Natural transformations of the composition of Weil and cotangent functors

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Abstract. We study geometrical properties of natural transformations $T^A T^* \to T^* T^A$ depending on a linear function defined on the Weil algebra $A$. We show that for many particular cases of $A$, all natural transformations $T^A T^* \to T^* T^A$ can be described in a uniform way by means of a simple geometrical construction.

1. Introduction. By Tulczyjew [15], and Modugno and Stefani [13], there is a natural equivalence $TT^* \to T^* T$ of second order tangent and cotangent functors. All natural transformations of this type were determined by Kolář and Radziszewski [11]. The tangent functor $T$ is a particular case of the functor $T^r_k$ of $k$-dimensional velocities of order $r$, which is defined by

\begin{equation}
T^r_k M = J^r_0(\mathbb{R}^k, M), \quad T^r_k f(j^r_0 g) = j^r_0(f \circ g)
\end{equation}

for all smooth manifolds $M$ and all smooth maps $f : M \to N$. Then Cantrijn, Crampin, Sarlet and Saunders [1] introduced a canonical natural equivalence $T^r_1 T^* \to T^* T^r_1$, which can be considered as a generalization of the natural equivalence $TT^* \to T^* T$. In [3] we have classified all natural transformations $T^2_1 T^* \to T^* T^2_1$ and in [4] we have determined all natural transformations $TTT^* \to TT^* T$, which is a similar problem.

In general, let $T^A$ be a Weil functor corresponding to a Weil algebra $A$. In the jet-like approach, a Weil functor $T^A$ can be interpreted as a generalization of the $(k, r)$-velocities functor $T^r_k$. By [10], Weil functors even represent a general model of all product preserving bundle functors. The aim of this paper is to study natural transformations $T^A T^* \to T^* T^A$. We first define natural transformations $s_f : T^A T^* \to T^* T^A$ depending on linear functions $f : A \to \mathbb{R}$ and describe some geometrical properties of such natural transformations. In particular, we discuss the role of $s_f$ in the theory of

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lifting of 1-forms and \((0, 2)\)-tensor fields to Weil bundles. We also consider the existence of a natural equivalence \(T^A T^* \to T^* T^A\). Finally we construct a fairly general model of natural transformations \(T^A T^* \to T^* T^A\), which simply characterizes all such natural transformations for some particular cases of the Weil algebra \(A\).

We remark that natural transformations \(T^r T^* \to T^* T^r\) are of fundamental importance in analytical mechanics [2], and a natural equivalence of this type enables us to introduce a symplectic structure on \(T^r T^* M\). In what follows we will use the theory of natural operations in differential geometry from [10]. All maps and manifolds are assumed to be infinitely differentiable.

2. Weil functors. We first recall the definition of a Weil functor \(T^A\) in a form generalizing the \((k, r)\)-velocities functor \(T^r_k\). Let \(\mathbb{R}[x_1, \ldots, x_k]\) be the algebra of all polynomials of \(k\) variables. A Weil ideal in \(\mathbb{R}[x_1, \ldots, x_k]\) is an arbitrary ideal \(A\) such that

\[
\langle x_1, \ldots, x_k \rangle^{r+1} \subset A \subset \langle x_1, \ldots, x_k \rangle^2
\]

where \(\langle x_1, \ldots, x_k \rangle \subset \mathbb{R}[x_1, \ldots, x_k]\) is the ideal of all polynomials without constant term and \(\langle x_1, \ldots, x_k \rangle^{r+1}\) is its \((r+1)\)th power, i.e. the ideal of all polynomials vanishing up to order \(r\) at 0. The factor algebra \(A = \mathbb{R}[x_1, \ldots, x_k]/A\) is then called the Weil algebra, the number \(k\) is said to be the width of \(A\) and the minimum of all \(r\)'s is called the depth of \(A\). If we replace \(\mathbb{R}[x_1, \ldots, x_k]\) by the algebra \(E(k)\) of all germs of smooth functions on \(\mathbb{R}^k\) at zero, then \(A\) generates an ideal \(\tilde{A} \subset E(k)\) and we have \(A = E(k)/\tilde{A}\) as well.

Let \(M\) be a manifold. Clearly, the jet space \(T^r_k M = J^r_0(\mathbb{R}^k, M)\) of all \(k\)-dimensional velocities of order \(r\) can also be defined as follows: Two maps \(g, h : \mathbb{R}^k \to M, g(0) = h(0) = x\), satisfy \(j^r_0 g = j^r_0 h\) if and only if

\[
\varphi \circ g - \varphi \circ h \in \langle x_1, \ldots, x_k \rangle^{r+1}
\]

for every germ \(\varphi \in C^\infty_x(M, \mathbb{R})\) of a smooth function on \(M\) at \(x\). The equivalence class of a mapping \(g : \mathbb{R}^k \to M\) is denoted by \(j^r_0 g\) and called the \(k\)-dimensional velocity of order \(r\). This algebraic definition of \(T^r_k M\) can be generalized in the following way.

\textbf{Definition.} Two maps \(g, h : \mathbb{R}^k \to M\) with \(g(0) = h(0) = x\) are said to be \(A\)-equivalent if for all germs \(\varphi \in C^\infty_x(M, \mathbb{R})\) we have \(\varphi \circ g - \varphi \circ h \in \tilde{A}\). The equivalence class of a mapping \(g : \mathbb{R}^k \to M\) will be denoted by \(j^A g\) and will be called the \(A\)-velocity of \(g\) at \(0\).

If we denote by \(T^A M\) the set of all \(A\)-velocities on \(M\), then \(T^A M\) is a fibered manifold over \(M\) with the projection \(p : T^A M \to M, p(j^A g) := g(0)\). It is easy to verify that \(T^A \mathbb{R} = A\). Further, for every \(f : M \to N\) we can define \(T^A f : T^A M \to T^A N\) by \(T^A f(j^A g) = j^A(f \circ g)\). Then
$T^A : \mathcal{M}f \to \mathcal{F}M$ is a functor from the category of all smooth manifolds and all smooth maps to the category of fibered manifolds, which is called the Weil functor corresponding to the Weil algebra $A$. For example, $\mathbb{D}_k^r = \mathbb{R}[x_1, \ldots, x_k] / \langle x_1, \ldots, x_k \rangle^{r+1}$ is the Weil algebra of the functor $T_k^r$. Then the tangent functor $T = T_1^1$ corresponds to $\mathbb{D} := \mathbb{D}_1^1 = \mathbb{R}[x] / \langle x^2 \rangle$, which is the algebra of dual numbers. Further, the tensor product $\mathbb{D} \otimes \mathbb{D}$ generates the iterated tangent functor $TT$. Now we briefly recall some important properties of Weil functors (see [10]).

(i) $T^A(M \times N) = T^A M \times T^A N$, so that the Weil functor $T^A$ preserves products. Conversely, every product preserving functor $F$ on $\mathcal{M}f$ is a Weil functor corresponding to the Weil algebra $A = FR$, i.e. $F = T^F R$.

(ii) The natural transformations $T^A \to T^B$ of two Weil functors are in a canonical bijection with the homomorphisms $A \to B$ of Weil algebras.

(iii) The iteration $T^A \circ T^B$ of two Weil functors is a Weil functor which corresponds to the tensor product $A \otimes B$ of the Weil algebras, i.e. $T^A(T^B M) = T^{A \otimes B} M$.

(iv) The exchange isomorphism $A \otimes B \to B \otimes A$ of Weil algebras induces a natural equivalence $\kappa : T^A \circ T^B \to T^B \circ T^A$, which generalizes the canonical involution of the second iterated tangent bundle $TTM$.

(v) There is an action of the elements of $A$ on the tangent vectors of $T^A M$, which can be introduced as follows. Let $\mu : \mathbb{R} \times TM \to TM$ be the multiplication of tangent vectors of $M$ by reals. Applying the functor $T^A$ we have $T^A \mu : A \times T^A TM \to T^A TM$. Using the exchange isomorphism $\kappa_M : TT^A M \to T^A TM$ we obtain the required action $A \times TT^A M \to TT^A M$.

3. Natural transformations $T^A T^* \to T^* T^A$. Let $A$ be a Weil algebra of width $k$. Given an arbitrary linear function $f : A \to \mathbb{R}$, we define a natural transformation $s_f : T^A T^* \to T^* T^A$ in the following way. Every $X \in T^A T^* M$ is an $A$-velocity $X = j^A g$, where $g : \mathbb{R}^k \to T^* M$. Denote by $q_M : T^* M \to M$, $p_M : T^A M \to M$ the bundle projections and by $\langle - , - \rangle : TM \times T^* M \to \mathbb{R}$ the evaluation mapping. Then $T^A q_M : T^A T^* M \to T^A M$, so that $v := T^A q_M(X) \in T^A M$. Take an arbitrary $Y \in T_v T^A M$. If $\kappa_M : TT^A M \to T^A TM$ is the canonical natural equivalence induced by the exchange isomorphism $\mathbb{D} \otimes A \to A \otimes \mathbb{D}$, then $\kappa_M(Y) \in T^A TM$ is an $A$-velocity of the form $\kappa_M(Y) = j^A h$ with $h : \mathbb{R}^k \to TM$. We have $(g, h) : \mathbb{R}^k \to \mathbb{R}$, $j^A ((g, h)) \in T^A \mathbb{R} = A$, so that $f \circ j^A ((g, h)) \in \mathbb{R}$. Now we can define a linear mapping $TT^A M \to \mathbb{R}$ by

$$Y \mapsto f \circ j^A ((g, h)).$$

Taking into account the identification of $T^* T^A M$ with linear maps $TT^A M \to \mathbb{R}$, we have constructed an element of $T^* T^A M$, which will be denoted by $(s_f)_M(X)$. Clearly, $s_f : T^A T^* \to T^* T^A$ is a natural transformation.
If $X \in T^AT^*M$, $v = T^Aq_M(X) \in T^AM$ and $Y \in T_vT^AM$, then $p_{T^*M}(X) \in T^*M$, $p_{TM}(\kappa_M(Y)) \in TM$. We have $\langle p_{T^*M}(X), p_{TM}(\kappa_M(Y)) \rangle \in \mathbb{R}$ and $T^A(\langle p_{T^*M}(X), p_{TM}(\kappa_M(Y)) \rangle) \in A$. Considering the identification of an element $(s_f)_M(X) \in T^*T^AM$ with a linear mapping $TT^AM \to \mathbb{R}$, we directly obtain

**Proposition 1.** Let $X \in T^AT^*M$, $v = T^Aq_M(X)$ and $Y \in T_vT^AM$. Then

$$(s_f)_M(X)(Y) = f \circ T^A(\langle p_{T^*M}(X), p_{TM}(\kappa_M(Y)) \rangle).$$

Denote by $S_A$ the space of all natural transformations $s_f : T^AT^* \to T^*T^A$ for linear functions $f : A \to \mathbb{R}$, i.e.

$$S_A = \{s_f : T^AT^* \to T^*T^A; \ f \in A^*\}.$$  

**Proposition 2.** $S_A$ is a vector space over $\mathbb{R}$ which is isomorphic to the dual vector space of $A$.

**Proof.** Let $s_f, s_g : T^AT^* \to T^*T^A$ be two natural transformations determined by linear functions $f, g : A \to \mathbb{R}$. For any $X \in T^AT^*M$ we have $q_{T^AM}(\langle (s_f)_M(X) \rangle) = q_{T^AM}(\langle (s_g)_M(X) \rangle)$, where $q_{T^AM} : T^*T^AM \to T^AM$ is the bundle projection. In this way we can define addition $(s_f + s_g)$ and multiplication by reals $(k \cdot s_f)$, $k \in \mathbb{R}$, by means of the corresponding operations on the vector bundle structure $T^*T^AM \to T^AM$. Obviously, the functions $f + g$ and $k \cdot f$, $k \in \mathbb{R}$, induce the natural transformations $s_f + s_g$ and $k \cdot s_f$, respectively.

**Example 1.** We describe a basis of the vector space $S_{D^1_r}$ of natural transformations $T^1TT^* \to T^*T^1$ depending on linear functions $D^1_r \to \mathbb{R}$. Consider some local coordinates $(x^i)$ on $M$ and denote by $(p_i)$ the additional coordinates on $T^*M$ and by $(y_1^i, \ldots, y_r^i)$ the additional coordinates on $T^1TT^*M$. Then the local coordinates on $T^1TT^*M$ are $(x^i, p_i, X_1^i, \ldots, X_r^i, P_i, \ldots, P_i, X_r^i)$. Further, using expressions $r_i dx^i + s_1^i dy_1^i + \ldots + s_r^i dy_r^i$, we have local coordinates $(x^i, y_1^i, \ldots, y_r^i, r_i, s_1^i, \ldots, s_r^i)$ on $T^*T^1TT^*M$. The Weil algebra of $T^r_1$ is $A = D^r_1 = \mathbb{R}[x]/\langle x^{r+1} \rangle$, so that elements of $D^r_1$ are of the form $a_0 + a_1 x + \ldots + a_r x^r$ and $\dim(D^r_1) = r + 1$. Consider now mappings $g : \mathbb{R} \to T^*M$ and $h : \mathbb{R} \to TM$ from the general definition of $s_f$. Using our local coordinates we obtain the coordinate form of $j^A(\langle g, h \rangle)$:

$$\langle g(t), h(t) \rangle|_0 = p_i dx^i,$$

$$\frac{d}{dt}\bigg|_0 \langle g(t), h(t) \rangle = P_{i,1} dx^i + p_i dX_1^i,$$

$$\ldots$$

$$\frac{d^r}{dt^r}\bigg|_0 \langle g(t), h(t) \rangle = \begin{pmatrix} r \cr 0 \end{pmatrix} P_{i,r} dx^i + \begin{pmatrix} r \cr 1 \end{pmatrix} P_{i,r-1} dX_1^i + \ldots + \begin{pmatrix} r \cr r \end{pmatrix} p_i dX_r^i.$$
In this way we have obtained $r + 1$ natural transformations $s_0, s_1, \ldots, s_r : T^*_1 T^* \to T^* T^*_1$ with coordinate forms

$$s_0 : r_i = p_i, \ s_1^\alpha = 0, \ldots, s^r_\gamma = 0,$$

$$s_1 : r_i = P_{i,1}, \ s_1^i = p_i, \ s_2^i = 0, \ldots, s^r_i = 0,$$

$$\ldots$$

$$s_r : r_i = P_{i,r}, \ s_1^i = \binom{r}{1} P_{i,r-1}, \ldots, s^{r-1}_i = \binom{r}{r-1} P_{i,1}, \ s^r_i = p_i.$$

We can see that every natural transformation $T^*_1 T^* \to T^* T^*_1$ and $s_0$ can be interpreted as a natural transformation $T^*_1 T^* \to T^* T^*_1$. To obtain a natural transformation $T^*_1 T^* \to T^* T^*_1$ from $T^*_1 T^* \to T^* T^*_1$, $k \leq r$ and from $T^*_1 T^* \to T^*$, we can use the inclusion

$$j_k : T^*_1 M \times_{T^*_1 M} T^* T^*_1 M \to T^* T^*_1 M, \quad 0 \leq k \leq r,$$

which is defined as follows. For $X \in T^*_1 M$ and $Y \in T^* T^*_1 M$ we have $j_k(X, Y) \in T^* T^*_1 M$, i.e. $j_k(X, Y) : T^*_1 M \to \mathbb{R}$. Taking an arbitrary $Z \in T_X T^*_1 M$ we put $j_k(X, Y) := (TP^r_M(Z), Y)$, where $TP^r_M : T^*_1 M \to T^*_1 M$ is the canonical projection.

**Example 2.** We show that the space $S_{D^k}$ of natural transformations $T^*_1 T^* \to T^* T^*_1$ is linearly generated by $k + 1$ natural transformations. The Weil algebra of $T^*_1$ is $D^1_k = \mathbb{R}[x_1, \ldots, x_k] / \langle x_1, \ldots, x_k \rangle^2$ with elements of the form $a_0 + a_1 x_1 + \ldots + a_k x_k$, so that $\dim(D^1_k) = k + 1$. Taking some local coordinates $(x^i)$ on $M$, we have the additional coordinates $(x^i_\alpha)$ on $T^*_1 M$, $\alpha = 1, \ldots, k$. Then the induced coordinates on $T^*_1 T^*_1 M$ are $(x^i, p_i, x^i_\alpha, p_i \alpha)$. Using expressions $r_i dx^i + s^\alpha_i dy^\alpha$ we obtain local coordinates $(x^i, y^\alpha_i, r_i, s^\alpha_i)$ on $T^* T^*_1 M$. For $g : \mathbb{R}^k \to T^* M$ and $h : \mathbb{R}^k \to TM$ we can write $\langle g(t_1, \ldots, t_k), h(t_1, \ldots, t_k) \rangle = p_i dx^i$ and $\frac{d}{dt_\gamma} \langle g, h \rangle = p_i \gamma dx^i + p_\gamma dx^i_\gamma$, $\gamma = 1, \ldots, k$. In this way we have obtained $k + 1$ natural transformations $s_0, s_1, \ldots, s_k : T^*_1 T^* \to T^* T^*_1$ with coordinate forms

$$s_0 : r_i = p_i, \ s_1^\alpha = 0 \quad \text{for all } \alpha = 1, \ldots, k,$$

$$s_\gamma : r_i = p_i \gamma, \ s_i^\gamma = p_i, \ s^\beta_i = 0 \quad \text{for all } \beta \neq \gamma, \ \gamma = 1, \ldots, k.$$

**Example 3.** The Weil algebra of the second iterated tangent functor $TT$ is $A = D \otimes D \cong \mathbb{R}[x_1, x_2] / \langle x_1^2, x_2^2 \rangle$ with elements $a + bx_1 + cx_2 + dx_1 x_2$. Since $\dim(A) = 4$, the vector space $S_A$ is linearly generated by four natural transformations.

**4. The existence of a natural equivalence $T^A T^* \to T^* T^A$.** The natural transformation $s_r : T^*_1 T^* \to T^* T^*_1$ from Example 1 is exactly the well known natural equivalence of Cauchy, Crampin, Sarlet and Saunders [1].
On the other hand, none of the natural transformations $s_0, \ldots, s_k : T_k^1 T^* \rightarrow T^* T_k^1$ from Example 2 is a natural equivalence. We first clarify under which conditions on a linear function $f : A \rightarrow \mathbb{R}$, the natural transformation $s_f : T^A T^* \rightarrow T^* T^A$ is an isomorphism. Given a linear function $f$ on the Weil algebra $A$, we have an associated symmetric bilinear mapping $\tilde{f} : A \times A \rightarrow \mathbb{R}$, $\tilde{f}(a, b) = f(a \cdot b)$. If we denote by $a_1, \ldots, a_p \in A$ a basis of $A$, the matrix $(a_{ij})$ of $\tilde{f}$ is defined as a real matrix with elements $a_{ij} = \tilde{f}(a_i, a_j)$.

**Definition.** A symmetric bilinear mapping $\varphi : A \times A \rightarrow \mathbb{R}$ is said to be nonsingular if the matrix of $\varphi$ is nonsingular.

Gancarzewicz, Mikulski and Pogoda [8] have studied relations between a product preserving functor $T^A$ and some operations on vector bundles. If $V$ is a free finite-dimensional $A$-module, then $V^*(A)$ denotes the $A$-module of all $A$-linear mappings $V \rightarrow A$. Analogously, if $\pi : E \rightarrow M$ is an $A$-module bundle, then the $A$-dual $A$-module bundle $E^*(A)$ is defined by $E^*(A) = \bigcup_{x \in M} E_{x}^*(A)$ (see [8]). By [8], every linear function $f : A \rightarrow \mathbb{R}$ defines a natural vector bundle homomorphism $\xi_E^f : E^*(A) \rightarrow E^*$, $\alpha \mapsto f \circ \alpha$. Moreover, this homomorphism is a vector bundle isomorphism if and only if the symmetric bilinear mapping $\tilde{f} : A \times A \rightarrow \mathbb{R}$, $\tilde{f}(a, b) = f(a \cdot b)$, associated with $f$ is nonsingular.

In our definition of $s_f : T^A T^* \rightarrow T^* T^A$, a linear function $f : A \rightarrow \mathbb{R}$ comes into play in (2), and $\xi_E^f$ from [8] is exactly the homomorphism $(T T^A M)^*(A) \rightarrow T^* T^A M$. Thus, Propositions 4.2 and 4.4 of [8] yield directly

**Proposition 3.** $s_f : T^A T^* \rightarrow T^* T^A$ is a natural equivalence if and only if the symmetric bilinear mapping $\tilde{f} : A \times A \rightarrow \mathbb{R}$, $\tilde{f}(a, b) = f(a \cdot b)$, is nonsingular.

Now we show that for $k \neq 1$ there is an obstruction to the existence of a natural equivalence $s_f : T_k^r T^* \rightarrow T^* T_k^r$.

**Proposition 4.** There is a natural equivalence $s_f : T_k^r T^* \rightarrow T^* T_k^r$ depending on a linear function $f : \mathbb{D}_k^r \rightarrow \mathbb{R}$ if and only if $k = 1$.

**Proof.** I. Consider first the case $k = 1$. We have $\mathbb{D}_1^r = \mathbb{R}[x]/\langle x^{r+1} \rangle$ and elements of $\mathbb{D}_1^r$ are of the form $a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r$. Hence the basis of $\mathbb{D}_1^r$ is $\{1, x, x^2, \ldots, x^r\}$ and multiplication in $\mathbb{D}_1^r$ has the form $(a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r) (b_0 + b_1 x + b_2 x^2 + \ldots + b_r x^r) = a_0 b_0 + \ldots + b_r x^r + a_1 x (b_0 + \ldots + b_{r-1} x^{r-1}) + \ldots + a_r x^r b_0$. If $f : \mathbb{D}_1^r \rightarrow \mathbb{R}$ is a linear function given by $f(a_0 + a_1 x + \ldots + a_r x^r) = a_r$, then the matrix of the associated symmetric bilinear function $\tilde{f}$ is
The corresponding natural equivalence \( s_f : T^r T^* \rightarrow T^* T^r \) is exactly \( s_r \) from Example 1, which is nothing else but the canonical isomorphism of Crampin, Sarlet and Saunders.

II. For \( r = 1 \) and any \( k \) we have \( A = \mathbb{D}_k^1 = \mathbb{R}[x_1, \ldots, x_k]/\langle x_1, \ldots, x_k \rangle^2 \) and multiplication in \( \mathbb{D}_k^1 \) has the form \((a_0+a_1 x_1 + \ldots + a_k x_k)(b_0+b_1 x_1 + \ldots + b_k x_k) = a_0(b_0+b_1 x_1 + \ldots + b_k x_k)+a_1 x_1 b_0 + a_2 x_2 b_0 + \ldots + a_k x_k b_0 \). If we denote by \( \{1, x_1, x_2, \ldots, x_k\} \) the basis of \( A \), the linear functions \( f_i : A \rightarrow \mathbb{R} \) defined by \( f_0(a_0+a_1 x_1 + \ldots + a_k x_k) = a_0, \ldots, f_k(a_0+a_1 x_1 + \ldots + a_k x_k) = a_k \) form a basis of \( A^* \). One finds easily that the matrix of each symmetric bilinear function \( f_0, f_1, \ldots, f_k \) is singular.

III. The Weil algebra of \( T^r \) is \( A = \mathbb{D}_k^r = \mathbb{R}[x_1, \ldots, x_k]/\langle x_1, \ldots, x_k \rangle^{r+1} \). Recall that a \( k \)-multiindex is a \( k \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_k) \) of nonnegative integers. We write \( |\alpha| = \alpha_1 + \ldots + \alpha_k \) and \( x^\alpha = (x_1^{\alpha_1}, \ldots, x_k^{\alpha_k}) \) for \( x = (x_1, \ldots, x_k) \) and \( x_i \in \mathbb{R} \). Then the elements of \( \mathbb{D}_k^r \) can be expressed in the form \( a_0 + a_\alpha x^\alpha \), where \( |\alpha| \leq r \) and \( a_0, a_\alpha \in \mathbb{R} \). If \( k > 1 \), then the basis of \( \mathbb{D}_k^r \) can be written as a set \( \{1, x^\alpha; |\alpha| \leq r\} \) and the corresponding dual basis is given by linear functions \( f_0, f_\alpha : \mathbb{D}_k^r \rightarrow \mathbb{R} \), \( f_0(a_0+a_\alpha x^\alpha) = a_0, f_\alpha(a_0+a_\alpha x^\alpha) = a_\alpha \), \( |\alpha| \leq r \). It is easy to verify that the matrix of each associated symmetric bilinear function \( \tilde{f}_0, \tilde{f}_\alpha \) is singular.

\[ \text{Proposition 5. For } k > 1 \text{ there is no natural equivalence } T^r T^* \rightarrow T^* T^r. \]

\textbf{Proof.} According to the general theory [10], natural transformations \( T^r T^* \rightarrow T^* T^r \) are in a canonical bijection with \( G^r m \)-equivariant maps of the corresponding standard fibers, where \( G^r_m \) means the group of all invertible \( r \)-jets of \( \mathbb{R}^m \) into \( \mathbb{R}^m \) with source and target zero. Denoting by \( (a^i_j) \) the canonical coordinates in \( G^1_m \), the coordinates of the inverse element will be denoted by \( (\tilde{a}^i_j) \). Further, denote by \( (x^i, p_i, x^i_\alpha, p_i_\alpha) \) the canonical coordinates on \( T^r_k T^* M \) and by \( (x^i, y^i_\alpha, r_i, s^\alpha_i) \) the canonical coordinates on \( T^* T^r_k M \), where \( \alpha \) is a \( k \)-multiindex with \( |\alpha| \leq r \). One calculates easily \( p_i = \tilde{a}^i_j p_j \) and

\[ (4) \quad x^i_\alpha = a^i_j x^j_\alpha + \ldots \]

Clearly, for \( |\alpha| = 1 \), the transformation law (4) is tensorial, while for \( |\alpha| > 1 \) there are terms with \( x^i_\beta \) on the right-hand side of (4), \( |\beta| < |\alpha| \). Analogously, \( \overline{y}^\alpha_i = a^i_j y^j_\alpha + \ldots \) and \( \overline{p}^\alpha_i = \tilde{a}^i_j p_j_\alpha + \ldots \) Finally, for all \( |\alpha| = r \) we find \( \Xi^\alpha_i = \tilde{a}^i_j s^\alpha_j \). This means that all \( s^\alpha_i \) with \( |\alpha| = r \) have a tensorial transformation.
law. On the other hand, among \((p_i, p_{i,\alpha})\) on the standard fibre \((T^*_k T^*)_0\), only \((p_i)\) have a tensorial transformation law. ■

5. Liftings of 1-forms and \((0, 2)\)-tensor fields to Weil bundles. In this section we investigate the role of natural transformations \(s_f : T^A T^* \to T^* T^A\) in the theory of lifting of 1-forms and \((0, 2)\)-tensor fields to Weil bundles. By a lifting of some tensor field \(G\) to a natural bundle \(F\) we understand a natural operator transforming the tensor field \(G\) on a manifold \(M\) into a tensor field of the same type on \(FM\).

Given a function \(\varphi : M \to \mathbb{R}\) and a function \(f : A \to \mathbb{R}\), we can define the \(f\)-lift \(\varphi^f : T^A M \to \mathbb{R}\) of \(\varphi\) to the bundle \(T^A M\) by \(\varphi^f := f \circ T^A \varphi\). Clearly, \(\varphi \mapsto \varphi^f\) defines a natural operator transforming functions on a manifold \(M\) into functions on \(T^A M\). If \(X : M \to TM\) is a vector field on \(M\), then \(T^A X : T^A M \to T^A TM\) and the composition \(T^A X := \kappa^{-1}_M \circ T^A X : T^A M \to TT^A M\) is a vector field on \(T^A M\). By [10], \(T^A X\) is exactly the flow prolongation of \(X\), it is also called the complete lift.

Let \(\omega : M \to T^* M\) be a 1-form on \(M\). Using the natural transformation \(s_f\) determined by a linear function \(f : A \to \mathbb{R}\), we can also define the \(f\)-lift of \(\omega\) to \(T^A M\). Indeed, \(T^A \omega : T^A M \to T^A T^* M\) and the composition with the natural transformation \((s_f)_M : T^A T^* M \to T^* T^A M\) gives rise to a 1-form \(\omega^f\) on \(T^A M\),

\[
\omega^f := (s_f)_M \circ T^A \omega : T^A M \to T^* T^A M.
\]

The \(f\)-lift of an evaluation mapping \(\langle \omega, X \rangle : M \to \mathbb{R}\) is a function \(\langle \omega, X \rangle^f : T^A M \to \mathbb{R}\). We have

**Proposition 6.** \(\langle \omega^f, T^A X \rangle = \langle \omega, X \rangle^f\).

**Proof.** Using Proposition 1 we obtain \(\langle \omega^f, T^A X \rangle = \langle (s_f)_M \circ T^A \omega, \kappa^{-1}_M \circ T^A X \rangle = f \circ T^A (\langle \omega, X \rangle)\) which is nothing else but \(\langle \omega, X \rangle^f\). ■

We remark that this formula has been proved in the particular case \(A = \mathbb{D}^2_1\) in [3].

A 1-form \(\omega : M \to T^* M\) on \(M\) can also be identified with a linear mapping \(\tilde{\omega} : TM \to \mathbb{R}\), \(\tilde{\omega}(X) = \langle \omega, X \rangle\). If \(f : A \to \mathbb{R}\) is a linear function on \(A\), then the map \(\tilde{\Omega} := f \circ T^A \tilde{\omega} \circ \kappa_M : TT^A M \to \mathbb{R}\) is linear, so that \(\tilde{\Omega}\) induces a 1-form \(\Omega : T^A M \to T^* T^A M\) on \(T^A M\). On the other hand, \(\omega^f := (s_f)_M \circ T^A \omega\) from (5) is also a 1-form on \(T^A M\). We have

**Proposition 7.** \(\Omega = \omega^f\).

**Proof.** Recall that there is a canonical action \(A \times TT^A M \to TT^A M\). If \(X\) is a vector field on \(M\) and \(a \in A\), then we can introduce the \(a\)-lift \(X^{(a)} : T^A M \to TT^A M\) of \(X\) to \(T^A M\) by \(X^{(a)} := a \cdot T^A M\). From [7] it follows that if \(G\) and \(H\) are two tensor fields of type \((0, k)\) or \((1, k)\) on
$T^A M$ satisfying $G(X_1^{(a_1)}, \ldots, X_k^{(a_k)}) = H(X_1^{(a_1)}, \ldots, X_k^{(a_k)})$ for all vector fields $X_1, \ldots, X_k$ on $M$ and all elements $a_1, \ldots, a_k$ from $A$, then $G = H$. By Proposition 6 we obtain natural equivalence $T^A X = f \circ T^A \bar{\omega} \circ (\kappa_M^{-1} \circ T^A X) = f \circ T^A \bar{\omega} \circ T^A X = f \circ T^A (\bar{\omega}(X)) = f \circ T^A (\langle \omega, X \rangle) = \langle \omega, X \rangle^f = \langle \omega^f, T^A X \rangle$. Using the $A$-linearity of both $f$ and $T^A \bar{\omega}$ we directly obtain $\bar{\omega}(X^{(a)}) = \langle \omega^f, X^{(a)} \rangle$ for all $a \in A$. ■

A $(0, 2)$-tensor field on $M$ can be interpreted as a linear mapping $G : TM \times_M TM \to \mathbb{R}$. Using the exchange isomorphism $\kappa_M : T^2 A M \to T^A TM$ and a linear function $f : A \to \mathbb{R}$, Gancarzewicz, Mikulski and Pogoda [7] introduced an $f$-lift $G^f$ of $G$ to the bundle $T^A M$ by

$$G^f := f \circ T^A G \circ (\kappa_M \times \kappa_M): T^2 A M \times_{T^A M} T^A M \to \mathbb{R}. $$

Further, each $(0, 2)$-tensor field $G$ on $M$ induces a linear mapping $G_L : TM \to T^* M$ by $\langle G_L(y), z \rangle = G(z, y), y, z \in T_x M$. If $G$ is a symplectic form on $M$, then $G_L$ is an isomorphism. Denote by $G^f_L : T^2 A M \to T^* T^A M$ the linear mapping corresponding to the $f$-lift $G^f$ of $G$.

**Proposition 8.** $G^f_L : T^2 A M \to T^* T^A M$ is of the form $G^f_L = (s_f)_M \circ T^A G_L \circ \kappa_M$.

**Proof.** Clearly,

$$G^f(T^A X, T^A Y) = f \circ T^A G \circ (\kappa_M \times \kappa_M)(T^A X, T^A Y)$$

$$= f \circ T^A G(T^A X, T^A Y)$$

$$= f \circ T^A (G(X, Y)) = (G(X, Y))^f. $$

Analogously to the proof of Proposition 7 we have

$$\langle (s_f)_M \circ T^A G_L \circ \kappa_M \rangle(T^A Y), T^A X \rangle = \langle (s_f)_M \circ T^A (G_L \circ Y), \kappa_M^{-1} \circ T^A X \rangle$$

$$= \langle G_L(Y), X \rangle^f = (G(X, Y))^f$$

$$= G^f(T^A X, T^A Y).$$

On the other hand, $\langle G^f_L(T^A Y), T^A X \rangle = G^f(T^A X, T^A Y)$. ■

We remark that the above assertion has been proved in [5] for $A = \mathbb{D}$. By [7], if $\omega$ is a 2-form on $M$, then $d \omega^f = (d \omega)^f$. We have

**Corollary.** Let $\omega = dp_i \wedge dx^i$ be the canonical symplectic form on $T^* M$ and $\omega^f$ be the $f$-lift of $\omega$ to $T^A(T^* M)$. If $s_f : T^2 A T^* \to T^* T^A$ is a natural equivalence, then $\omega^f$ is a symplectic form on $T^A T^* M$.

**6. General description of some natural transformations** $T^A T^* \to T^* T^A$. In this section we show that for some particular cases of a Weil algebra $A$, the space of all natural transformations $T^A T^* \to T^* T^A$ can be
characterized by means of a general geometrical description. It is our belief that this description works also for many other Weil algebras.

**Definition.** A natural function $g$ on a natural bundle $F$ is defined as a system of functions $g_M : FM \to M$ for any $m$-dimensional manifold $M$ satisfying $g_M = g_N \circ Ff$ for every local diffeomorphism $f : M \to N$. A natural (or absolute) vector field $X$ on $F$ is a system of vector fields $X_M : FM \to TFM$ for every $m$-dimensional manifold $M$ satisfying $TFf \circ X_M = X_N \circ Ff$ for every local diffeomorphism $f : M \to N$.

On the other hand, the space of all natural transformations from $T^AT^*$ into $T^*T^A$ is a $C^\infty(T^AT^*)$-module.

**Remark 1.** By the general theory [10], absolute vector fields on $T^AM$ correspond to one-parameter groups of natural transformations of $T^A$ into itself. In particular, the natural transformations $T^k_k \to T^p_p$ are in bijection with the elements of $J_0^1(\mathbb{R}^k, \mathbb{R}^k)_0$ and each of them has the form of a reparametrization $X \mapsto X \circ P$, $X \in T^k_k$, $P \in J_0^1(\mathbb{R}^k, \mathbb{R}^k)_0$. For example, all natural transformations of $TM$ into itself are homotheties $X \mapsto kX$, $k \in \mathbb{R}$, and the vector field tangent to them is the classical Liouville vector field. In the case of an arbitrary Weil functor $T^A$, denote by $\text{Aut}(A)$ the Lie algebra associated with the Lie group of all algebra automorphisms of the Weil algebra $A$. In [10] it is proved that all absolute vector fields on $T^AM$ are the generalized Liouville vector fields determined by all elements $D \in \text{Aut}(A)$.

**Remark 2.** We remark that the problem of finding all natural functions on $T^*T^A$ for an arbitrary Weil algebra $A$ is rather complicated. First, Kolář [9] has determined all natural functions on $T^*T^1_1$. Recently Tomáš [14] has described all natural functions on $T^*T^A$ for some particular cases of $A$.

**Example 4.** We describe all natural functions on $T^r_1T^*$. Denote by $L$ the generalized Liouville vector field on $T^r_1M$ induced by the reparametrizations $x(t) \mapsto x(kt)$, $0 \neq k \in \mathbb{R}$, of a curve $x : \mathbb{R} \to M$. By Kolář [9], all absolute vector fields on $T^r_1$ are linearly generated by $L_1 = L$, $L_2 = Q \circ L$, $\ldots$, $L_r = Q^{r-1} \circ L$, where $Q : TT^1_1 \to TT^r_1$ is a natural linear morphism (affinor) defined by de León and Rodrigues [2], whose coordinate expression is $(dx^1, dy^1_1, dy^2_2, \ldots, dy^r_r) \mapsto (0, dx^1, dy^1_1, \ldots, dy^r_{r-1})$. Let $s_r : TT^*T^* \to T^*T^r_1M$ be the natural equivalence from Example 1. Denoting by $q_M : T^*M \to M$ the bundle projection, we have $q_{TT^1_1M}(s_r(Y)) \in T^r_1M$ for all $Y \in TT^1_1T^*M$. Then every absolute vector field $L_i$ determines a natural function $\varphi_i : T^1_1T^r_1M \to \mathbb{R}$,

$$\varphi_i(Y) = \langle s_r(Y), L_i(q_{TT^1_1M}(s_r(Y))) \rangle.$$
By [9], all natural functions on $T_1^*T^*$ are of the form $\varphi(\varphi_1, \ldots, \varphi_r)$, where $\varphi : \mathbb{R}^r \rightarrow \mathbb{R}$ is an arbitrary smooth function of $r$ variables.

In general, let $(s_f)_M : T^A T^* M \rightarrow T^* T^A M$ be a natural transformation induced by a linear function $f : A \rightarrow \mathbb{R}$. For $Y \in T^A T^* M$ we have $q_{T^A M}((s_f)_M(Y)) \in T^A M$ and each absolute vector field $X : T^A M \rightarrow TT^A M$ on $T^A M$ determines a natural function $\varphi_{X,f} : T^A T^* M \rightarrow \mathbb{R}$ by

$$\varphi_{X,f}(Y) = \langle (s_f)_M(Y), X(q_{T^A M}((s_f)_M(Y))) \rangle.$$  

Denote by $\varphi_1, \ldots, \varphi_l$ all such functions determined by all functions $f \in A^*$ and all absolute vector fields $X$ on $T^A M$ and let $\varphi : \mathbb{R}^l \rightarrow \mathbb{R}$ be an arbitrary smooth function. We have

**Proposition 9.** Let $A = \mathbb{D}$ or $A = \mathbb{D} \otimes \mathbb{D}$ or $A = \mathbb{D}_k^1$ or $A = \mathbb{D}_k^1$. Then all natural functions on $T^A T^*$ are of the form $\varphi(\varphi_1, \ldots, \varphi_l)$.

**Proof.** For $A = \mathbb{D}_k^1$ and $A = \mathbb{D}$ this follows from Example 4. Consider now $A = \mathbb{D}_k^1$ and write equations of all natural functions on $T^A M$. If $(x^i, p_i, x^i_\alpha, p_i, \alpha = 1, \ldots, k)$ are the canonical coordinates on $T^A T^* M$, then all natural functions on $T^A T^* M$ are of the form $\varphi(p_i x^i_\alpha, \alpha = 1, \ldots, k)$ with $\varphi : \mathbb{R}^k \rightarrow \mathbb{R}$ being any smooth function (see [6]). On the other hand, by Remark 1 we find easily the coordinate form of absolute vector fields on $T^A M$, $L^i_\alpha = x^i_\alpha \partial / \partial x^i_\alpha$, $\alpha, \beta = 1, \ldots, k$. Now the assertion for $A = \mathbb{D}_k^1$ follows from Example 2. For $A = \mathbb{D} \otimes \mathbb{D}$ all natural functions on $TTT^*$ are determined in [4] and the rest of the proof is quite similar to that for $A = \mathbb{D}_k^1$.

Finally we describe all natural transformations $T^A T^* \rightarrow T^* T^A$ for some particular cases of $A$ by means of a simple and universal geometrical construction. We will proceed in the following steps.

I. Denote by $\mathcal{B}$ the basis of $A^*$. For every $f \in \mathcal{B}$ we have a natural transformation $s_f : T^A T^* \rightarrow T^* T^A$.

II. Let $\text{Tr}(A)$ be the space of all natural transformations $T^A M \rightarrow T^A M$.

III. Let $\text{V}(A)$ be the space of all absolute vector fields on $T^A M$ (see Remark 1).

IV. Natural transformations $s_f$ from I and absolute vector fields $X \in \text{V}(A)$ from III determine natural functions $\varphi_{X,f} : T^A T^* M \rightarrow \mathbb{R}$ (see (6)). If $\varphi_1, \ldots, \varphi_l$ are all such functions for all $f \in \mathcal{B}$ and all absolute vector fields on $T^A M$, then $\varphi(\varphi_1, \ldots, \varphi_l)$ is a natural function on $T^A T^* M$ for each smooth function $\varphi : \mathbb{R}^l \rightarrow \mathbb{R}$.

V. Let $s_1, \ldots, s_r : T^A T^* \rightarrow T^* T^A$ be a basis of the vector space $S_\text{A}$ (see Proposition 2). Write $s := k_1 s_1 + \ldots + k_r s_r$, where on the right-hand side we have the sum in the vector bundle structure $T^* T^A M \rightarrow T^A M$ and $k_i : T^A T^* M \rightarrow \mathbb{R}$ are natural functions from IV of the form $\varphi(\varphi_1, \ldots, \varphi_l)$.  


VI. All natural transformations $T^A N \rightarrow T^A T^* N$ from $N = T^* M$ determine a system of natural transformations $T^A T^* M \rightarrow T^A T^* M$ over the identity of $T^* M$. This system depends on certain real parameters. If we replace them by arbitrary natural functions $\varphi(\varphi_1, \ldots, \varphi_l) : T^A T^* M \rightarrow \mathbb{R}$, we obtain a new system $\tau$ of natural transformations $T^A T^* M \rightarrow T^A T^* M$.

VII. Write

$$t := s \circ \tau : T^A T^* M \rightarrow T^* T^A M.$$  

**Proposition 10.** Let $A = D$ or $A = D \otimes D$ or $A = D^2_1$ or $A = D^1_k$. Then all natural transformations $T^A T^* \rightarrow T^* T^A$ are of the form (7).

**Proof.** Consider first $A = D^1_k$ and denote by $(x^i, y^i_\alpha, r_i, s^\alpha_i, \alpha = 1, \ldots, k)$ the canonical coordinates on $T^1_k T^* M$. By [6], the coordinate form of all natural transformations $T^1_k T^* \rightarrow T^* T^1_k$ is $y^i_\alpha = A^\alpha_\beta x^\beta_i$, $s^\alpha_i = B^\alpha p_i$ and $r_i = A^\beta_\alpha B^\alpha_{i, \beta} + C p_i$. Clearly, this is the coordinate form of $t$ described in item VII. For $A = D$ the assertion follows from [11], for $A = D \otimes D$ from [4] and finally for $A = D^2_1$ from [3].

**References**


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