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Normal families of bicomplex meromorphic functions

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Abstract. We introduce the extended bicomplex plane $\overline{\mathbb{T}}$, its geometric model: the bicomplex Riemann sphere, and the bicomplex chordal metric that enables us to talk about convergence of sequences of bicomplex meromorphic functions. Hence the concept of normality of a family of bicomplex meromorphic functions on bicomplex domains emerges. Besides obtaining a normality criterion for such families, the bicomplex analog of the Montel theorem for meromorphic functions and the fundamental normality tests for families of bicomplex holomorphic functions and bicomplex meromorphic functions are also obtained.

1. Introduction. The concept of normality of a family of bicomplex holomorphic functions was introduced in [CRS], and we now intend to study the same property for bicomplex meromorphic functions. A family $\boldsymbol{F}$ of meromorphic functions on a domain $D \subset \mathbb{C}$ is said to be normal in D if every sequence in $\boldsymbol{F}$ contains a subsequence which converges uniformly on compact subsets of D; the limit function is either meromorphic in D or identically equal to ∞. Of course, the convergence in this situation is with respect to the chordal metric on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (cf. [Sc]). Unfortunately, the one complex variable case does not admit any simple generalization to the bicomplex case.

In order to discuss the convergence of sequences of bicomplex meromorphic functions on bicomplex plane domains, we introduce the extended bicomplex plane $\overline{\mathbb{T}}$, its geometric model, viz. the bicomplex Riemann sphere, the bicomplex chordal metric on the bicomplex Riemann sphere, and the idea of convergence on $\overline{\mathbb{T}}$. In turn, these developments enable the intro-

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duction of the concept of normality of a family of bicomplex meromorphic functions on bicomplex domains. This is the content of Section 3.

In Section 4, a normality criterion for families of bicomplex meromorphic functions, the bicomplex analog of the Montel theorem for meromorphic functions and the fundamental normality tests for families of bicomplex holomorphic functions and bicomplex meromorphic functions are obtained.

2. Preliminaries. As in [RS] (see also [CR] and [CRS]), the algebra of *bicomplex numbers*

$$\mathbb{T} := \{z_1 + z_2\mathbf{i}_2 : z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)\} \tag{2.1}$$

is a space isomorphic to $\mathbb{R}^4$ via the map

$$z_1 + z_2\mathbf{i}_2 = x_0 + x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{j} \mapsto (x_0, x_1, x_2, x_3) \in \mathbb{R}^4,$$

with multiplication defined using the following rules:

$$\mathbf{i}_1^2 = \mathbf{i}_2^2 = -1, \qquad \mathbf{i}_1\mathbf{i}_2 = \mathbf{i}_2\mathbf{i}_1 = \mathbf{j} \quad \text{so that } \mathbf{j}^2 = 1.$$

Here $\mathbb{C}(\mathbf{i}_k) := \{x + y\mathbf{i}_k : \mathbf{i}_k^2 = -1 \text{ and } x, y \in \mathbb{R}\}$ for $k = 1, 2$. It is easy to see that multiplication of bicomplex numbers is commutative. In fact, the bicomplex numbers

$$\mathbb{T} \cong \mathrm{Cl}_{\mathbb{C}}(1, 0) \cong \mathrm{Cl}_{\mathbb{C}}(0, 1)$$

are *unique* among the *complex Clifford algebras* (see [BDS, DSS] and [Ry]) in that they are commutative but not a division algebra. Also, since the map $z_1 + z_2\mathbf{i}_2 \mapsto (z_1, z_2)$ gives a natural isomorphism between the $\mathbb{C}$-vector spaces $\mathbb{T}$ and $\mathbb{C}^2$, we have $\mathbb{T} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. That is, we can view the algebra $\mathbb{T}$ as the complexified $\mathbb{C}(\mathbf{i}_1)$ exactly the way $\mathbb{C}$ is complexified $\mathbb{R}$. In particular, in (2.1), if we put $z_1 = x$ and $z_2 = y\mathbf{i}_1$ with $x, y \in \mathbb{R}$, then we obtain the subalgebra of *hyperbolic numbers*, also called duplex numbers (see e.g. [RS], [So]):

$$\mathbb{D} := \{x + y\mathbf{j} : \mathbf{j}^2 = 1,\ x, y \in \mathbb{R}\} \cong \mathrm{Cl}_{\mathbb{R}}(0, 1).$$

The two projection maps $\mathcal{P}_1, \mathcal{P}_2 : \mathbb{T} \to \mathbb{C}(\mathbf{i}_1)$ defined by

$$\mathcal{P}_1(z_1 + z_2\mathbf{i}_2) = z_1 - z_2\mathbf{i}_1 \quad \text{and} \quad \mathcal{P}_2(z_1 + z_2\mathbf{i}_2) = z_1 + z_2\mathbf{i}_1, \tag{2.2}$$

are used extensively in the following.

The complex (square) norm $\mathrm{CN}(w)$ of the bicomplex number w is the complex number $z_1^2 + z_2^2$; writing $w^* = z_1 - z_2\mathbf{i}_2$, we see that $\mathrm{CN}(w) = ww^*$. Then a bicomplex number $w = z_1 + z_2\mathbf{i}_2$ is invertible if and only if $\mathrm{CN}(w) \neq 0$. Precisely,

$$w^{-1} = \frac{w^*}{\mathrm{CN}(w)}.$$

The set of units in the algebra $\mathbb{T}$ is a multiplicative group which we denote by $\mathbb{T}_*$ (see [BW]). Unlike $\mathbb{C}$, the bicomplex algebra $\mathbb{T}$ has zero divisors given by

$$\mathcal{NC} = \{w \in \mathbb{T} : \mathrm{CN}(w) = 0\} = \{z(1 \pm \mathbf{j}) : z \in \mathbb{C}(\mathbf{i}_1)\},$$

which we may call the *null-cone*. Note that, using the orthogonal idempotents

$$\mathbf{e}_1 = \frac{1+\mathbf{j}}{2}, \quad \mathbf{e}_2 = \frac{1-\mathbf{j}}{2}, \quad \text{in } \mathcal{NC},$$

each bicomplex number $w = z_1 + z_2\mathbf{i}_2 \in \mathbb{T}$ can be uniquely expressed as

$$w = \mathcal{P}_1(w)\mathbf{e}_1 + \mathcal{P}_2(w)\mathbf{e}_2,$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the projection maps defined in (2.2). This representation of $\mathbb{T}$ as $\mathbb{C} \oplus \mathbb{C}$ helps to do addition, multiplication and division term-by-term. With this representation we can set

$$|w|_{\mathbf{j}} := |\mathcal{P}_1(w)|\mathbf{e}_1 + |\mathcal{P}_2(w)|\mathbf{e}_2,$$

which will be referred to as the $\mathbf{j}$-*modulus* of $w = z_1 + z_2\mathbf{i}_2 \in \mathbb{T}$ (see [RS]). Moreover, the usual Euclidean norm of $\mathbb{R}^4$ can be rewritten as

$$\|w\| = \sqrt{\frac{|\mathcal{P}_1(w)|^2 + |\mathcal{P}_2(w)|^2}{2}}.$$

DEFINITION 2.1. Let X_1 and X_2 be subsets of $\mathbb{C}(\mathbf{i_1})$. Then the set

$$X_1 \times_e X_2 := \{w = z_1 + z_2\mathbf{i}_2 \in \mathbb{T} : \mathcal{P}_1(w) \in X_1 \text{ and } \mathcal{P}_2(w) \in X_2\}$$

is called the $\mathbb{T}$-*cartesian set* determined by X_1 and X_2, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the projections (2.2).

It is easy to see that if X_1 and X_2 are domains (open and connected) of $\mathbb{C}(\mathbf{i}_1)$ then $X_1 \times_e X_2$ is also a domain of $\mathbb{T}$. We define the "disc" with center $a = a_1 + a_2\mathbf{i}_2$ of radii r_1 and r_2 of $\mathbb{T}$ as follows [P]:

$$\begin{aligned} D(a; r_1, r_2) &= B^1(a_1 - a_2\mathbf{i}_1, r_1) \times_e B^1(a_1 + a_2\mathbf{i}_1, r_2) \\ &= \{w_1\mathbf{e}_1 + w_2\mathbf{e}_2 : |w_1 - (a_1 - a_2\mathbf{i}_1)| < r_1,\ |w_2 - (a_1 + a_2\mathbf{i}_1)| < r_2\}, \end{aligned}$$

where $B^n(z, r)$ is the open ball with center $z \in \mathbb{C}^n(\mathbf{i}_1)$ and radius $r > 0$. In the particular case where $r = r_1 = r_2$, $D(a; r, r)$ will be called the $\mathbb{T}$-*disc* with center a and radius r. In particular, we define

$$\overline{D}(a; r_1, r_2) := \overline{B^1(a_1 - a_2\mathbf{i}_1, r_1)} \times_e \overline{B^1(a_1 + a_2\mathbf{i}_1, r_2)} \subset \overline{D(a; r_1, r_2)}.$$

We remark that $D(0; r, r)$ is, in fact, the *Lie ball* (see [A]) of radius r in $\mathbb{T}$.

Further, the projections defined in (2.2) help to understand bicomplex holomorphic functions in terms of the following Ringleb Decomposition Lemma [R].

THEOREM 2.2. *Let $\Omega \subset \mathbb{T}$ be an open set. A function $f : \Omega \to \mathbb{T}$ is $\mathbb{T}$-holomorphic on Ω if and only if the two natural functions $f_{e1} : \mathcal{P}_1(\Omega) \to \mathbb{C}(\mathbf{i}_1)$ and $f_{e2} : \mathcal{P}_2(\Omega) \to \mathbb{C}(\mathbf{i}_1)$ are holomorphic,where*

$$f(w) = f_{e1}(\mathcal{P}_1(w))\mathbf{e}_1 + f_{e2}(\mathcal{P}_2(w))\mathbf{e}_2, \qquad \forall w = z_1 + z_2\mathbf{i}_2 \in \Omega,$$

Ringleb's Lemma for bicomplex meromorphic functions is as follows [CR].

THEOREM 2.3. *Let $\Omega \subset \mathbb{T}$ be an open set. A function $f : \Omega \to \mathbb{T}$ is bicomplex meromorphic on Ω if and only if the functions $f_{e1} : \mathcal{P}_1(\Omega) \to \mathbb{C}(\mathbf{i}_1)$ and $f_{e2} : \mathcal{P}_2(\Omega) \to \mathbb{C}(\mathbf{i}_1)$ are meromorphic.*

DEFINITION 2.4. Let $f : \Omega \to \mathbb{T}$ be a bicomplex meromorphic function on the open set $\Omega \subset \mathbb{T}$. Then we say that $w = \mathcal{P}_1(w)\mathbf{e}_1 + \mathcal{P}_2(w)\mathbf{e}_2 \in \Omega$ is a *pole* (resp. *strong pole*) for f if $\mathcal{P}_1(w)$ or (resp. and) $\mathcal{P}_2(w)$ is a pole for f_{e1} and f_{e2}, respectively.

REMARK 2.5. Poles of bicomplex meromorphic functions are not isolated singularities. It is also easy to obtain the following characterization of poles.

PROPOSITION 2.6. *Let $f : X \to \mathbb{T}$ be a bicomplex meromorphic function on the open set $\Omega \subset \mathbb{T}$. If $w_0 \in \Omega$ then w_0 is a pole of f if and only if*

$$\lim_{w \to w_0} \|f(w)\| = \infty.$$

A classical example of bicomplex meromorphic function is the *bicomplex Riemann zeta* function introduced by Rochon [Ro1].

3. The extended bicomplex plane $\overline{\mathbb{T}}$. Since the range of a bicomplex meromorphic function lies beyond the bicomplex plane, we need the *extended bicomplex plane* to study bicomplex meromorphic functions. Further, it would help to study the limit points of unbounded sets in the bicomplex plane. We obtain this extended bicomplex plane by using the extended $\mathbb{C}(\mathbf{i}_1)$-plane.

We consider the set

$$\begin{aligned}\overline{\mathbb{C}(\mathbf{i}_1)} \times_e \overline{\mathbb{C}(\mathbf{i}_1)} &= (\mathbb{C}(\mathbf{i}_1) \cup \{\infty\}) \times_e (\mathbb{C}(\mathbf{i}_1) \cup \{\infty\}) \\ &= (\mathbb{C}(\mathbf{i}_1) \times_e \mathbb{C}(\mathbf{i}_1)) \cup (\mathbb{C}(\mathbf{i}_1) \times_e \{\infty\}) \cup (\{\infty\} \times_e \mathbb{C}(\mathbf{i}_1)) \cup \{\infty\} \\ &= \mathbb{T} \cup I_\infty,\end{aligned}$$

writing I_∞ for $(\mathbb{C}(\mathbf{i}_1) \times_e \{\infty\}) \cup (\{\infty\} \times_e \mathbb{C}(\mathbf{i}_1)) \cup \{\infty\}$. Clearly, any unbounded sequence in $\mathbb{T}$ will have a limit point in I_∞.

DEFINITION 3.1. The set $\overline{\mathbb{T}} = \overline{\mathbb{C}(\mathbf{i}_1)} \times_e \overline{\mathbb{C}(\mathbf{i}_1)}$ is called the *extended bicomplex plane*. That is,

$$\overline{\mathbb{T}} = \mathbb{T} \cup I_\infty, \qquad \text{with} \qquad I_\infty = \{w \in \overline{\mathbb{T}} : \|w\| = \infty\}.$$

It is of importance to observe that formation of the extended bicomplex plane $\overline{\mathbb{T}}$ requires us to add an infinity set, viz. I_∞, which we may call the *bicomplex infinity set*.

We need some definitions in order to give a characterization of this set.

DEFINITION 3.2. An element $w \in I_\infty$ is said to be a $\mathcal{P}_1$*-infinity* (resp. $\mathcal{P}_2$*-infinity*) *element* if $\mathcal{P}_1(w) = \infty$ (resp. $\mathcal{P}_2(w) = \infty$) and $\mathcal{P}_2(w) \neq \infty$ (resp. $\mathcal{P}_1(w) \neq \infty$).

DEFINITION 3.3. The set of all $\mathcal{P}_1$-infinity elements is called the I_1*-infinity set*. It is denoted by $I_{1,\infty}$. Therefore,

$$I_{1,\infty} = \{w \in \overline{\mathbb{T}} : \mathcal{P}_1(w) = \infty,\ \mathcal{P}_2(w) \neq \infty\}.$$

Similarly, the I_2*-infinity set* is

$$I_{2,\infty} = \{w \in \overline{\mathbb{T}} : \mathcal{P}_1(w) \neq \infty,\ \mathcal{P}_2(w) = \infty\}.$$

DEFINITION 3.4. An element $w \in \overline{\mathbb{T}}$ is said to be a $\mathcal{P}_1$*-zero* (resp. $\mathcal{P}_2$*-zero*) *element* if $\mathcal{P}_1(w) = 0$ (resp. $\mathcal{P}_2(w) = 0$) and $\mathcal{P}_2(w) \neq 0$ (resp. $\mathcal{P}_1(w) \neq 0$).

DEFINITION 3.5. The set of all $\mathcal{P}_1$-zero elements is called the I_1*-zero set*; it is denoted by $I_{1,0}$. That is, $I_{1,0} = \{w \in \overline{\mathbb{T}} : \mathcal{P}_1(w) = 0,\ \mathcal{P}_2(w) \neq 0\}$. Similarly, the I_2*-zero set* is $\{w \in \overline{\mathbb{T}} : \mathcal{P}_1(w) \neq 0,\ \mathcal{P}_2(w) = 0\}$.

We now construct the following two new sets:

$$I_\infty^- = I_{1,\infty} \cup I_{2,\infty}, \qquad I_0^- = I_{1,0} \cup I_{2,0},$$

so that $I_\infty = I_\infty^- \cup \{\infty\}$ and $\mathcal{NC} = I_0^- \cup \{0\}$. With these definitions, each element in the null-cone has an inverse in I_∞ and vice versa. One can easily check that the elements of I_∞^- do not satisfy all the properties as satisfied by the $\mathbb{C}(\mathbf{i}_1)$-infinity but the element $\infty = \infty\mathbf{e}_1 + \infty\mathbf{e}_2$ does. We may call I_∞^- the *weak bicomplex infinity set*, and the element $\infty = \infty\mathbf{e}_1 + \infty\mathbf{e}_2$ the *strong infinity*. This nature of the set I_∞ generates the idea of weak and strong poles for bicomplex meromorphic functions (see [CR]). Now, in order to work in the extended bicomplex plane, it is desirable to have a geometric model wherein the elements of $\overline{\mathbb{T}}$ have a concrete representative so that the points of I_∞ are as good as any other point of $\overline{\mathbb{T}}$. To obtain such a model, one can use the usual stereographic projections of $\overline{\mathbb{C}(\mathbf{i}_1)}$ as two components in the idempotent decomposition to get a one-to-one and onto correspondence between the points of $S \times S$, where S is the unit sphere in $\mathbb{R}^3$, and $\overline{\mathbb{T}}$. Hence, we can visualize the extended bicomplex plane directly in $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$. With this representation, we call $\overline{\mathbb{T}}$ the *bicomplex Riemann sphere*.

Observe that what is done above is basically a compactification of $\mathbb{C}^2$, using the bicomplex setting. That is, suitable points at infinity are added to $\mathbb{T}$ to get the extended bicomplex plane $\overline{\mathbb{T}}$. In higher dimensions such compactifications are well known under the name of conformal compactifications. In fact, such compactifications are obtained as homogeneous spaces of Lie groups (see [BW] and [BE]).

3.1. The chordal metric on $\overline{\mathbb{T}}$. To study normal families of bicomplex meromorphic functions, we first have to extend the chordal distance to the extended bicomplex plane, to be able to introduce convergence of sequences and continuity of bicomplex meromorphic functions. The chordal metric on $\overline{\mathbb{C}(\mathbf{i}_1)}$ can be used to define a distance on $\overline{\mathbb{T}}$.

Proposition 3.6. *Let $\chi : \overline{\mathbb{C}(\mathbf{i}_1)} \times \overline{\mathbb{C}(\mathbf{i}_1)} \to \mathbb{R}$ be the chordal metric on $\overline{\mathbb{C}(\mathbf{i}_1)}$. Then the mapping $\chi_e : \overline{\mathbb{T}} \times \overline{\mathbb{T}} \to \mathbb{R}$ defined as*

$$\chi_e(z,w) = \sqrt{\frac{\chi^2(\mathcal{P}_1(z),\mathcal{P}_1(w)) + \chi^2(\mathcal{P}_2(z),\mathcal{P}_2(w))}{2}}$$

is a metric on $\overline{\mathbb{T}}$.

Proof. It is easy to verify that for all $z, w \in \overline{\mathbb{T}}$ we have: $\chi_e(z,w) \geq 0$; $\chi_e(z,w) = 0$ iff $z = w$; $\chi_e(z,w) = \chi_e(w,z)$. Now, we show that χ also satisfies the triangle inequality. Let $z, w, v \in \overline{\mathbb{T}}$.

Using Minkowski's inequality, we obtain

$$\begin{aligned}
&\chi_e(z,w)\\
&\quad= \sqrt{\frac{\chi^2(\mathcal{P}_1(z),\mathcal{P}_1(w)) + \chi^2(\mathcal{P}_2(z),\mathcal{P}_2(w))}{2}}\\
&\quad\leq \sqrt{\frac{\{\chi(\mathcal{P}_1(z),\mathcal{P}_1(v)) + \chi(\mathcal{P}_1(v),\mathcal{P}_1(w))\}^2 + \{\chi(\mathcal{P}_2(z),\mathcal{P}_2(v)) + \chi(\mathcal{P}_2(v),\mathcal{P}_2(w))\}^2}{2}}\\
&\quad\leq \sqrt{\frac{\chi^2(\mathcal{P}_1(z),\mathcal{P}_1(v)) + \chi^2(\mathcal{P}_2(z),\mathcal{P}_2(v))}{2}}\\
&\qquad+ \sqrt{\frac{\chi^2(\mathcal{P}_1(v),\mathcal{P}_1(w)) + \chi^2(\mathcal{P}_2(v),\mathcal{P}_2(w))}{2}}\\
&\quad= \chi_e(z,v) + \chi_e(v,w). \ \blacksquare
\end{aligned}$$

We call χ_e the *bicomplex chordal metric.* The virtue of this metric is that it allows $w \in I_\infty$ to be treated like any other point. Hence, we are now able to analyse the behavior of bicomplex meromorphic functions in the extended bicomplex plane, especially on the set I_∞.

Remark 3.7. As for the **j**-modulus, let us define

$$\chi_{\mathbf{j}}(z,w) := \chi(\mathcal{P}_1(z),\mathcal{P}_1(w))\mathbf{e}_1 + \chi(\mathcal{P}_2(z),\mathcal{P}_2(w))\mathbf{e}_2.$$

Then $\mathrm{Re}(\chi_{\mathbf{j}}^2(z,w)) = \chi_e^2(z,w)$ and thus

$$\chi_e(z,w) = \sqrt{\mathrm{Re}(\chi_{\mathbf{j}}^2(z,w))}$$

where

$$\chi_{\mathbf{j}}(z,w) = \frac{|z-w|_{\mathbf{j}}}{\sqrt{1+|z|_{\mathbf{j}}}\sqrt{1+|w|_{\mathbf{j}}}} \quad \text{if } z,w \in \mathbb{T}.$$

Some properties of the bicomplex chordal metric are discussed in the following result.

THEOREM 3.8. *If $z = z_1\mathbf{e}_1 + z_2\mathbf{e}_2$ and $w = w_1\mathbf{e}_1 + w_2\mathbf{e}_2$ are any two elements in the extended bicomplex plane, then:*

1. $\chi_e(z,w) \le 1$;
2. $\chi_e(0,\infty) = 1$;
3. $\chi_e(z,w) = \frac{1}{\sqrt{2}}\chi(z_1,\infty)$ *if* $\mathcal{P}_2(z) = \mathcal{P}_2(w) = 0$ *and* $\mathcal{P}_1(w) = \infty$;
4. $\chi_e(z,w) = \frac{1}{\sqrt{2}}\chi(z_1,w_1)$ *if* $\mathcal{P}_2(z) = \mathcal{P}_2(w) = \infty$;
5. $\chi_e(z,\infty) = \frac{1}{\sqrt{2}}\chi(z_2,\infty)$ *if* $\mathcal{P}_1(z) = \infty$;
6. $\chi_e(z,w) = \chi_e(z^{-1},w^{-1})$;
7. $\chi_e(z,w) = \chi(z,w)$ *if* $z,w \in \overline{\mathbb{C}(\mathbf{i}_1)}$;
8. $\chi_e(z,w) \le \|z-w\|$ *if* $z,w \in \mathbb{T}$;
9. $\chi_e(z,w)$ *is a continuous function on* $\mathbb{T}$.

The implication $\|z\| \le \|w\| \Rightarrow \chi_e(0,z) \le \chi_e(0,w)$ need not be true:

EXAMPLE 3.9. Let

$$z = (1+2\mathbf{i}_1)\mathbf{e}_1 + (2+3\mathbf{i}_1)\mathbf{e}_2 \quad \text{and} \quad w = (1+\mathbf{i}_1)\mathbf{e}_1 + (3+3\mathbf{i}_1)\mathbf{e}_2.$$

Then $\|z\| \le \|w\|$, but $\chi_e(0,z) = \sqrt{0.88}$ and $\chi_e(0,w) = \sqrt{0.80}$.

EXAMPLE 3.10. Let

$$z = (4+\mathbf{i}_1)\mathbf{e}_1 + (2+3\mathbf{i}_1)\mathbf{e}_2 \quad \text{and} \quad w = (1+2\mathbf{i}_1)\mathbf{e}_1 + (3+4\mathbf{i}_1)\mathbf{e}_2.$$

Then $\|z\| = \|w\|$, but $\chi_e(0,z) = \sqrt{0.93}$ and $\chi_e(0,w) = \sqrt{0.89}$.

However, we can prove the following result.

PROPOSITION 3.11. *Let $z,w \in \mathbb{T}$. If $\|z\| \le \|w\|$ then*

$$\chi_e(0,z) \le \chi_e(0,\sqrt{2}\,\|w\|).$$

Proof. By definition,

$$\begin{aligned}\chi_e(0,z) &= \sqrt{\frac{\chi^2(0,\mathcal{P}_1(z)) + \chi^2(0,\mathcal{P}_2(z))}{2}} \\ &= \sqrt{\frac{1}{2}\left\{\frac{|\mathcal{P}_1(z)|^2}{1+|\mathcal{P}_1(z)|^2} + \frac{|\mathcal{P}_2(z)|^2}{1+|\mathcal{P}_2(z)|^2}\right\}}.\end{aligned}$$

Since, $|\mathcal{P}_i(z)| \le \sqrt{2}\|z\| \le \sqrt{2}\,\|w\|$ for $i = 1,2$ we have $\chi(0,\mathcal{P}_i(z)) = \chi(0,|\mathcal{P}_i(z)|) \le \chi(0,\sqrt{2}\|w\|)$ for $i = 1,2$. Hence, $\chi_e(0,z) \le \chi_e(0,\sqrt{2}\|w\|)$. ∎

3.2. Convergence in $\overline{\mathbb{T}}$

Definition 3.12. A sequence $\{f_n\}$ of functions on $\mathbb{T}$ converges *bispherically uniformly* to a function f on a set $E \subset \mathbb{T}$ if, for any $\epsilon > 0$, there is a number n_0 such that $n \geq n_0$ implies

$$\chi_e(f_n(w), f(w)) < \epsilon \quad \text{for all } w \in E.$$

Note that if $\{f_n\}$ converges uniformly to f on $E \subset \mathbb{T}$, then it also converges spherically uniformly to f on E. The converse holds if the limit function is bounded.

Lemma 3.13.

$$\chi_e(z, w) \geq \frac{\|z - w\|}{\sqrt{1 + 2\|z\|^2}\sqrt{1 + 2\|w\|^2}}, \qquad \textit{if } z, w \in \mathbb{T}.$$

Proof. We shall establish this inequality by obtaining an equivalent inequality that holds trivially. For $z, w \in \mathbb{T}$, put $\mathcal{P}_1(z) = a$, $\mathcal{P}_2(z) = b$, $\mathcal{P}_1(w) = c$, and $\mathcal{P}_2(w) = d$. Then

$$\begin{aligned}
&\chi_e(z, w) \geq \frac{\|z - w\|}{\sqrt{1 + 2\|z\|^2}\sqrt{1 + 2\|w\|^2}} \\
&\quad \Leftrightarrow \ \chi_e^2(z, w) \geq \frac{\|z - w\|^2}{(1 + 2\|z\|^2)(1 + 2\|w\|^2)} \\
&\quad \Leftrightarrow \ \chi^2(a, c) + \chi^2(b, d) \geq \frac{|a - c|^2 + |b - d|^2}{(1 + |a|^2 + |b|^2)(1 + |c|^2 + |d|^2)} \\
&\quad \Leftrightarrow \ \frac{|a - c|^2}{(1 + |a|^2)(1 + |c|^2)} + \frac{|b - d|^2}{(1 + |b|^2)(1 + |d|^2)} \\
&\qquad \geq \frac{|a - c|^2}{(1 + |a|^2 + |b|^2)(1 + |c|^2 + |d|^2)} + \frac{|b - d|^2}{(1 + |a|^2 + |b|^2)(1 + |c|^2 + |d|^2)} \\
&\quad \Leftrightarrow \ |a - c|^2 \left[\frac{1}{(1 + |a|^2)(1 + |c|^2)} - \frac{1}{(1 + |a|^2 + |b|^2)(1 + |c|^2 + |d|^2)} \right] \\
&\qquad \geq |b - d|^2 \left[\frac{1}{(1 + |a|^2 + |b|^2)(1 + |c|^2 + |d|^2)} - \frac{1}{(1 + |b|^2)(1 + |d|^2)} \right] \\
&\quad \Leftrightarrow \ \frac{|a - c|^2[|d|^2 + |b|^2 + |a|^2|d|^2 + |b|^2|c|^2 + |b|^2|d|^2]}{(1 + |a|^2)(1 + |c|^2)} \\
&\qquad \geq \frac{|b - d|^2[-\{|c|^2 + |a|^2 + |a|^2|c|^2 + |a|^2|d|^2 + |b|^2|c|^2\}]}{(1 + |b|^2)(1 + |d|^2)}.
\end{aligned}$$

The left hand side of the last inequality is positive, while the right hand side is negative, so the inequality holds trivially. ∎

Theorem 3.14. *If $\{f_n\}$ converges bispherically uniformly to a bounded function f on $E \subset \mathbb{T}$, then $\{f_n\}$ converges uniformly to f on E.*

Proof. By assumption, for every $\epsilon > 0$ there is n_0 such that for all $n \geq n_0$, we have

$$\chi_e(f_n(w), f(w)) < \epsilon.$$

Now the definition of χ_e implies that $\mathcal{P}_i(f_n(w))$ converges uniformly to $\mathcal{P}_i(f(w))$ on $\mathcal{P}_i(E)$, $i = 1, 2$. Further, since f is bounded on E, $\mathcal{P}_i(f(w))$ is bounded on $\mathcal{P}_i(E)$, $i = 1, 2$, and hence $\mathcal{P}_i(f_n(w))$ is bounded on $\mathcal{P}_i(E)$, $i = 1, 2$, for all but finitely many n. This implies that there is a positive constant L such that

$$\|f_n(w)\| < L \quad \forall n \geq n_0,$$

on $\mathcal{P}_1(E) \times_e \mathcal{P}_2(E) \supseteq E$. By Lemma 3.13, we have

$$\|f_n(w) - f(w)\| \leq \sqrt{1 + 2\|f_n(w)\|^2}\sqrt{1 + 2\|f(w)\|^2}\, \chi_e(f_n(w), f(w))$$

for all $n \geq n_0$ and $w \in E$. But f is bounded on E and $\{f_n\}$ is bounded on E for all $n \geq n_0$, so $\{f_n\}$ converges uniformly to f on E. ∎

The notion of continuity with respect to the bicomplex chordal metric is given in the following definition.

DEFINITION 3.15. A function f is *bispherically continuous* at a point $w_0 \in \mathbb{T}$ if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\chi_e(f(w), f(w_0)) < \epsilon \quad \text{whenever} \quad \|w - w_0\| < \delta.$$

In the case of *bicomplex meromorphic functions* we have the following result.

THEOREM 3.16. *If f is a bicomplex meromorphic function in a domain $E \subset \mathbb{T}$, then f is bispherically continuous in E.*

Proof. By Theorem 2.3 there exist meromorphic functions $f_{e1} : E_1 \to \mathbb{C}(\mathbf{i}_1)$ and $f_{e2} : E_2 \to \mathbb{C}(\mathbf{i}_1)$ with $E_1 = \mathcal{P}_1(E)$ and $E_2 = \mathcal{P}_2(E)$ such that

$$f(z_1 + z_2\mathbf{i}_2) = f_{e1}(z_1 - z_2\mathbf{i}_1)\mathbf{e}_1 + f_{e2}(z_1 + z_2\mathbf{i}_1)\mathbf{e}_2 \quad \forall z_1 + z_2\mathbf{i}_2 \in E.$$

If f is $\mathbb{T}$-holomorphic at $w_0 \in E$, then f_{ei} is holomorphic on E_i for $i = 1, 2$. Hence, it is bispherically continuous on E since

$$\chi_e(f(w), f(w_0)) \leq \|f(w) - f(w_0)\|. \tag{3.1}$$

If w_0 is a strong pole, then $1/f_{e1}$ and $1/f_{e2}$ are continuous at $\mathcal{P}_1(w_0)$ and $\mathcal{P}_2(w_0)$ respectively. Moreover, noting that

$$\begin{aligned}
&\chi_e(f(w), f(w_0)) \\
&\quad = \chi_e\left(\frac{1}{f(w)}, \frac{1}{f(w_0)}\right) \\
&\quad = \sqrt{\frac{\chi^2\left(\frac{1}{f_{e1}(\mathcal{P}_1(w))}, \frac{1}{f_{e1}(\mathcal{P}_1(w_0))}\right) + \chi^2\left(\frac{1}{f_{e2}(\mathcal{P}_2(w))}, \frac{1}{f_{e2}(\mathcal{P}_2(w_0))}\right)}{2}},
\end{aligned}$$

the result follows as in the preceding case. If w_0 is a weak pole, then $1/f_{e1}$ or $1/f_{e2}$ is continuous at $\mathcal{P}_1(w_0)$ or $\mathcal{P}_2(w_0)$ respectively. Suppose, without loss of generality, that $1/f_{e1}$ is continuous at $\mathcal{P}_1(w_0)$ with f_{e2} continuous at $\mathcal{P}_2(w_0)$. Then

$$\chi_e(f(w), f(w_0)) = \sqrt{\frac{\chi^2\Big(\frac{1}{f_{e1}(\mathcal{P}_1(w))}, \frac{1}{f_{e1}(\mathcal{P}_1(w_0))}\Big) + \chi^2(f_{e2}(\mathcal{P}_2(w), f_{e2}(\mathcal{P}_2(w_0)))}{2}},$$

and the result follows by using (3.1) in the complex plane (in $\mathbf{i}_1$). ∎

DEFINITION 3.17. A family $\boldsymbol{F}$ of bicomplex functions defined on a domain $\Omega \subset \mathbb{T}$ is said to be *bispherically equicontinuous* at a point $w_0 \in \Omega$ if for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon, w_0)$ such that

$$\chi_e(f(w), f(w_0)) < \epsilon \; \forall f \in \boldsymbol{F} \quad \text{whenever} \quad \|w - w_0\| < \delta.$$

Moreover, $\boldsymbol{F}$ is bispherically equicontinuous on a subset $E \subset \Omega$ if it is bispherically equicontinuous at each point of E.

REMARK 3.18. By (3.1), equicontinuity with respect to the euclidean metric implies bispherical equicontinuity.

4. Normal families of bicomplex meromorphic functions

4.1. Basic results

DEFINITION 4.1. A family $\boldsymbol{F}$ of bicomplex meromorphic functions in a domain $\Omega \subset \mathbb{T}$ is *normal* in Ω if every sequence $\{f_n\} \subset \boldsymbol{F}$ contains a subsequence which converges bispherically uniformly on compact subsets of Ω.

That the limit function is either bicomplex meromorphic in Ω or in the set I_∞^- or identically ∞ is a consequence of Corollary 4.4. That the limit can actually be identically ∞ is shown by the following example.

EXAMPLE 4.2. Let $f_n(w) = n/w$, $n = 1, 2, \ldots$, on the Lie ball $D(0; r, r)$. Then each f_n is bicomplex meromorphic and $\{f_n\}$ converges bispherically uniformly to ∞ in $D(0; r, r)$.

THEOREM 4.3. *A family $\boldsymbol{F}$ of bicomplex meromorphic functions is normal in a domain Ω if and only if the family of meromorphic functions $F_{ei} = \mathcal{P}_i(\boldsymbol{F})$ is normal in $\mathcal{P}_i(\Omega)$ for $i = 1, 2$ with respect to the chordal metric.*

Proof. Suppose that $\boldsymbol{F}$ is normal in Ω with respect to the bicomplex chordal metric. Let $\{(f_n)_1\}$ be a sequence in $\boldsymbol{F}_{e1} = \mathcal{P}_1(\boldsymbol{F})$. We want to prove, without loss of generality, that the family of meromorphic functions $\{(f_n)_1\}$ contains a subsequence which converges spherically locally uniformly on $\mathcal{P}_1(\Omega)$. By definition, we can find a sequence $\{f_n\}$ in $\boldsymbol{F}$ such that $\{\mathcal{P}_1(f_n)\} = \{(f_n)_1\}$. Moreover, for any $z_0 \in \mathcal{P}_1(\Omega)$, we can find a $w_0 \in \Omega$ such that $\mathcal{P}_1(w_0) = z_0$. Now, consider a closed $\mathbb{T}$-disc $\overline{D}(w_0; r, r)$ in Ω. By hypotheses, the sequence $\{f_n\}$ contains a subsequence $\{f_{n_k}\}$ which converges bispherically uniformly on $\overline{D}(w_0; r, r)$. Hence, $\mathcal{P}_1(f_{n_k}) = (f_{n_k})_1$ converges spherically uniformly on $\overline{B^1(z_0, r)} \subset \mathcal{P}_1(\Omega)$.

Conversely, suppose that $\boldsymbol{F}_{ei} = \mathcal{P}_i(\boldsymbol{F})$ is normal on $\mathcal{P}_i(\Omega) = \Omega_i$ for $i = 1, 2$. We want to show that $\boldsymbol{F}$ is normal in Ω. Let $\{f_n\}$ be any sequence in $\boldsymbol{F}$ and K be any compact subset of Ω. Then $\{\mathcal{P}_1(f_n)\} = \{(f_n)_1\}$ is a sequence in $\boldsymbol{F}_{\mathbf{e}_1} = \mathcal{P}_1(\boldsymbol{F})$. Since $\boldsymbol{F}_{\mathbf{e}_1} = \mathcal{P}_1(\boldsymbol{F})$ is normal in $\mathcal{P}_1(\Omega)$, $\{(f_n)_1\}$ has a subsequence $\{(f_{n_k})_1\}$ which converges spherically uniformly on $\mathcal{P}_1(K)$ to a $\overline{\mathbb{C}(\mathbf{i}_1)}$-function. Now, consider $\{f_{n_k}\}$ in $\boldsymbol{F}$. Then $\{\mathcal{P}_2(f_{n_k})\} = \{(f_{n_k})_2\}$ is a sequence in $\boldsymbol{F}_{e2} = \mathcal{P}_2(\boldsymbol{F})$. Since $\boldsymbol{F}_{e2} = \mathcal{P}_2(\boldsymbol{F})$ is normal in $\mathcal{P}_1(\Omega)$, $\{(f_{n_k})_2\}$ has a subsequence $\{(f_{n_{k_l}})_2\}$ which converges spherically uniformly on $\mathcal{P}_2(K)$ to a $\overline{\mathbb{C}(\mathbf{i}_1)}$-function. This implies that $\{(f_{n_{k_l}})_1\mathbf{e}_1 + (f_{n_{k_l}})_2\mathbf{e}_2\}$ is a subsequence of $\{f_n\}$ which converges bispherically uniformly on $\mathcal{P}_1(K) \times_e \mathcal{P}_2(K) \supseteq K$ to a $\overline{\mathbb{T}}$-function, showing that $\boldsymbol{F}$ is normal in Ω. ∎

Since the limit function of a locally convergent sequence of meromorphic functions is either meromorphic or identically equal to ∞, we have automatically the following result as a direct consequence of Theorems 2.3 and 4.3.

Corollary 4.4. *Let $\{f_n\}$ be a sequence of bicomplex meromorphic functions on Ω which converges bispherically uniformly on compact subsets to f. Then f is either a bicomplex meromorphic function on Ω or in the set I_∞^- or identically ∞.*

Moreover, from the fact that a family of analytic functions is normal with respect to the usual metric if and only if the family is normal with respect to the chordal metric (see [Sc, Cor. 3.1.7]) and from the characterization of the notion of normality for a family of bicomplex holomorphic functions (see [CRS, Thm. 8]), we obtain the following result as a consequence of Theorems 2.2 and 4.3.

Corollary 4.5. *A family $\boldsymbol{F}$ of $\mathbb{T}$-holomorphic functions is normal in a domain Ω with respect to the Euclidean metric if and only if $\boldsymbol{F}$ is normal in Ω with respect to the bicomplex chordal metric.*

4.2. Bicomplex Montel theorem. In this subsection, we will give a proof of a bicomplex version of the Montel theorem for a family of bicomplex meromorphic functions. We start with the following results.

Lemma 4.6. *If $\{f_n\}$ is a sequence of bispherically continuous functions which converges bispherically uniformly to a function f on a compact subset $E \subset \mathbb{T}$, then f is uniformly bispherically continuous on E and the functions $\{f_n\}$ are bispherically equicontinuous on E.*

Proof. The proof is the same, with necessary changes, as the one complex variable analogue (see [Sc, Prop. 1.6.2]). ∎

Lemma 4.7. *The bicomplex Riemann sphere is a compact metric space.*

Proof. We will prove that $\overline{\mathbb{T}}$ is sequentially compact. Let $\{w_n\}$ be a sequence in $\overline{\mathbb{T}}$. Then $\{\mathcal{P}_i(w_n)\}$ is a sequence in $\overline{\mathbb{C}(\mathbf{i}_1)}$ for $i = 1, 2$. Since the Riemann sphere is the one-point compactification of the complex plane, $\{\mathcal{P}_1(w_n)\}$ has a spherically convergent subsequence $\{\mathcal{P}_1(w_{n_k})\}$ in $\overline{\mathbb{C}(\mathbf{i}_1)}$ and $\{\mathcal{P}_2(w_{n_k})\}$ also has a spherically convergent subsequence $\{\mathcal{P}_2(w_{n_{k_l}})\}$ in $\overline{\mathbb{C}(\mathbf{i}_1)}$ such that $\{\mathcal{P}_i(w_{n_{k_l}})\}$ converges spherically in $\overline{\mathbb{C}(\mathbf{i}_1)}$ for $i = 1, 2$. Hence, $\{w_{n_{k_l}}\}$ converges bispherically in $\overline{\mathbb{T}}$. ∎

As for one complex variable, in discussing the normality of a family of bicomplex meromorphic functions, the concept of local boundedness is not entirely relevant. However, bispherical equicontinuity can be substituted in the following counterpart of Montel's theorem.

Theorem 4.8. *A family $\boldsymbol{F}$ of bicomplex meromorphic functions in a bicomplex domain $\Omega \subset \mathbb{T}$ is normal if and only if $\boldsymbol{F}$ is bispherically equicontinuous in Ω.*

Proof. Suppose $\boldsymbol{F}$ is normal but not bispherically equicontinuous in Ω. Then there is a point $w_0 \in \Omega$, some $\epsilon > 0$, a sequence $\{w_n\}$ converging to w_0 and a sequence $\{f_n\} \subset \boldsymbol{F}$ such that

$$\chi_e\left(f_n(w_0), f_n(w_n)\right) > \epsilon, \qquad n = 1, 2, \ldots. \tag{4.1}$$

Since $\boldsymbol{F}$ is normal, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ converging bisherically uniformly on compact subsets of Ω and in particular on a compact subset containing $\{w_n\}$. By Lemma 4.6, this implies that $\{f_{n_k}\}$ is bispherically equicontinuous at w_0. This contradicts (4.1). Therefore $\boldsymbol{F}$ is bispherically equicontinuous.

Conversely, let $\boldsymbol{F}$ be a bispherically equicontinuous family of bicomplex meromorphic functions defined on Ω. To show that $\boldsymbol{F}$ is normal in Ω we need to extract a locally bispherically uniformly convergent subsequence from every sequence in $\boldsymbol{F}$. Let $\{f_n\}$ be any sequence in $\boldsymbol{F}$ and let E be a countable dense subset of Ω, for example we can take $E \cap \Omega$ where $E = \{w_n = w_{1,n}\mathbf{e}_1 + w_{2,n}\mathbf{e}_2 : w_{j,n} = x_{j,n} + \mathbf{i}_1 y_{j,n}$ where $x_{j,n}, y_{j,n} \in \mathbb{Q}, j = 1, 2\}$. Take any sequence $\{f_n\} \subset \boldsymbol{F}$ and consider the sequence of bicomplex numbers $\{f_n(w_1)\}$. Since the bicomplex Riemann sphere is a *compact metric space* (see Lemma 4.7), $\{f_n\}$ has a subsequence $\{f_{n,1}\}$ converging bispherically at w_1. Next, we can find a subsequence $\{f_{n,2}\}$ of $\{f_{n,1}\}$ such that $\{f_{n,2}(w_2)\}$ converges bispherically at w_2. Since $\{f_{n,2}\}$ is a subsequence of $\{f_{n,1}\}$, $\{f_{n,2}(w_1)\}$ also converges bispherically at w_1. Therefore, $\{f_{n,2}\}$ converges bispherically at w_1 and w_2. Continuing this process, for each $k \geq 1$ we obtain a subsequence $\{f_{n,k}\}$ that converges bispherically at $w_1, \ldots, w_k$ and $\{f_{n,k}\} \subset \{f_{n,k-1}\}$. Now by Cantor's diagonal process we define a sequence $\{g_n\}$ as

$$g_n(w) = f_{n,n}(w), \quad n \in \mathbb{N}.$$

Hence, $\{g_n(w_k)\}$ is a subsequence of the bispherically convergent sequence $\{f_{n,k}(w_k)\}_{n\geq k}$ and hence converges for each $w_k \in E$. Now, by hypothesis, $\{g_n\}$ is bispherically equicontinuous on every compact subset of Ω. So for every $\epsilon > 0$ and every compact subset K of Ω there is a $\delta > 0$ such that

$$\chi_e(g_n(w), g_n(w')) < \frac{\epsilon}{3}, \ \forall n \in \mathbb{N} \text{ and } \forall w, w' \in K \text{ with } \|w - w'\| < \delta. \tag{4.2}$$

Since K is compact, we can cover it by a finite collection of balls, say

$$K \subset \bigcup_{j=1}^{p} \{B^2(\varsigma_j, \delta/2) : \varsigma_j \in E\}.$$

Since $\varsigma_j \in E$, $\{g_n(\varsigma_j)\}$ converges for each j with $1 \leq j \leq p$, which further implies that $\{g_n(\varsigma_j)\}$ is a Cauchy sequence. That is, there is a positive integer n_0 such that

$$\chi_e(g_n(\varsigma_j), g_m(\varsigma_j)) < \epsilon/3, \quad \forall m, n \geq n_0 \ (1 \leq j \leq p). \tag{4.3}$$

Finally, for any $w \in K$, $w \in B^2(\varsigma_{j_0}, \delta/2)$ for some $1 \leq j_0 \leq p$. Thus, from (4.2) and (4.3), we have

$$\begin{aligned}
&\chi_e(g_n(w), g_m(w)) \\
&\qquad \leq \chi_e(g_n(w), g_n(\varsigma_{j_0})) + \chi_e(g_n(\varsigma_{j_0}), g_m(\varsigma_{j_0})) + \chi_e(g_n(\varsigma_{j_0}), g_m(w)) \\
&\qquad < \epsilon/3 + \epsilon/3 + \epsilon/3, \quad \forall m, n \geq n_0.
\end{aligned}$$

Therefore, by construction, $\{g_n\}$ is locally bispherically uniformly Cauchy and hence converges locally bispherically uniformly on Ω. ∎

4.3. Fundamental normality test. Finally, we prove the bicomplex version of the fundamental normality test for meromorphic functions. First, we prove it for bicomplex holomorphic functions.

THEOREM 4.9. *Let $\boldsymbol{F}$ be a family of bicomplex holomorphic functions in a domain $\Omega \subset \mathbb{T}$. Suppose there are $\alpha, \beta \in \mathbb{T}$ such that*

(a) *$\alpha - \beta$ is invertible, and*
(b) *$S \cap \mathcal{R}(f) = \emptyset$ for all $f \in \boldsymbol{F}$, where $S = \{w \in \mathbb{T} : w - \alpha \in \mathcal{NC}\} \cup \{w \in \mathbb{T} : w - \beta \in \mathcal{NC}\}$ and $\mathcal{R}(f)$ denotes the range of f.*

Then $\boldsymbol{F}$ is a normal family in Ω.

Proof. Conditions (a) and (b) imply that for each $f \in \boldsymbol{F}$ the projection $\mathcal{P}_i(f)$ does not assume $\mathcal{P}_i(\alpha)$ and $\mathcal{P}_i(\beta)$, where $\mathcal{P}_i(\alpha) \neq \mathcal{P}_i(\beta)$, for $i = 1, 2$. Then by the fundamental normality test for holomorphic functions (see [Sc]), it follows that $\mathcal{P}_i(\boldsymbol{F})$ is normal in $\mathcal{P}_i(\Omega)$ for $i = 1, 2$. Now by Theorem 11 of [CRS] we conclude that $\boldsymbol{F}$ is normal in Ω. ∎

Following [Ro2], one can easily obtain a bicomplex version of Picard's Little Theorem for meromorphic functions.

THEOREM 4.10. *Let f be a bicomplex meromorphic function in $\mathbb{T}$. Suppose there exist $\alpha, \beta, \gamma \in \mathbb{T}$ such that*

(a) *$\alpha - \beta$, $\beta - \gamma$, $\gamma - \alpha$ are invertible, and*
(b) *$S \cap \mathcal{R}(f) = \emptyset$, $\forall f \in \boldsymbol{F}$, where $S = \{w \in \mathbb{T} : w - \alpha \in \mathcal{NC}\} \cup \{w \in \mathbb{T} : w - \beta \in \mathcal{NC}\} \cup \{w \in \mathbb{T} : w - \gamma \in \mathcal{NC}\}$ and $\mathcal{R}(f)$ denotes the range of f.*

Then f is a constant function.

THEOREM 4.11. *Let $\boldsymbol{F}$ be a family of bicomplex meromorphic functions defined in a domain $\Omega \subset \mathbb{T}$. Suppose there exist $\alpha, \beta, \gamma \in \mathbb{T}$ such that*

(a) *$\alpha - \beta$, $\beta - \gamma$, $\gamma - \alpha$ are invertible, and*
(b) *$S \cap \mathcal{R}(f) = \emptyset$, $\forall f \in \boldsymbol{F}$, where $S = \{w \in \mathbb{T} : w - \alpha \in \mathcal{NC}\} \cup \{w \in \mathbb{T} : w - \beta \in \mathcal{NC}\} \cup \{w \in \mathbb{T} : w - \gamma \in \mathcal{NC}\}$ and $\mathcal{R}(f)$ denotes the range of f.*

Then $\boldsymbol{F}$ is normal in Ω.

Proof. Following the method of proof of Theorem 4.9 and applying Theorem 4.3 and the fundamental normality test for meromorphic functions ([Sc, p. 74]) we can easily conclude that $\boldsymbol{F}$ is normal in Ω. ∎

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