^k \dot{\partial}_k. \tag{3.18}$$

If $(M, g)$ is a locally flat manifold, then $R_{aij}{}^k = 0$. Hence from (3.18), we deduce $\mathcal{V}\widetilde{\nabla}_{\delta_i}\delta_j = 0$, i.e., $HTM$ is a totally geodesic distribution in $TTM$. Conversely, if $\mathcal{V}\widetilde{\nabla}_{\delta_i}\delta_j = 0$ then by (3.18) we deduce

$$y^a R_{aij}{}^k = 0. \tag{3.19}$$

Applying $\dot{\partial}_h$ to this equation, we obtain $R_{hij}{}^k = 0$. Hence $(M, g)$ is locally flat. ■

**4. Conformal vector fields on $TM$.** In this section, we study the conformal vector fields on $TM$ with respect to the metric $G$. Let us first recall the following fact.

Lemma 4.1 ([PH]). *Let $\widetilde{X} = \widetilde{X}^i \delta_i + \dot{\widetilde{X}}^i \dot{\partial}_i$ be a vector field on $TM$. Then $\widetilde{X}$ is fiber-preserving vector field on $TM$ if and only if the $\widetilde{X}^i$ are functions on $M$.*

In the fiber-preserving case, we denote $\widetilde{X}^i$ by $X^i$. Therefore, every fiber-preserving vector field $\widetilde{X}$ on $TM$ induces a vector field $X = X^i \partial_i$ on $M$.

Definition 4.2 ([YI]). Let $X$ be a vector field on $M$ with components $X^i$. The following vector fields on $TM$ are called respectively *complete,* *horizontal* and *vertical* lifts of $X$:

$$X^C = X^i\delta_i + y^j(\nabla_j X^i)\dot{\partial}_i, \tag{4.1}$$

$$X^H = X^i\delta_i, \tag{4.2}$$

$$X^V = X^i\dot{\partial}_i. \tag{4.3}$$

Recall that the Lie derivative of $G$ with respect to $\widetilde{X}$ is given by

$$(\pounds_{\widetilde{X}}G)(\widetilde{Y},\widetilde{Z}) = G(\widetilde{\nabla}_{\widetilde{Y}}\widetilde{X},\widetilde{Z}) + G(\widetilde{Y},\widetilde{\nabla}_{\widetilde{Z}}\widetilde{X}). \tag{4.4}$$

By using (3.6) and (3.9), we obtain

$$\begin{aligned} G(\widetilde{\nabla}_{v\widetilde{Y}}\widetilde{X}, v\widetilde{Z}) &= \dot{\widetilde{Y}}^i\dot{\widetilde{Z}}^j\{G((\partial_i\widetilde{X}^k)\delta_k,\dot{\partial}_j) + \widetilde{X}^kG(\widetilde{\nabla}_{\dot{\partial}_i}\delta_k,\dot{\partial}_j) \\ &\quad + G((\dot{\partial}_i\dot{\widetilde{X}}^k)\dot{\partial}_k,\dot{\partial}_j) + G(\dot{\widetilde{X}}^k\widetilde{\nabla}_{\dot{\partial}_i}\dot{\partial}_k,\dot{\partial}_j)\} \\ &= a(\tau^2)\dot{\widetilde{Y}}^i\dot{\widetilde{Z}}^j\{\dot{\partial}_i(\widetilde{X}^s)g_{sj} + K\widetilde{X}^k(g_{kj}y_i - g_{ik}y_j) \\ &\quad + \dot{\partial}_i(\dot{\widetilde{X}}^s)g_{js} + K\dot{\widetilde{X}}^k(y_ig_{kj} - y_jg_{ik} + y_kg_{ij})\}, \end{aligned} \tag{4.5}$$

where $v\widetilde{Y} = \dot{\widetilde{Y}}^i\dot{\partial}_i$. Similarly,

$$\begin{aligned} G(v\widetilde{Y}, \widetilde{\nabla}_{v\widetilde{Z}}\widetilde{X}) &= a(\tau^2)\dot{\widetilde{Y}}^i\dot{\widetilde{Z}}^j\{\dot{\partial}_j(\widetilde{X}^s)g_{si} + K\widetilde{X}^k(g_{ki}y_j - g_{jk}y_i) \\ &\quad + \dot{\partial}_j(\dot{\widetilde{X}}^s)g_{is} + K\dot{\widetilde{X}}^k(y_jg_{ki} - y_ig_{jk} + y_kg_{ij})\}. \end{aligned} \tag{4.6}$$

From (4.4)–(4.6) we deduce

$$\begin{aligned} (\pounds_{\widetilde{X}}G)(v\widetilde{Y}, v\widetilde{Z}) &= a(\tau^2)\dot{\widetilde{Y}}^i\dot{\widetilde{Z}}^j\{\dot{\partial}_i(\widetilde{X}^k)g_{kj} + \dot{\partial}_j(\widetilde{X}^k)g_{ik} + \dot{\partial}_i(\dot{\widetilde{X}}^k)g_{kj} \\ &\quad + \dot{\partial}_j(\dot{\widetilde{X}}^k)g_{ki} + 2K\dot{\widetilde{X}}^ky_kg_{ij}\}. \end{aligned} \tag{4.7}$$

Similarly, we can prove that

$$\begin{aligned} (\pounds_{\widetilde{X}}G)(h\widetilde{Y}, v\widetilde{Z}) &= a(\tau^2)\widetilde{Y}^i\dot{\widetilde{Z}}^j\{g_{kj}\nabla_i\widetilde{X}^k + \widetilde{X}^ky^aR_{jaik} + g_{kj}\nabla_i\dot{\widetilde{X}}^k \\ &\quad + \dot{\partial}_j(\dot{\widetilde{X}}^k)g_{ki} + 2K\dot{\widetilde{X}}^ky_kg_{ij}\}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} (\pounds_{\widetilde{X}}G)(h\widetilde{Y}, h\widetilde{Z}) &= a(\tau^2)\widetilde{Y}^i\widetilde{Z}^j\{\dot{\widetilde{X}}^ky^aR_{aikj} + \dot{\widetilde{X}}^ky^aR_{ajki} + g_{kj}\nabla_i\dot{\widetilde{X}}^k \\ &\quad + g_{ki}\nabla_j\dot{\widetilde{X}}^k\}. \end{aligned} \tag{4.9}$$

By using (4.7)–(4.9) we obtain:

Proposition 4.3. *Let $(M,g)$ be a Riemannian manifold and $G$ be the pseudo-Riemannian metric on $TM$ defined by (3.2). Then $\widetilde{X} = \widetilde{X}^i\delta_i + \dot{\widetilde{X}}^i\dot{\partial}_i$ is a conformal vector field on $TM$ with respect to $G$ if and only if the fol-*

*lowing relations hold:*

$$\dot\partial_i(\widetilde{X}^k)g_{kj}+\dot\partial_j(\widetilde{X}^k)g_{ik}+\dot\partial_i(\dot{\widetilde{X}}^k)g_{kj}+\dot\partial_j(\dot{\widetilde{X}}^k)g_{ki}+2K\dot{\widetilde{X}}^k y_k g_{ij}=2\widetilde{\rho}g_{ij}, \tag{4.10}$$

$$g_{kj}\nabla_i\widetilde{X}^k+\widetilde{X}^k y^r R_{jrik}+g_{kj}\nabla_i\dot{\widetilde{X}}^k+\dot\partial_j(\dot{\widetilde{X}}^k)g_{ki}+2K\dot{\widetilde{X}}^k y_k g_{ij}=2\widetilde{\rho}g_{ij}, \tag{4.11}$$

$$\widetilde{X}^k y^r R_{rikj}+\widetilde{X}^k y^r R_{rjki}+g_{kj}\nabla_i\dot{\widetilde{X}}^k+g_{ki}\nabla_j\dot{\widetilde{X}}^k=0, \tag{4.12}$$

*where $\widetilde{\rho}$ is a function on $TM$.*

Now, let $\widetilde{X}=X^i\delta_i+\dot{\widetilde{X}}^i\dot\partial_i$ be a fiber-preserving conformal vector field on $TM$. Then by interchanging $i$ and $j$ in (4.11) and adding the new relation to (4.11), we infer that

$$\begin{aligned} g_{kj}\nabla_i X^k+g_{ki}\nabla_j X^k+X^k y^r R_{jrik}+X^k y^r R_{irjk}+g_{kj}\nabla_i\dot{\widetilde{X}}^k\\ +g_{ki}\nabla_j\dot{\widetilde{X}}^k+\dot\partial_j(\dot{\widetilde{X}}^k)g_{ki}+\dot\partial_i(\dot{\widetilde{X}}^k)g_{kj}+4K\dot{\widetilde{X}}^k y_k g_{ij}=4\widetilde{\rho}g_{ij}. \end{aligned} \tag{4.13}$$

Since $\dot\partial_i(X^k)=0$, by using (4.10) and (4.12) in (4.13) we derive

$$g_{kj}\nabla_i X^k+g_{ki}\nabla_j X^k+2K\dot{\widetilde{X}}^k y_k g_{ij}=2\widetilde{\rho}g_{ij}. \tag{4.14}$$

If we suppose $\widetilde{\varphi}=\widetilde{\rho}-K\dot{\widetilde{X}}^k y_k$, then (4.14) gives

$$\nabla_i X_j+\nabla_j X_i=2\widetilde{\varphi}g_{ij}, \tag{4.15}$$

where $X_j=g_{kj}X^k$. It is easy to see that the vector field $X=X^i\partial_i$ on $M$ satisfies

$$\pounds_X g_{ij}=\nabla_i X_j+\nabla_j X_i. \tag{4.16}$$

The relations (4.15) and (4.16) imply

$$\pounds_X g_{ij}=2\widetilde{\varphi}g_{ij}. \tag{4.17}$$

This shows that the function $\widetilde{\varphi}$ on $TM$ depends only on the variables $(x^h)$ in the induced coordinates $(x^h, y^h)$. Thus we can regard $\widetilde{\varphi}$ as a function on $M$. In this case, we write $\varphi$ instead of $\widetilde{\varphi}$. Therefore, we have the following result.

PROPOSITION 4.4. *Let $(M,g)$ be a Riemannian manifold and $G$ be the pseudo-Riemannian metric on $TM$ defined by (3.2). Then if $\widetilde{X}=X^i\delta_i+\dot{\widetilde{X}}^i\dot\partial_i$ is a fiber-preserving conformal vector field on $TM$ with respect to $G$ then $X=X^i\partial_i$ is a conformal vector field on $M$.*

LEMMA 4.5. *The vertical components $\dot{\widetilde{X}}^k$ of $X$ can be written in the form*

$$\dot{\widetilde{X}}^k=y^r A_r^k+B^k, \tag{4.18}$$

*where $A_r^k$ and $B^k$ are the components of certain tensor fields $A$ and $B$ on $M$, respectively.*

*Proof.* From (4.10) we obtain

$$\dot{\partial}_i(\dot{\tilde{X}}^k)g_{kj} + \dot{\partial}_j(\dot{\tilde{X}}^k)g_{ki} = 2\varphi g_{ij}. \tag{4.19}$$

Applying $\dot{\partial}_r$ to the above equality, we obtain

$$g_{kj}\dot{\partial}_r\dot{\partial}_i(\dot{\tilde{X}}^k) + g_{ki}\dot{\partial}_r\dot{\partial}_j(\dot{\tilde{X}}^k) = 0. \tag{4.20}$$

Hence

$$\begin{aligned} g_{ki}\dot{\partial}_r\dot{\partial}_j(\dot{\tilde{X}}^k) &= -g_{kj}\dot{\partial}_r\dot{\partial}_i(\dot{\tilde{X}}^k) = -\dot{\partial}_i(g_{kj}\dot{\partial}_r(\dot{\tilde{X}}^k)) \\ &= -\dot{\partial}_i(-g_{kr}\dot{\partial}_j(\dot{\tilde{X}}^k) + 2\varphi g_{jr}) = g_{kr}\dot{\partial}_i\dot{\partial}_j(\dot{\tilde{X}}^k) \\ &= \dot{\partial}_j(g_{kr}\dot{\partial}_i(\dot{\tilde{X}}^k)) = \dot{\partial}_j(-g_{ki}\dot{\partial}_r(\dot{\tilde{X}}^k) + 2\varphi g_{ri}) \\ &= -g_{ki}\dot{\partial}_j\dot{\partial}_r(\dot{\tilde{X}}^k) = -g_{ki}\dot{\partial}_r\dot{\partial}_j(\dot{\tilde{X}}^k), \end{aligned} \tag{4.21}$$

which implies that

$$g_{ki}\dot{\partial}_r\dot{\partial}_j(\dot{\tilde{X}}^k) = 0.$$

This shows that $\dot{\partial}_j(\dot{\tilde{X}}^k)$ depends only on the variables $(x^k)$ and consequently $\dot{\tilde{X}}^k$ can be written as (4.18). ∎

LEMMA 4.6. *The components $A_r^k$ and $B^k$ of the tensor fields $A$ and $B$ satisfy*

$$A_{ij} + A_{ji} = 2\varphi g_{ij}, \tag{4.22}$$

$$X^k R_{hikj} + \nabla_j A_{hi} = 0, \tag{4.23}$$

$$\nabla_j B_i - \nabla_i X_j + A_{ij} = 0, \tag{4.24}$$

*where $X_j = g_{jk}X^k$, $B_i = g_{ik}B^k$ and $A_{ij} = g_{jk}A_i^k$.*

*Proof.* Inserting (4.18) in (4.19) we get

$$\dot{\partial}_i(y^r A_r^k + B^k)g_{kj} + \dot{\partial}_j(y^r A_r^k + B^k)g_{ki} = 2\varphi g_{ij}.$$

Since $A_r^k$ and $B^k$ are functions of $(x^k)$, the above implies

$$A_i^k g_{kj} + A_j^k g_{ki} = 2\varphi g_{ij}.$$

This yields (4.22). From (4.11), we have

$$g_{kj}\nabla_i X^k + X^k y^r R_{jrik} + g_{kj}\nabla_i\dot{\tilde{X}}^k + \dot{\partial}_j(\dot{\tilde{X}}^k)g_{ki} = 2\varphi g_{ij},$$

Applying (4.18) in the above equation yields

$$\nabla_i X_j + X^k y^r R_{jrik} + y^r\nabla_i A_{rj} + \nabla_i B_j + A_{ji} = 2\varphi g_{ij}. \tag{4.25}$$

Inserting (4.18) in (4.12) we also obtain

$$X^k y^r R_{rikj} + X^k y^r R_{rjki} + y^r\nabla_i A_{rj} + \nabla_i B_j + y^r\nabla_j A_{ri} + \nabla_j B_i = 0. \tag{4.26}$$

From (4.25) and (4.26), we get

$$X^k y^r R_{rikj} + y^r\nabla_j A_{ri} + \nabla_j B_i + 2\varphi g_{ij} - \nabla_i X_j - A_{ji} = 0.$$

Using (4.22) in the above equation implies that

$$X^k y^r R_{rikj} + y^r \nabla_j A_{ri} + \nabla_j B_i - \nabla_i X_j + A_{ij} = 0. \tag{4.27}$$

Taking the derivatives of both sides with respect to $\dot{\partial}_h$ gives (4.23). By applying (4.23) in (4.27) we deduce (4.24). ■

By interchanging $i$ and $j$ in (4.24) and adding the new equation to (4.24) we get

$$\nabla_i B_j + \nabla_j B_i - \nabla_i X_j - \nabla_j X_i + A_{ij} + A_{ji} = 0.$$

The relations (4.15) and (4.22) now imply that

$$\pounds_B g_{ij} = \nabla_i B_j + \nabla_j B_i = 0.$$

Hence, we have the following result.

Lemma 4.7. *The vector field $B = B^i \partial_i$ is a Killing vector field on $M$.*

By using (4.22) we obtain

$$\nabla_h A_{ij} + \nabla_h A_{ji} = 2g_{ij} \nabla_h \varphi.$$

The above relation and (4.23) give

$$X^k R_{ijkh} + X^k R_{jikh} = 2g_{ij} \nabla_h \varphi.$$

Since $R_{ijkh} = -R_{jikh}$, we infer that $\nabla_h \varphi = 0$. Consequently, $\partial_h \varphi = 0$, i.e., $\varphi$ is constant on $M$. Therefore we have the following result.

Lemma 4.8. *The vector field $X = X^i \partial_i$ is a homothetic vector field on $M$.*

From Proposition 4.4 and Lemmas 4.7 and 4.8, we have the following.

Theorem 4.9. *Let $(M, g)$ be a Riemannian manifold and $G$ be the pseudo-Riemannian metric on $TM$ defined by (3.2). Then every fiber-preserving conformal vector field on $TM$ with respect to $G$ induces a homothetic vector field and a Killing vector field on $M$.*

Now, let $X = X^i \partial_i$ be a homothetic vector field on $M$ with respect to the constant function $\varphi$. Then

$$\pounds_X g_{ij} = \nabla_i X_j + \nabla_j X_i = 2\varphi g_{ij}. \tag{4.28}$$

If $\pounds_X \Gamma^h_{ij}$ are the components of the tensor field $\pounds_X \nabla$, then ([F], [K])

$$\pounds_X \Gamma^h_{ij} = \nabla_i \nabla_j X^h + R_{rij}{}^h X^r = 0,$$

or

$$\nabla_i \nabla_j X_k + R_{rijk} X^r = 0. \tag{4.29}$$

We define the vector field $\widetilde{X}$ on $TM$ by

$$\widetilde{X} = X^i \delta_i + y^r (\nabla_r X^i) \dot{\partial}_i. \tag{4.30}$$

By using (4.9), (4.29) and (4.30) we obtain

(4.31) $$(\pounds_{\widetilde{X}}G)(\delta_i,\delta_j) = ay^r(X^kR_{rikj}+X^kR_{rjki}+\nabla_i\nabla_rX_j+\nabla_j\nabla_rX_i) = 0.$$

The relations (4.8), (4.28), (4.29) and (4.30) give

$$(\pounds_{\widetilde{X}}G)(\delta_i,\dot{\partial}_j) = 2a(\varphi + Ky^r(\nabla_rX^k)y_k)g_{ij}.$$

Hence,

(4.32) $$(\pounds_{\widetilde{X}}G)(\delta_i,\dot{\partial}_j) = 2a\widetilde{\rho}g_{ij},$$

where $\widetilde{\rho} = \varphi + Ky^r(\nabla_rX^k)y_k$. Using (4.7), (4.28) and (4.30) we also obtain

(4.33) $$(\pounds_{\widetilde{X}}G)(\dot{\partial}_i,\dot{\partial}_j) = 2a(\varphi + Ky^r(\nabla_rX^k)y_k)g_{ij} = 2\widetilde{\rho}g_{ij}.$$

By using (4.31), (4.32) and (4.33), we get $\pounds_{\widetilde{X}}G = 2\widetilde{\rho}G$. This yields the following theorem.

THEOREM 4.10. *Let $M$ be an $n$-dimensional Riemannian manifold, and $TM$ be its tangent bundle with the metric $G$. Then every infinitesimal homothetic vector field $X$ on $M$ induces an infinitesimal fiber-preserving conformal vector field on $TM$.*

If $X$ is a vector field on $M$, then its *complete lift* defined by

$$X^C = X^i\delta_i + y^r(\nabla_rX^i)\dot{\partial}_i$$

is a fiber-preserving vector field on $TM$. Obviously, the vector field $\widetilde{X}$ defined by (4.30) is the complete lift of $X = X^i\partial_i$. Therefore from Theorems 4.9 and 4.10, we deduce the following result.

THEOREM 4.11. *Let $(M,g)$ be a Riemannian manifold, $X$ a vector field on $M$, and $X^C$ the complete lift of $X$ to $TM$. If we endow $TM$ with the metric $G$, then $X^C$ is a conformal vector field on $TM$ if and only if $X$ is homothetic on $M$.*

PROPOSITION 4.12. *Let $M$ be an $n$-dimensional Riemannian manifold, and $TM$ be its tangent bundle with the metric $G$. Then every infinitesimal horizontal inessential conformal vector field on $TM$ induces an infinitesimal conformal vector field on $M$.*

*Proof.* Let $\widetilde{X} = \widetilde{X}^i\delta_i$ be a horizontal inessential conformal vector field on $TM$. Then there exists a function $\rho(x)$ on $M$ such that $\pounds_{\widetilde{X}}G = 2\rho G$. By using (4.10)–(4.12), we obtain

(4.34) $$\dot{\partial}_i(\widetilde{X}^k)g_{kj} + \dot{\partial}_j(\widetilde{X}^k)g_{ki} = 2\rho g_{ij},$$

(4.35) $$\nabla_i\widetilde{X}_j + \widetilde{X}^ky^aR_{jaik} = 2\rho g_{ij},$$

(4.36) $$\widetilde{X}^ky^aR_{aikj} + \widetilde{X}^ky^aR_{ajki} = 0.$$

Differentiating (4.34) with respect to $\dot{\partial}_h$ and using (4.21), we get

(4.37) $$\dot{\partial}_h\dot{\partial}_i(\widetilde{X}^k) = 0,$$

or

(4.38) $$\widetilde{X}^k = y^rA^k_r + B^k.$$

By interchanging $i$ and $j$ in (4.35), and adding the new relation to (4.35) and using (4.36) we deduce

$$\nabla_i \widetilde{X}_j + \nabla_j \widetilde{X}_i = 4\rho g_{ij}, \tag{4.39}$$

where $\widetilde{X}_j = g_{jk}\widetilde{X}^k$. Putting (4.38) into (4.39) implies that

$$y^r \nabla_i A_{rj} + y^r \nabla_j A_{ri} + \nabla_i B_j + \nabla_j B_i = 4\rho g_{ij}. \tag{4.40}$$

Since the first two terms depend only on $y$, we have

$$\nabla_i A_{hj} + \nabla_j A_{hi} = 0. \tag{4.41}$$

From (4.40) and (4.41), we obtain

$$\nabla_i B_j + \nabla_j B_i = 4\rho g_{ij}. \tag{4.42}$$

Since $\rho$ is a function on $M$, $B = B^i\partial_i$ is a conformal vector field on $M$. ∎

PROPOSITION 4.13. *Let $M$ be an $n$-dimensional Riemannian manifold, and $TM$ be its tangent bundle with the metric $G$. Then every horizontal lift conformal vector field on $TM$ is a Killing vector field on $M$ and it induces a Killing vector field on $M$.*

*Proof.* Let $X^H = X^i\delta_i$ be the horizontal lift vector field of $X = X^i\partial_i$. If $X^H$ is a conformal vector field on $TM$, then by using (4.10) we infer that $\widetilde{\rho} = 0$. Hence $X^H$ is Killing on $TM$. Using (4.39) we also deduce that $X = X^i\partial_i$ is a Killing vector field on $M$. ∎

PROPOSITION 4.14. *Let $M$ be an $n$-dimensional Riemannian manifold, and $TM$ be its tangent bundle with the metric $G$. Then every vertical lift conformal vector field on $TM$ induces a Killing vector field on $M$.*

*Proof.* Let $X^V = X^i\dot{\partial}_i$ be the vertical lift of $X = X^i\partial_i$. If $X^V$ is a conformal vector field on $TM$, then by using (4.12) we get $\nabla_i X_j + \nabla_j X_i = 0$. Hence $X = X^i\partial_i$ is a Killing vector field on $M$. ∎

## 5. Riemannian curvature tensor

LEMMA 5.1. *Let $(M, g)$ be a Riemannian manifold. Then the coefficients of the Riemannian curvature tensor with respect to the metric $G$ are*

$$\begin{aligned}\widetilde{R}(\dot{\partial}_i, \dot{\partial}_j)\dot{\partial}_k = &\Big[(2K' - K^2)(g_{ik}y_j - g_{jk}y_i)y^s + K(g_{ik}\delta_j^s - g_{jk}\delta_i^s) \\ &+ \frac{K}{2}y^l y^r g_{jk}R_{lir}{}^s - \frac{K}{2}y^l y^r g_{ik}R_{ljr}{}^s\Big]\delta_s \\ &+ \Big[(2K' - K^2)(y_i\delta_j^s - y_j\delta_i^s)y_k - \frac{K}{2}y^l y^r g_{jk}R_{lir}{}^s \\ &+ K(g_{ik}\delta_j^s - g_{jk}\delta_i^s) + \frac{K}{2}y^l y^r g_{ik}R_{ljr}{}^s\Big]\dot{\partial}_s,\end{aligned} \tag{5.1}$$

(5.2)

$$\begin{aligned}\widetilde{R}(\dot{\partial}_i,\dot{\partial}_j)\delta_k = \Big[&(2K'-K^2)(g_{ik}y_j-g_{jk}y_i)y^s+K(g_{ik}\delta_j^s-g_{jk}\delta_i^s)-R_{ijk}{}^s\\&+\frac{K}{2}y^hy^rg_{jk}R_{hir}{}^s-\frac{K}{2}y^hy^rg_{ik}R_{hjr}{}^s+\frac{1}{4}y^ly^hR_{ljk}{}^rR_{hir}{}^s\\&-\frac{1}{4}y^ly^hR_{lik}{}^rR_{hjr}{}^s\Big]\delta_s\\&+\Big[-\frac{1}{4}y^ly^hR_{ljk}{}^rR_{hir}{}^s+\frac{K}{2}y^hy^rg_{ik}R_{hjr}{}^s\\&+R_{ijk}{}^s+\frac{1}{4}y^ly^hR_{lik}{}^rR_{hjr}{}^s+\frac{K}{2}y^ly_rR_{ljk}{}^r\delta_i^s\\&-\frac{K}{2}y^ly_rR_{lik}{}^r\delta_j^s-\frac{K}{2}y^hy^rg_{jk}R_{hir}{}^s\Big]\dot{\partial}_s,\end{aligned}$$

(5.3)

$$\begin{aligned}\widetilde{R}(\delta_i,\dot{\partial}_j)\dot{\partial}_k = \Big[&\frac{K}{2}y^ly^rg_{jk}R_{lir}{}^s+\frac{K}{2}y^ly^rg_{jk}R_{lri}{}^s+2K'(g_{ik}y^s-y_k\delta_i^s)y_j\\&+K(g_{ik}\delta_j^s-g_{jk}\delta_i^s)+\frac{1}{2}R_{jki}{}^s-K^2g_{ik}y_jy^s-\frac{K}{2}y^hy^rg_{ik}R_{hjr}{}^s\\&-\frac{1}{4}y^ly^hR_{lki}{}^rR_{hjr}{}^s+K^2y_ky_j\delta_i^s\Big]\delta_s\\&+\Big[-Ky^ly^rg_{jk}R_{lir}{}^s-\frac{1}{2}R_{jki}{}^s\\&+\frac{K}{2}y^hy^rg_{ik}R_{hjr}{}^s+\frac{1}{4}y^ly^hR_{lki}{}^rR_{hjr}{}^s-\frac{K}{2}y^ly_rR_{lki}{}^r\delta_j^s\Big]\dot{\partial}_s,\end{aligned}$$

(5.4)

$$\begin{aligned}\widetilde{R}(\delta_i,\dot{\partial}_j)\delta_k = \Big[&-\frac{1}{2}y^l\nabla_iR_{ljk}{}^s+\frac{K}{2}y^hy^rg_{jk}R_{hir}{}^s+\frac{1}{4}y^ly^hR_{ljk}{}^rR_{hir}{}^s\\&+\frac{K}{2}y^hy^rg_{jk}R_{hri}{}^s+\frac{K}{2}y^ly_rR_{ljk}{}^r\delta_i^s+\frac{1}{2}R_{jik}{}^s+\frac{1}{2}R_{jki}{}^s\\&-\frac{1}{4}y^ly^hR_{lik}{}^rR_{hjr}{}^s-\frac{1}{4}y^ly^hR_{lki}{}^rR_{hjr}{}^s\Big]\delta_s\\&+\Big[\frac{1}{2}y^l\nabla_iR_{ljk}{}^s-Ky^hy^rR_{hir}{}^sg_{jk}-\frac{1}{2}y^ly^hR_{ljk}{}^rR_{hir}{}^s\\&+\frac{1}{4}y^ly^hR_{ljk}{}^rR_{hri}{}^s-R_{jik}{}^s+\frac{1}{4}y^ly^hR_{lik}{}^rR_{hjr}{}^s\\&+\frac{1}{4}y^ly^hR_{lki}{}^rR_{hjr}{}^s-y^ly_rR_{lik}{}^r\delta_j^s\Big]\dot{\partial}_s,\end{aligned}$$

(5.5)

$$\begin{aligned}\widetilde{R}(\delta_i,\delta_j)\dot{\partial}_k = &\Big[\frac{1}{2}y^l(\nabla_j R_{lki}{}^s - \nabla_i R_{lkj}{}^s) + \frac{K}{2}y^h y^r g_{jk} R_{hir}{}^s - \frac{K}{2}y^h y^r g_{ik} R_{hjr}{}^s \\ &+ \frac{1}{4}y^l y^h R_{lkj}{}^r R_{hir}{}^s - \frac{1}{4}y^l y^h R_{lki}{}^r R_{hjr}{}^s + \frac{K}{2}y^h y^r g_{jk} R_{hri}{}^s \\ &- \frac{K}{2}y^h y^r g_{ik} R_{hrj}{}^s + \frac{K}{2}y^l y_r R_{lkj}{}^r \delta_i^s - \frac{K}{2}y^l y_r R_{lki}{}^r \delta_j^s\Big]\delta_s \\ &+ \Big[\frac{1}{2}y^l(\nabla_i R_{lkj}{}^s - \nabla_j R_{lki}{}^s) - K y^h y^r g_{jk} R_{hir}{}^s \\ &+ K y^h y^r g_{ik} R_{hjr}{}^s \\ &- \frac{1}{2}y^l y^h R_{lkj}{}^r R_{hir}{}^s + \frac{1}{2}y^l y^h R_{lki}{}^r R_{hjr}{}^s + \frac{1}{4}y^l y^h R_{lkj}{}^r R_{hri}{}^s \\ &- \frac{1}{4}y^l y^h R_{lki}{}^r R_{hrj}{}^s + R_{ijk}{}^s + K y^l y_r R_{ijl}{}^r \delta_k^s\Big]\dot{\partial}_s,\end{aligned}$$

(5.6)

$$\begin{aligned}\widetilde{R}(\delta_i,\delta_j)\delta_k = &\Big[R_{ijk}{}^s + \frac{1}{2}y^l(\nabla_j R_{lik}{}^s - \nabla_i R_{ljk}{}^s) + \frac{1}{2}y^l(\nabla_j R_{lki}{}^s - \nabla_i R_{lkj}{}^s) \\ &+ \frac{1}{4}y^l y^h R_{ljk}{}^r R_{hir}{}^s - \frac{1}{4}y^l y^h R_{lik}{}^r R_{hjr}{}^s + \frac{1}{4}y^l y^h R_{lkj}{}^r R_{hir}{}^s \\ &- \frac{1}{4}y^l y^h R_{lki}{}^r R_{hjr}{}^s - \frac{1}{4}y^l y^h R_{kjl}{}^r R_{hri}{}^s + \frac{1}{2}y^l y^h R_{ijl}{}^r R_{hrk}{}^s \\ &+ K y^l y_r (R_{ljk}{}^r \delta_i^s - R_{lik}{}^r \delta_j^s) \\ &- K y^l y_r R_{ijl}{}^r \delta_k^s + 2K y^l y^s R_{ijlk}\Big]\delta_s \\ &+ \Big[y^l(\nabla_i R_{ljk}{}^s - \nabla_j R_{lik}{}^s) - \frac{1}{2}y^l y^h R_{ljk}{}^r R_{hir}{}^s \\ &- \frac{1}{2}y^l y^h R_{lkj}{}^r R_{hir}{}^s + \frac{1}{2}y^l y^h R_{lki}{}^r R_{hjr}{}^s + \frac{1}{2}y^l y^h R_{ljk}{}^r R_{hri}{}^s \\ &- \frac{1}{2}y^l y^h R_{lik}{}^r R_{hrj}{}^s - \frac{1}{2}y^l y^h R_{ijl}{}^r R_{hrk}{}^s\Big]\dot{\partial}_s.\end{aligned}$$

*Proof.* We only prove (5.1). Since $[\dot{\partial}_i, \dot{\partial}_j] = 0$, by using (2.1) we have

$$\widetilde{R}(\dot{\partial}_i,\dot{\partial}_j)\dot{\partial}_k = \widetilde{\nabla}_{\dot{\partial}_i}\widetilde{\nabla}_{\dot{\partial}_j}\dot{\partial}_k - \widetilde{\nabla}_{\dot{\partial}_j}\widetilde{\nabla}_{\dot{\partial}_i}\dot{\partial}_k. \tag{5.7}$$

From (3.6) and (3.9), we obtain

$$\begin{aligned}\widetilde{\nabla}_{\dot{\partial}_i}\widetilde{\nabla}_{\dot{\partial}_j}\dot{\partial}_k &= \widetilde{\nabla}_{\dot{\partial}_i}(-K y^r g_{jk}\delta_r + K(y_j\delta_k^r + y_k\delta_j^r)\dot{\partial}_r) \\ &= \dot{\partial}_i(-K y^r g_{jk})\delta_r + \dot{\partial}_i(K(y_j\delta_k^r + y_k\delta_j^r))\dot{\partial}_r \\ &\quad - K y^r g_{jk}\widetilde{\nabla}_{\dot{\partial}_i}\delta_r + K(y_j\delta_k^r + y_k\delta_j^r)\widetilde{\nabla}_{\dot{\partial}_i}\dot{\partial}_r\end{aligned} \tag{5.8}$$

$$= \Big[-2K' g_{jk} y_i y^s - K g_{jk}\delta_i^s + \frac{K}{2} y^l y^r g_{jk} R_{lir}{}^s - K^2 g_{ik} y_j y^s - K^2 g_{ij} y_k y^s\Big]\delta_s + \Big[K^2 y_i y_j \delta_k^s + K^2 y_i y_k \delta_j^s + 2K' y_i (y_j \delta_k^s + y_k \delta_j^s) + K(g_{ij}\delta_k^s + g_{ik}\delta_j^s) - \frac{K}{2} y^l y^r g_{jk} R_{lir}{}^s + 2K^2 y_k y_j \delta_i^s\Big]\dot{\partial}_s.$$

Interchanging $i$ and $j$ in the above equation gives

$$\widetilde{\nabla}_{\dot{\partial}_j}\widetilde{\nabla}_{\dot{\partial}_i}\dot{\partial}_k = \Big[-2K' g_{ik} y_j y^s - K g_{ik}\delta_j^s + \frac{K}{2} y^l y^r g_{ik} R_{ljr}{}^s - K^2 g_{jk} y_i y^s - K^2 g_{ij} y_k y^s\Big]\delta_s + \Big[K^2 y_i y_j \delta_k^s + K^2 y_j y_k \delta_i^s + 2K' y_j (y_i \delta_k^s + y_k \delta_i^s) + K(g_{ij}\delta_k^s + g_{jk}\delta_i^s) - \frac{K}{2} y^l y^r g_{ik} R_{ljr}{}^s + 2K^2 y_k y_i \delta_j^s\Big]\dot{\partial}_s. \tag{5.9}$$

Putting (5.8) and (5.9) in (5.7) implies (5.1). ∎

Now, let $\widetilde{R} = 0$. Then by using (5.6), we obtain

$$0 = (\widetilde{R}(\delta_i, \delta_j)\delta_k)_{(x,0)} = R_{ijk}{}^s \delta_s.$$

Hence $R_{ijk}{}^s$=0, i.e., $R = 0$. Conversely, if $R = 0$, then by using Lemma 5.1, we obtain $\widetilde{R}(\delta_i, \dot{\partial}_j)\delta_k = \widetilde{R}(\delta_i, \delta_j)\dot{\partial}_k = \widetilde{R}(\delta_i, \delta_j)\delta_k = 0$ and

$$\widetilde{R}(\dot{\partial}_i, \dot{\partial}_j)\dot{\partial}_k = [(2K' - K^2)(g_{ik} y_j - g_{jk} y_i) y^s + K(g_{ik}\delta_j^s - g_{jk}\delta_i^s)]\delta_s + [(2K' - K^2)(y_i \delta_j^s - y_j \delta_i^s) y_k + K(g_{ik}\delta_j^s - g_{jk}\delta_i^s)]\dot{\partial}_s, \tag{5.10}$$

$$\widetilde{R}(\dot{\partial}_i, \dot{\partial}_j)\delta_k = [(2K' - K^2)(g_{ik} y_j - g_{jk} y_i) y^s + K(g_{ik}\delta_j^s - g_{jk}\delta_i^s)]\delta_s, \tag{5.11}$$

$$\widetilde{R}(\delta_i, \dot{\partial}_j)\dot{\partial}_k = [K^2 y_k y_j \delta_i^s + 2K' y_j (g_{ik} y^s - y_k \delta_i^s) - K^2 g_{ik} y_j y^s + K(g_{ik}\delta_j^s - g_{jk}\delta_i^s)]\delta_s. \tag{5.12}$$

If $K = 0$, then from the above equations, we derive that

$$\widetilde{R}(\dot{\partial}_i, \dot{\partial}_j)\dot{\partial}_k = \widetilde{R}(\dot{\partial}_i, \dot{\partial}_j)\delta_k = \widetilde{R}(\delta_i, \dot{\partial}_j)\dot{\partial}_k = 0,$$

and consequently $\widetilde{R} = 0$. But if $\widetilde{R} = 0$, then by considering (5.11) we deduce that

$$(2K' - K^2)(g_{ik} y_j - g_{jk} y_i) y^s + K(g_{ik}\delta_j^s - g_{jk}\delta_i^s) = 0.$$

By contracting the above equation with $y_k$, we conclude that $K = 0$. Therefore we have the following result.

THEOREM 5.2. *Let $(M, g)$ be a Riemannian manifold and $TM$ be its tangent bundle with the metric $G$. Then we have the following assertions:*

(i) *If $TM$ is flat then $M$ is flat.*
(ii) *If $M$ is flat then $TM$ is flat if and only if $K=0$.*

REMARK. In [Ko], it is proved that the tangent bundle of a Riemannian manifold $M$ with the Sasaki metric is flat if and only if $M$ is flat. On the other hand the tangent bundle of $M$ with the Cheeger–Gromoll metric cannot be flat [Se].

Next, we assume $\widetilde{\nabla}\widetilde{R}=0$. Then by Lemma 5.1, we obtain

$$(5.13)\qquad 0=(\widetilde{\nabla}_{\delta_m}\widetilde{R})(\delta_i,\delta_j)\delta_k=\widetilde{\nabla}_{\delta_m}(\widetilde{R}(\delta_i,\delta_j)\delta_k)-\widetilde{R}(\widetilde{\nabla}_{\delta_m}\delta_i,\delta_j)\delta_k - \widetilde{R}(\delta_i,\widetilde{\nabla}_{\delta_m}\delta_j)\delta_k-\widetilde{R}(\delta_i,\delta_j)\widetilde{\nabla}_{\delta_m}\delta_k.$$

If we restrict ourselves to the zero section of $TM$, then by Lemma 5.1 and (3.6)–(3.9) we get

$$(5.14)\qquad (\widetilde{\nabla}_{\delta_m}(\widetilde{R}(\delta_i,\delta_j)\delta_k))_{(x,0)}=R_{ijk}{}^{s}\Gamma^l_{ms}\delta_l,$$

$$(5.15)\qquad (\widetilde{R}(\widetilde{\nabla}_{\delta_m}\delta_i,\delta_j)\delta_k)_{(x,0)}=R_{rjk}{}^{l}\Gamma^r_{mi}\delta_l,$$

$$(5.16)\qquad (\widetilde{R}(\delta_i,\widetilde{\nabla}_{\delta_m}\delta_j)\delta_k)_{(x,0)}=R_{rik}{}^{l}\Gamma^r_{mj}\delta_l,$$

$$(5.17)\qquad (\widetilde{R}(\delta_i,\delta_j)\widetilde{\nabla}_{\delta_m}\delta_k)_{(x,0)}=R_{ijr}{}^{l}\Gamma^r_{mk}\delta_l.$$

Applying the above equations in (5.13) yields

$$0=(R_{ijk}{}^{s}\Gamma^l_{ms}-R_{rjk}{}^{l}\Gamma^r_{mi}-R_{rik}{}^{l}\Gamma^r_{mj}-R_{ijr}{}^{l}\Gamma^r_{mk})\partial_l=(\nabla_{\partial_m}R)(\partial_i,\partial_j)\partial_k,$$

that is, $\nabla R=0$. Hence we have

THEOREM 5.3. *Let $(M,g)$ be a Riemannian manifold and $TM$ be its tangent bundle with the metric $G$. If $TM$ is locally symmetric, then so is $M$.*

It is remarkable that, in [Ko], it is proved that if the tangent bundle $TM$ of the Riemannian manifold $M$ with the Sasaki metric $g^S$ is locally symmetric, then $M$ is flat and hence $TM$ is also flat. Further, if $TM$ with the Cheeger–Gromoll metric is locally symmetric, then so is $M$ [AM].

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