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Natural transformations of the composition of Weil and cotangent functors

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Abstract. We study geometrical properties of natural transformations $T^A T^* \to T^* T^A$ depending on a linear function defined on the Weil algebra A. We show that for many particular cases of A, all natural transformations $T^A T^* \to T^* T^A$ can be described in a uniform way by means of a simple geometrical construction.

1. Introduction. By Tulczyjew [15], and Modugno and Stefani [13], there is a natural equivalence $TT^* \to T^*T$ of second order tangent and cotangent functors. All natural transformations of this type were determined by Kolář and Radziszewski [11]. The tangent functor T is a particular case of the functor T^r_k of k-dimensional velocities of order r, which is defined by

$$T^r_k M = J^r_0(\mathbb{R}^k, M), \quad T^r_k f(j^r_0 g) = j^r_0(f \circ g) \tag{1}$$

for all smooth manifolds M and all smooth maps $f : M \to N$. Then Cantrijn, Crampin, Sarlet and Saunders [1] introduced a canonical natural equivalence $T^r_1 T^* \to T^* T^r_1$, which can be considered as a generalization of the natural equivalence $TT^* \to T^*T$. In [3] we have classified all natural transformations $T^2_1 T^* \to T^* T^2_1$ and in [4] we have determined all natural transformations $TTT^* \to TT^*T$, which is a similar problem.

In general, let T^A be a Weil functor corresponding to a Weil algebra A. In the jet-like approach, a Weil functor T^A can be interpreted as a generalization of the (k, r)-velocities functor T^r_k. By [10], Weil functors even represent a general model of all product preserving bundle functors. The aim of this paper is to study natural transformations $T^A T^* \to T^* T^A$. We first define natural transformations $s_f : T^A T^* \to T^* T^A$ depending on linear functions $f : A \to \mathbb{R}$ and describe some geometrical properties of such natural transformations. In particular, we discuss the role of s_f in the theory of

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lifting of 1-forms and $(0,2)$-tensor fields to Weil bundles. We also consider the existence of a natural equivalence $T^A T^* \to T^* T^A$. Finally we construct a fairly general model of natural transformations $T^A T^* \to T^* T^A$, which simply characterizes all such natural transformations for some particular cases of the Weil algebra A.

We remark that natural transformations $T_1^r T^* \to T^* T_1^r$ are of fundamental importance in analytical mechanics [2], and a natural equivalence of this type enables us to introduce a symplectic structure on $T_1^r T^* M$. In what follows we will use the theory of natural operations in differential geometry from [10]. All maps and manifolds are assumed to be infinitely differentiable.

2. Weil functors. We first recall the definition of a Weil functor T^A in a form generalizing the (k,r)-velocities functor T_k^r. Let $\mathbb{R}[x_1,\dots,x_k]$ be the algebra of all polynomials of k variables. A *Weil ideal* in $\mathbb{R}[x_1,\dots,x_k]$ is an arbitrary ideal $\mathcal{A}$ such that

$$\langle x_1,\dots,x_k\rangle^{r+1} \subset \mathcal{A} \subset \langle x_1,\dots,x_k\rangle^2$$

where $\langle x_1,\dots,x_k\rangle \subset \mathbb{R}[x_1,\dots,x_k]$ is the ideal of all polynomials without constant term and $\langle x_1,\dots,x_k\rangle^{r+1}$ is its $(r+1)$th power, i.e. the ideal of all polynomials vanishing up to order r at 0. The factor algebra $A = \mathbb{R}[x_1,\dots,x_k]/\mathcal{A}$ is then called the *Weil algebra*, the number k is said to be the *width* of A and the minimum of all r's is called the *depth* of A. If we replace $\mathbb{R}[x_1,\dots,x_k]$ by the algebra $E(k)$ of all germs of smooth functions on $\mathbb{R}^k$ at zero, then $\mathcal{A}$ generates an ideal $\widetilde{\mathcal{A}} \subset E(k)$ and we have $A = E(k)/\widetilde{\mathcal{A}}$ as well.

Let M be a manifold. Clearly, the jet space $T_k^r M = J_0^r(\mathbb{R}^k, M)$ of all k-dimensional velocities of order r can also be defined as follows: Two maps $g,h : \mathbb{R}^k \to M$, $g(0) = h(0) = x$, satisfy $j_0^r g = j_0^r h$ if and only if

$$\varphi \circ g - \varphi \circ h \in \langle x_1,\dots,x_k\rangle^{r+1}$$

for every germ $\varphi \in C_x^\infty(M,\mathbb{R})$ of a smooth function on M at x. The equivalence class of a mapping $g : \mathbb{R}^k \to M$ is denoted by $j_0^r g$ and called the *k-dimensional velocity* of order r. This algebraic definition of $T_k^r M$ can be generalized in the following way.

Definition. Two maps $g,h : \mathbb{R}^k \to M$ with $g(0) = h(0) = x$ are said to be *A-equivalent* if for all germs $\varphi \in C_x^\infty(M,\mathbb{R})$ we have $\varphi \circ g - \varphi \circ h \in \widetilde{\mathcal{A}}$. The equivalence class of a mapping $g : \mathbb{R}^k \to M$ will be denoted by $j^A g$ and will be called the *A-velocity* of g at 0.

If we denote by $T^A M$ the set of all A-velocities on M, then $T^A M$ is a fibered manifold over M with the projection $p : T^A M \to M$, $p(j^A g) := g(0)$. It is easy to verify that $T^A \mathbb{R} = A$. Further, for every $f : M \to N$ we can define $T^A f : T^A M \to T^A N$ by $T^A f(j^A g) = j^A(f \circ g)$. Then

$T^A : \mathcal{M}f \to \mathcal{FM}$ is a functor from the category of all smooth manifolds and all smooth maps to the category of fibered manifolds, which is called the *Weil functor* corresponding to the Weil algebra A. For example, $\mathbb{D}_k^r = \mathbb{R}[x_1, \dots, x_k] / \langle x_1, \dots, x_k \rangle^{r+1}$ is the Weil algebra of the functor T_k^r. Then the tangent functor $T = T_1^1$ corresponds to $\mathbb{D} := \mathbb{D}_1^1 = \mathbb{R}[x] / \langle x^2 \rangle$, which is the algebra of dual numbers. Further, the tensor product $\mathbb{D} \otimes \mathbb{D}$ generates the iterated tangent functor TT. Now we briefly recall some important properties of Weil functors (see [10]).

(i) $T^A(M \times N) = T^A M \times T^A N$, so that the Weil functor T^A preserves products. Conversely, every product preserving functor F on $\mathcal{M}f$ is a Weil functor corresponding to the Weil algebra $A = F\mathbb{R}$, i.e. $F = T^{F\mathbb{R}}$.

(ii) The natural transformations $T^A \to T^B$ of two Weil functors are in a canonical bijection with the homomorphisms $A \to B$ of Weil algebras.

(iii) The iteration $T^A \circ T^B$ of two Weil functors is a Weil functor which corresponds to the tensor product $A \otimes B$ of the Weil algebras, i.e. $T^A(T^B M) = T^{A \otimes B} M$.

(iv) The exchange isomorphism $A \otimes B \to B \otimes A$ of Weil algebras induces a natural equivalence $\kappa : T^A \circ T^B \to T^B \circ T^A$, which generalizes the canonical involution of the second iterated tangent bundle TTM.

(v) There is an action of the elements of A on the tangent vectors of $T^A M$, which can be introduced as follows. Let $\mu : \mathbb{R} \times TM \to TM$ be the multiplication of tangent vectors of M by reals. Applying the functor T^A we have $T^A \mu : A \times T^A TM \to T^A TM$. Using the exchange isomorphism $\kappa_M : TT^A M \to T^A TM$ we obtain the required action $A \times TT^A M \to TT^A M$.

3. Natural transformations $T^A T^* \to T^* T^A$. Let A be a Weil algebra of width k. Given an arbitrary linear function $f : A \to \mathbb{R}$, we define a natural transformation $s_f : T^A T^* \to T^* T^A$ in the following way. Every $X \in T^A T^* M$ is an A-velocity $X = j^A g$, where $g : \mathbb{R}^k \to T^* M$. Denote by $q_M : T^* M \to M$, $p_M : T^A M \to M$ the bundle projections and by $\langle -, - \rangle : TM \times T^* M \to \mathbb{R}$ the evaluation mapping. Then $T^A q_M : T^A T^* M \to T^A M$, so that $v := T^A q_M(X) \in T^A M$. Take an arbitrary $Y \in T_v T^A M$. If $\kappa_M : TT^A M \to T^A TM$ is the canonical natural equivalence induced by the exchange isomorphism $\mathbb{D} \otimes A \to A \otimes \mathbb{D}$, then $\kappa_M(Y) \in T^A TM$ is an A-velocity of the form $\kappa_M(Y) = j^A h$ with $h : \mathbb{R}^k \to TM$. We have $\langle g, h \rangle : \mathbb{R}^k \to \mathbb{R}$, $j^A(\langle g, h \rangle) \in T^A \mathbb{R} = A$, so that $f \circ j^A(\langle g, h \rangle) \in \mathbb{R}$. Now we can define a linear mapping $TT^A M \to \mathbb{R}$ by

$$Y \mapsto f \circ j^A(\langle g, h \rangle). \qquad (2)$$

Taking into account the identification of $T^* T^A M$ with linear maps $TT^A M \to \mathbb{R}$, we have constructed an element of $T^* T^A M$, which will be denoted by $(s_f)_M(X)$. Clearly, $s_f : T^A T^* \to T^* T^A$ is a natural transformation.

If $X \in T^AT^*M$, $v = T^Aq_M(X) \in T^AM$ and $Y \in T_vT^AM$, then $p_{T^*M}(X) \in T^*M$, $p_{TM}(\kappa_M(Y)) \in TM$. We have $\langle p_{T^*M}(X), p_{TM}(\kappa_M(Y))\rangle \in \mathbb{R}$ and $T^A\left(\langle p_{T^*M}(X), p_{TM}(\kappa_M(Y))\rangle\right) \in A$. Considering the identification of an element $(s_f)_M(X) \in T^*T^AM$ with a linear mapping $TT^AM \to \mathbb{R}$, we directly obtain

Proposition 1. *Let $X \in T^AT^*M$, $v = T^Aq_M(X)$ and $Y \in T_vT^AM$. Then*

$$(s_f)_M(X)(Y) = f \circ T^A\left(\langle p_{T^*M}(X), p_{TM}(\kappa_M(Y))\rangle\right).$$

Denote by S_A the space of all natural transformations $s_f : T^AT^* \to T^*T^A$ for linear functions $f : A \to \mathbb{R}$, i.e.

$$S_A = \{s_f : T^AT^* \to T^*T^A;\ f \in A^*\}. \tag{3}$$

Proposition 2. *S_A is a vector space over $\mathbb{R}$ which is isomorphic to the dual vector space of A.*

Proof. Let $s_f, s_g : T^AT^* \to T^*T^A$ be two natural transformations determined by linear functions $f, g : A \to \mathbb{R}$. For any $X \in T^AT^*M$ we have $q_{T^AM}\left((s_f)_M(X)\right) = q_{T^AM}\left((s_g)_M(X)\right)$, where $q_{T^AM} : T^*T^AM \to T^AM$ is the bundle projection. In this way we can define addition $(s_f + s_g)$ and multiplication by reals $(k \cdot s_f)$, $k \in \mathbb{R}$, by means of the corresponding operations on the vector bundle structure $T^*T^AM \to T^AM$. Obviously, the functions $f + g$ and $k \cdot f$, $k \in \mathbb{R}$, induce the natural transformations $s_f + s_g$ and $k \cdot s_f$, respectively. ∎

Example 1. We describe a basis of the vector space $S_{\mathbb{D}_1^r}$ of natural transformations $T_1^rT^* \to T^*T_1^r$ depending on linear functions $\mathbb{D}_1^r \to \mathbb{R}$. Consider some local coordinates (x^i) on M and denote by (p_i) the additional coordinates on T^*M and by $(y_1^i, \dots, y_r^i)$ the additional coordinates on T_1^rM. Then the local coordinates on $T_1^rT^*M$ are $(x^i, p_i, X_1^i, \dots, X_r^i, P_{i,1}, \dots, P_{i,r})$. Further, using expressions $r_idx^i + s_i^1dy_1^i + \dots + s_i^rdy_r^i$ we have local coordinates $(x^i, y_1^i, \dots, y_r^i, r_i, s_i^1, \dots, s_i^r)$ on $T^*T_1^rM$. The Weil algebra of T_1^r is $A = \mathbb{D}_1^r = \mathbb{R}[x]/\langle x^{r+1}\rangle$, so that elements of $\mathbb{D}_1^r$ are of the form $a_0 + a_1x + \dots + a_rx^r$ and $\dim(\mathbb{D}_1^r) = r + 1$. Consider now mappings $g : \mathbb{R} \to T^*M$ and $h : \mathbb{R} \to TM$ from the general definition of s_f. Using our local coordinates we obtain the coordinate form of $j^A(\langle g, h\rangle)$:

$$\begin{aligned}
\langle g(t), h(t)\rangle|_0 &= p_idx^i, \\
\left.\frac{d}{dt}\right|_0 \langle g(t), h(t)\rangle &= P_{i,1}dx^i + p_idX_1^i, \\
&\dots \\
\left.\frac{d^r}{dt^r}\right|_0 \langle g(t), h(t)\rangle &= \binom{r}{0}P_{i,r}dx^i + \binom{r}{1}P_{i,r-1}dX_1^i + \dots + \binom{r}{r}p_idX_r^i.
\end{aligned}$$

In this way we have obtained $r+1$ natural transformations $s_0, s_1, \dots, s_r : T_1^r T^* \to T^* T_1^r$ with coordinate forms

$$s_0 : r_i = p_i,\ s_i^1 = 0, \dots, s_i^r = 0,$$
$$s_1 : r_i = P_{i,1},\ s_i^1 = p_i,\ s_i^2 = 0, \dots, s_i^r = 0,$$
$$\dots$$
$$s_r : r_i = P_{i,r},\ s_i^1 = \binom{r}{1} P_{i,r-1}, \dots, s_i^{r-1} = \binom{r}{r-1} P_{i,1},\ s_i^r = p_i.$$

We can see that every s_k, $1 \le k \le r$, can also be interpreted as a natural transformation $T_1^r T^* \to T^* T_1^k$ and s_0 can be interpreted as a natural transformation $T_1^r T^* \to T^*$. To obtain a natural transformation $T_1^r T^* \to T^* T_1^r$ from $T_1^r T^* \to T^* T_1^k$, $k \le r$ and from $T_1^r T^* \to T^*$, we can use the inclusion

$$j_k : T_1^r M \times_{T_1^k M} T^* T_1^k M \to T^* T_1^r M, \quad 0 \le k \le r,$$

which is defined as follows. For $X \in T_1^r M$ and $Y \in T^* T_1^k M$ we have $j_k(X,Y) \in T^* T_1^r M$, i.e. $j_k(X,Y) : T T_1^r M \to \mathbb{R}$. Taking an arbitrary $Z \in T_X T_1^r M$ we put $j_k(X,Y) := \langle T p_M^{r,k}(Z), Y \rangle$, where $p_M^{r,k} : T_1^r M \to T_1^k M$ is the canonical projection.

Example 2. We show that the space $S_{\mathbb{D}_k^1}$ of natural transformations $T_k^1 T^* \to T^* T_k^1$ is linearly generated by $k+1$ natural transformations. The Weil algebra of T_k^1 is $\mathbb{D}_k^1 = \mathbb{R}[x_1, \dots, x_k] / \langle x_1, \dots, x_k \rangle^2$ with elements of the form $a_0 + a_1 x_1 + \dots + a_k x_k$, so that $\dim(\mathbb{D}_k^1) = k+1$. Taking some local coordinates (x^i) on M, we have the additional coordinates (x_α^i) on $T_k^1 M$, $\alpha = 1, \dots, k$. Then the induced coordinates on $T_k^1 T^* M$ are $(x^i, p_i, x_\alpha^i, p_{i,\alpha})$. Using expressions $r_i dx^i + s_i^\alpha dy_\alpha^i$ we obtain local coordinates $(x^i, y_\alpha^i, r_i, s_i^\alpha)$ on $T^* T_k^1 M$. For $g : \mathbb{R}^k \to T^* M$ and $h : \mathbb{R}^k \to TM$ we can write $\langle g(t_1, \dots, t_k), h(t_1, \dots, t_k) \rangle|_0 = p_i dx^i$ and $\frac{d}{dt_\gamma}\Big|_0 \langle g, h \rangle = p_{i,\gamma} dx^i + p_i dx_\gamma^i$, $\gamma = 1, \dots, k$. In this way we have obtained $k+1$ natural transformations $s_0, s_1, \dots, s_k : T_k^1 T^* \to T^* T_k^1$ with coordinate forms

$$s_0 : r_i = p_i,\ s_i^\alpha = 0 \quad \text{for all } \alpha = 1, \dots, k,$$
$$s_\gamma : r_i = p_{i,\gamma},\ s_i^\gamma = p_i,\ s_i^\beta = 0 \quad \text{for all } \beta \ne \gamma,\ \gamma = 1, \dots, k.$$

Example 3. The Weil algebra of the second iterated tangent functor TT is $A = \mathbb{D} \otimes \mathbb{D} \cong \mathbb{R}[x_1, x_2] / \langle x_1^2, x_2^2 \rangle$ with elements $a + b x_1 + c x_2 + d x_1 x_2$. Since $\dim(A) = 4$, the vector space S_A is linearly generated by four natural transformations.

4. The existence of a natural equivalence $T^A T^* \to T^* T^A$. The natural transformation $s_r : T_1^r T^* \to T^* T_1^r$ from Example 1 is exactly the well known natural equivalence of Cantrijn, Crampin, Sarlet and Saunders [1].

On the other hand, none of the natural transformations $s_0, \dots, s_k : T^1_k T^* \to T^* T^1_k$ from Example 2 is a natural equivalence. We first clarify under which conditions on a linear function $f : A \to \mathbb{R}$, the natural transformation $s_f : T^A T^* \to T^* T^A$ is an isomorphism. Given a linear function f on the Weil algebra A, we have an associated symmetric bilinear mapping $\widetilde{f} : A \times A \to \mathbb{R}$, $\widetilde{f}(a,b) = f(a \cdot b)$. If we denote by $a_1, \dots, a_p \in A$ a basis of A, the matrix (a_{ij}) of $\widetilde{f}$ is defined as a real matrix with elements $a_{ij} = \widetilde{f}(a_i, a_j)$.

Definition. A symmetric bilinear mapping $\varphi : A \times A \to \mathbb{R}$ is said to be *nonsingular* if the matrix of φ is nonsingular.

Gancarzewicz, Mikulski and Pogoda [8] have studied relations between a product preserving functor T^A and some operations on vector bundles. If V is a free finite-dimensional A-module, then $V^{*(A)}$ denotes the A-module of all A-linear mappings $V \to A$. Analogously, if $\pi : E \to M$ is an A-module bundle, then the A-dual A-module bundle $E^{*(A)}$ is defined by $E^{*(A)} = \bigcup_{x \in M} E_x^{*(A)}$ (see [8]). By [8], every linear function $f : A \to \mathbb{R}$ defines a natural vector bundle homomorphism $\xi_E^f : E^{*(A)} \to E^*$, $\alpha \mapsto f \circ \alpha$. Moreover, this homomorphism is a vector bundle isomorphism if and only if the symmetric bilinear mapping $\widetilde{f} : A \times A \to \mathbb{R}$, $\widetilde{f}(a,b) = f(a \cdot b)$, associated with f is nonsingular.

In our definition of $s_f : T^A T^* \to T^* T^A$, a linear function $f : A \to \mathbb{R}$ comes into play in (2), and ξ_E^f from [8] is exactly the homomorphism $(TT^A M)^{*(A)} \to T^* T^A M$. Thus, Propositions 4.2 and 4.4 of [8] yield directly

Proposition 3. *$s_f : T^A T^* \to T^* T^A$ is a natural equivalence if and only if the symmetric bilinear mapping $\widetilde{f} : A \times A \to \mathbb{R}$, $\widetilde{f}(a,b) = f(a \cdot b)$, is nonsingular.*

Now we show that for $k \neq 1$ there is an obstruction to the existence of a natural equivalence $s_f : T^r_k T^* \to T^* T^r_k$.

Proposition 4. *There is a natural equivalence $s_f : T^r_k T^* \to T^* T^r_k$ depending on a linear function $f : \mathbb{D}^r_k \to \mathbb{R}$ if and only if $k = 1$.*

Proof. I. Consider first the case $k = 1$. We have $\mathbb{D}^r_1 = \mathbb{R}[x] / \langle x^{r+1} \rangle$ and elements of $\mathbb{D}^r_1$ are of the form $a_0 + a_1 x + a_2 x^2 + \dots + a_r x^r$. Hence the basis of $\mathbb{D}^r_1$ is $\{1, x, x^2, \dots, x^r\}$ and multiplication in $\mathbb{D}^r_1$ has the form $(a_0 + a_1 x + a_2 x^2 + \dots + a_r x^r)(b_0 + b_1 x + b_2 x^2 + \dots + b_r x^r) = a_0(b_0 + \dots + b_r x^r) + a_1 x (b_0 + \dots + b_{r-1} x^{r-1}) + \dots + a_r x^r b_0$. If $f : \mathbb{D}^r_1 \to \mathbb{R}$ is a linear function given by $f(a_0 + a_1 x + \dots + a_r x^r) = a_r$, then the matrix of the associated symmetric bilinear function $\widetilde{f}$ is

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The corresponding natural equivalence $s_f : T^r_1 T^* \to T^* T^r_1$ is exactly s_r from Example 1, which is nothing else but the canonical isomorphism of Cantrijn, Crampin, Sarlet and Saunders.

II. For $r = 1$ and any k we have $A = \mathbb{D}^1_k = \mathbb{R}[x_1, \dots, x_k]/\langle x_1, \dots, x_k\rangle^2$ and multiplication in $\mathbb{D}^1_k$ has the form $(a_0 + a_1 x_1 + \ldots + a_k x_k)(b_0 + b_1 x_1 + \ldots + b_k x_k) = a_0(b_0 + b_1 x_1 + \ldots + b_k x_k) + a_1 x_1 b_0 + a_2 x_2 b_0 + \ldots + a_k x_k b_0$. If we denote by $\{1, x_1, x_2, \dots, x_k\}$ the basis of A, the linear functions $f_i : A \to \mathbb{R}$ defined by $f_0(a_0 + a_1 x_1 + \ldots + a_k x_k) = a_0, \dots, f_k(a_0 + a_1 x_1 + \ldots + a_k x_k) = a_k$ form a basis of A^*. One finds easily that the matrix of each symmetric bilinear function $\widetilde{f}_0, \widetilde{f}_1, \dots, \widetilde{f}_k$ is singular.

III. The Weil algebra of T^r_k is $A = \mathbb{D}^r_k = \mathbb{R}\,[x_1, \dots, x_k]\,/\langle x_1, \dots, x_k\rangle^{r+1}$. Recall that a k-multiindex is a k-tuple $\alpha = (\alpha_1, \dots, \alpha_k)$ of nonnegative integers. We write $|\alpha| = \alpha_1 + \ldots + \alpha_k$ and $x^\alpha = (x_1^{\alpha_1}, \dots, x_k^{\alpha_k})$ for $x = (x_1, \dots, x_k)$, $x_i \in \mathbb{R}$. Then the elements of $\mathbb{D}^r_k$ can be expressed in the form $a_0 + a_\alpha x^\alpha$, where $|\alpha| \le r$ and $a_0, a_\alpha \in \mathbb{R}$. If $k > 1$, then the basis of $\mathbb{D}^r_k$ can be written as a set $\{1, x^\alpha;\ |\alpha| \le r\}$ and the corresponding dual basis is given by linear functions $f_0, f_\alpha : \mathbb{D}^r_k \to \mathbb{R}$, $f_0(a_0 + a_\alpha x^\alpha) = a_0$, $f_\alpha(a_0 + a_\alpha x^\alpha) = a_\alpha$, $|\alpha| \le r$. It is easy to verify that the matrix of each associated symmetric bilinear function $\widetilde{f}_0, \widetilde{f}_\alpha$ is singular. ∎

PROPOSITION 5. *For $k > 1$ there is no natural equivalence $T^r_k T^* \to T^* T^r_k$.*

Proof. According to the general theory [10], natural transformations $T^r_k T^* \to T^* T^r_k$ are in a canonical bijection with G^{r+1}_m-equivariant maps of the corresponding standard fibers, where G^r_m means the group of all invertible r-jets of $\mathbb{R}^m$ into $\mathbb{R}^m$ with source and target zero. Denoting by (a^i_j) the canonical coordinates in G^1_m, the coordinates of the inverse element will be denoted by $(\widetilde{a}^j_i)$. Further, denote by $(x^i, p_i, x^i_\alpha, p_{i,\alpha})$ the canonical coordinates on $T^r_k T^* M$ and by $(x^i, y^i_\alpha, r_i, s^\alpha_i)$ the canonical coordinates on $T^* T^r_k M$, where α is a k-multiindex with $|\alpha| \le r$. One calculates easily $\overline{p}_i = \widetilde{a}^j_i p_j$ and

$$\overline{x}^i_\alpha = a^i_j x^j_\alpha + \dots \tag{4}$$

Clearly, for $|\alpha| = 1$, the transformation law (4) is tensorial, while for $|\alpha| > 1$ there are terms with x^j_β on the right-hand side of (4), $|\beta| < |\alpha|$. Analogously, $\overline{y}^i_\alpha = a^i_j y^j_\alpha + \dots$ and $\overline{p}^\alpha_i = \widetilde{a}^j_i p_{j,\alpha} + \dots$ Finally, for all $|\alpha| = r$ we find $\overline{s}^\alpha_i = \widetilde{a}^j_i s^\alpha_j$. This means that all s^α_i with $|\alpha| = r$ have a tensorial transformation law. On the other hand, among $(p_i, p_{i,\alpha})$ on the standard fibre $(T^r_k T^*)_0$, only (p_i) have a tensorial transformation law. ■

5. Liftings of 1-forms and $(0,2)$-tensor fields to Weil bundles. In this section we investigate the role of natural transformations $s_f : T^A T^* \to T^* T^A$ in the theory of lifting of 1-forms and $(0,2)$-tensor fields to Weil bundles. By a *lifting* of some tensor field G to a natural bundle F we understand a natural operator transforming the tensor field G on a manifold M into a tensor field of the same type on FM.

Given a function $\varphi : M \to \mathbb{R}$ and a function $f : A \to \mathbb{R}$, we can define the *f-lift* $\varphi^f : T^A M \to \mathbb{R}$ of φ to the bundle $T^A M$ by $\varphi^f := f \circ T^A \varphi$. Clearly, $\varphi \mapsto \varphi^f$ defines a natural operator transforming functions on a manifold M into functions on $T^A M$. If $X : M \to TM$ is a vector field on M, then $T^A X : T^A M \to T^A TM$ and the composition $\mathcal{T}^A X := \kappa_M^{-1} \circ T^A X : T^A M \to TT^A M$ is a vector field on $T^A M$. By [10], $\mathcal{T}^A X$ is exactly the flow prolongation of X, it is also called the *complete lift*.

Let $\omega : M \to T^* M$ be a 1-form on M. Using the natural transformation s_f determined by a linear function $f : A \to \mathbb{R}$, we can also define the f-lift of ω to $T^A M$. Indeed, $T^A \omega : T^A M \to T^A T^* M$ and the composition with the natural transformation $(s_f)_M : T^A T^* M \to T^* T^A M$ gives rise to a 1-form ω^f on $T^A M$,

$$\omega^f := (s_f)_M \circ T^A \omega : T^A M \to T^* T^A M. \tag{5}$$

The f-lift of an evaluation mapping $\langle \omega, X \rangle : M \to \mathbb{R}$ is a function $\langle \omega, X \rangle^f : T^A M \to \mathbb{R}$. We have

PROPOSITION 6. $\langle \omega^f, \mathcal{T}^A X \rangle = \langle \omega, X \rangle^f$.

Proof. Using Proposition 1 we obtain $\langle \omega^f, \mathcal{T}^A X \rangle = \langle (s_f)_M \circ T^A \omega, \kappa_M^{-1} \circ T^A X \rangle = f \circ T^A (\langle \omega, X \rangle)$ which is nothing else but $\langle \omega, X \rangle^f$. ■

We remark that this formula has been proved in the particular case $A = \mathbb{D}^2_1$ in [3].

A 1-form $\omega : M \to T^* M$ on M can also be identified with a linear mapping $\widetilde{\omega} : TM \to \mathbb{R}$, $\widetilde{\omega}(X) = \langle \omega, X \rangle$. If $f : A \to \mathbb{R}$ is a linear function on A, then the map $\widetilde{\Omega} := f \circ T^A \widetilde{\omega} \circ \kappa_M : TT^A M \to \mathbb{R}$ is linear, so that $\widetilde{\Omega}$ induces a 1-form $\Omega : T^A M \to T^* T^A M$ on $T^A M$. On the other hand, $\omega^f := (s_f)_M \circ T^A \omega$ from (5) is also a 1-form on $T^A M$. We have

PROPOSITION 7. $\Omega = \omega^f$.

Proof. Recall that there is a canonical action $A \times TT^A M \to TT^A M$. If X is a vector field on M and $a \in A$, then we can introduce the a-lift $X^{(a)} : T^A M \to TT^A M$ of X to $T^A M$ by $X^{(a)} := a \cdot \mathcal{T}^A M$. From [7] it follows that if G and H are two tensor fields of type $(0,k)$ or $(1,k)$ on

T^AM satisfying $G(X_1^{(a_1)},\dots,X_k^{(a_k)}) = H(X_1^{(a_1)},\dots,X_k^{(a_k)})$ for all vector fields $X_1,\dots,X_k$ on M and all elements $a_1,\dots,a_k$ from A, then $G=H$. By Proposition 6 we obtain $\widetilde{\Omega}(\mathcal{T}^AX) = (f\circ T^A\widetilde{\omega}\circ\kappa_M)(\kappa_M^{-1}\circ T^AX) = f\circ T^A\widetilde{\omega}\circ T^AX = f\circ T^A(\widetilde{\omega}(X)) = f\circ T^A(\langle\omega,X\rangle) = \langle\omega,X\rangle^f = \langle\omega^f,\mathcal{T}^AX\rangle$. Using the A-linearity of both f and $T^A\widetilde{\omega}$ we directly obtain $\widetilde{\Omega}(X^{(a)}) = \langle\omega^f, X^{(a)}\rangle$ for all $a\in A$. ■

A $(0,2)$-tensor field on M can be interpreted as a linear mapping $G: TM\times_M TM\to\mathbb{R}$. Using the exchange isomorphism $\kappa_M: TT^AM\to T^ATM$ and a linear function $f: A\to\mathbb{R}$, Gancarzewicz, Mikulski and Pogoda [7] introduced an f-lift G^f of G to the bundle T^AM by

$$G^f := f\circ T^AG\circ(\kappa_M\times\kappa_M): TT^AM\times_{T^AM}TT^AM\to\mathbb{R}.$$

Further, each $(0,2)$-tensor field G on M induces a linear mapping $G_L: TM\to T^*M$ by $\langle G_L(y),z\rangle = G(z,y)$, $y,z\in T_xM$. If G is a symplectic form on M, then G_L is an isomorphism. Denote by $G_L^f: TT^AM\to T^*T^AM$ the linear mapping corresponding to the f-lift G^f of G.

PROPOSITION 8. *$G_L^f: TT^AM\to T^*T^AM$ is of the form $G_L^f = (s_f)_M\circ T^AG_L\circ\kappa_M$.*

Proof. Clearly,

$$\begin{aligned} G^f(\mathcal{T}^AX,\mathcal{T}^AY) &= f\circ T^AG\circ(\kappa_M\times\kappa_M)(\mathcal{T}^AX,\mathcal{T}^AY)\\ &= f\circ T^AG(T^AX,T^AY)\\ &= f\circ T^A(G(X,Y)) = (G(X,Y))^f.\end{aligned}$$

Analogously to the proof of Proposition 7 we have

$$\begin{aligned}\langle(s_f)_M\circ T^AG_L\circ\kappa_M)(\mathcal{T}^AY),\mathcal{T}^AX\rangle &= \langle(s_f)_M\circ T^A(G_L\circ Y),\kappa_M^{-1}\circ T^AX\rangle\\ &= \langle G_L(Y),X\rangle^f = (G(X,Y))^f\\ &= G^f(\mathcal{T}^AX,\mathcal{T}^AY).\end{aligned}$$

On the other hand, $\langle G_L^f(\mathcal{T}^AY),\mathcal{T}^AX\rangle = G^f(\mathcal{T}^AX,\mathcal{T}^AY)$. ■

We remark that the above assertion has been proved in [5] for $A=\mathbb{D}$. By [7], if ω is a 2-form on M, then $d\omega^f = (d\omega)^f$. We have

COROLLARY. *Let $\omega = dp_i\wedge dx^i$ be the canonical symplectic form on T^*M and ω^f be the f-lift of ω to $T^A(T^*M)$. If $s_f: T^AT^*\to T^*T^A$ is a natural equivalence, then ω^f is a symplectic form on T^AT^*M.*

6. General description of some natural transformations $T^AT^*\to T^*T^A$. In this section we show that for some particular cases of a Weil algebra A, the space of all natural transformations $T^AT^*\to T^*T^A$ can be characterized by means of a general geometrical description. It is our belief that this description works also for many other Weil algebras.

Definition. A *natural function* g on a natural bundle F is defined as a system of functions $g_M : FM \to M$ for any m-dimensional manifold M satisfying $g_M = g_N \circ Ff$ for every local diffeomorphism $f : M \to N$. A *natural* (or *absolute*) *vector field* X on F is a system of vector fields $X_M : FM \to TFM$ for every m-dimensional manifold M satisfying $TFf \circ X_M = X_N \circ Ff$ for every local diffeomorphism $f : M \to N$.

On the other hand, the space of all natural transformations from T^AT^* into T^*T^A is a $C^\infty(T^AT^*)$-module.

Remark 1. By the general theory [10], absolute vector fields on T^AM correspond to one-parameter groups of natural transformations of T^A into itself. In particular, the natural transformations $T^r_k \to T^r_k$ are in bijection with the elements of $J^r_0(\mathbb{R}^k, \mathbb{R}^k)_0$ and each of them has the form of a reparametrization $X \mapsto X \circ P$, $X \in T^r_kM$, $P \in J^r_0(\mathbb{R}^k, \mathbb{R}^k)_0$. For example, all natural transformations of TM into itself are homotheties $X \mapsto kX$, $k \in \mathbb{R}$, and the vector field tangent to them is the classical Liouville vector field. In the case of an arbitrary Weil functor T^A, denote by $\mathrm{Aut}(A)$ the Lie algebra associated with the Lie group of all algebra automorphisms of the Weil algebra A. In [10] it is proved that all absolute vector fields on T^AM are the generalized Liouville vector fields determined by all elements $D \in \mathrm{Aut}(A)$.

Remark 2. We remark that the problem of finding all natural functions on T^*T^A for an arbitrary Weil algebra A is rather complicated. First, Kolář [9] has determined all natural functions on $T^*T^r_1$. Recently Tomáš [14] has described all natural functions on T^*T^A for some particular cases of A.

Example 4. We describe all natural functions on $T^r_1T^*$. Denote by L the generalized Liouville vector field on T^r_1M induced by the reparametrizations $x(t) \mapsto x(kt)$, $0 \neq k \in \mathbb{R}$, of a curve $x : \mathbb{R} \to M$. By Kolář [9], all absolute vector fields on T^r_1 are linearly generated by $L_1 = L$, $L_2 = Q \circ L, \dots, L_r = Q^{r-1} \circ L$, where $Q : TT^r_1M \to TT^r_1M$ is a natural linear morphism (affinor) defined by de León and Rodrigues [2], whose coordinate expression is $(dx^i, dy^i_1, dy^i_2, \dots, dy^i_r) \mapsto (0, dx^i, dy^i_1, \dots, dy^i_{r-1})$. Let $s_r : T^r_1T^*M \to T^*T^r_1M$ be the natural equivalence from Example 1. Denoting by $q_M : T^*M \to M$ the bundle projection, we have $q_{T^r_1M}(s_r(Y)) \in T^r_1M$ for all $Y \in T^r_1T^*M$. Then every absolute vector field L_i determines a natural function $\varphi_i : T^r_1T^*M \to \mathbb{R}$,

$$\varphi_i(Y) = \langle s_r(Y), L_i(q_{T^r_1M}(s_r(Y)))\rangle.$$

By [9], all natural functions on $T^r_1T^*$ are of the form $\varphi(\varphi_1,\dots,\varphi_r)$, where $\varphi:\mathbb{R}^r\to\mathbb{R}$ is an arbitrary smooth function of r variables.

In general, let $(s_f)_M : T^AT^*M \to T^*T^AM$ be a natural transformation induced by a linear function $f : A\to\mathbb{R}$. For $Y\in T^AT^*M$ we have $q_{T^AM}((s_f)_M(Y))\in T^AM$ and each absolute vector field $X : T^AM\to TT^AM$ on T^AM determines a natural function $\varphi_{X,f}: T^AT^*M\to\mathbb{R}$ by

$$\varphi_{X,f}(Y) = \langle (s_f)_M(Y), X(q_{T^AM}((s_f)_M(Y)))\rangle. \tag{6}$$

Denote by $\varphi_1,\dots,\varphi_l$ all such functions determined by all functions $f\in A^*$ and all absolute vector fields X on T^AM and let $\varphi:\mathbb{R}^l\to\mathbb{R}$ be an arbitrary smooth function. We have

PROPOSITION 9. *Let $A=\mathbb{D}$ or $A=\mathbb{D}\otimes\mathbb{D}$ or $A=\mathbb{D}^r_1$ or $A=\mathbb{D}^1_k$. Then all natural functions on T^AT^* are of the form $\varphi(\varphi_1,\dots,\varphi_l)$.*

Proof. For $A=\mathbb{D}^r_1$ and $A=\mathbb{D}$ this follows from Example 4. Consider now $A=\mathbb{D}^1_k$ and write equations of all natural functions on $T^1_kT^*$. If $(x^i,p_i,x^i_\alpha,p_{i,\alpha},\alpha=1,\dots,k)$ are the canonical coordinates on $T^1_kT^*M$, then all natural functions on $T^1_kT^*$ are of the form $\varphi(p_ix^i_\alpha,\alpha=1,\dots,k)$ with $\varphi:\mathbb{R}^k\to\mathbb{R}$ being any smooth function (see [6]). On the other hand, by Remark 1 we find easily the coordinate form of absolute vector fields on T^1_kM, $L^\alpha_\beta = x^i_\beta\partial/\partial x^i_\alpha$, $\alpha,\beta=1,\dots,k$. Now the assertion for $A=\mathbb{D}^1_k$ follows from Example 2. For $A=\mathbb{D}\otimes\mathbb{D}$ all natural functions on TTT^* are determined in [4] and the rest of the proof is quite similar to that for $A=\mathbb{D}^1_k$. ■

Finally we describe all natural transformations $T^AT^*\to T^*T^A$ for some particular cases of A by means of a simple and universal geometrical construction. We will proceed in the following steps.

I. Denote by $\mathcal{B}$ the basis of A^*. For every $f\in\mathcal{B}$ we have a natural transformation $s_f: T^AT^*\to T^*T^A$.

II. Let $\mathrm{Tr}(A)$ be the space of all natural transformations $T^AM\to T^AM$.

III. Let $\mathrm{V}(A)$ be the space of all absolute vector fields on T^AM (see Remark 1).

IV. Natural transformations s_f from I and absolute vector fields $X\in\mathrm{V}(A)$ from III determine natural functions $\varphi_{X,f}: T^AT^*M\to\mathbb{R}$ (see (6)). If $\varphi_1,\dots,\varphi_l$ are all such functions for all $f\in\mathcal{B}$ and all absolute vector fields on T^AM, then $\varphi(\varphi_1,\dots,\varphi_l)$ is a natural function on T^AT^*M for each smooth function $\varphi:\mathbb{R}^l\to\mathbb{R}$.

V. Let $s_1,\dots,s_r: T^AT^*\to T^*T^A$ be a basis of the vector space S_A (see Proposition 2). Write $s:=k_1s_1+\dots+k_rs_r$, where on the right-hand side we have the sum in the vector bundle structure $T^*T^AM\to T^AM$ and $k_i: T^AT^*M\to\mathbb{R}$ are natural functions from IV of the form $\varphi(\varphi_1,\dots,\varphi_l)$.

VI. All natural transformations $T^AN \to T^AN$ from II applied to $N = T^*M$ determine a system of natural transformations $T^AT^*M \to T^AT^*M$ over the identity of T^*M. This system depends on certain real parameters. If we replace them by arbitrary natural functions $\varphi(\varphi_1, \dots, \varphi_l) : T^AT^*M \to \mathbb{R}$, we obtain a new system $\overline{s}$ of natural transformations $T^AT^*M \to T^AT^*M$.

VII. Write

$$t := s \circ \overline{s} : T^AT^*M \to T^*T^AM. \tag{7}$$

PROPOSITION 10. *Let* $A = \mathbb{D}$ *or* $A = \mathbb{D} \otimes \mathbb{D}$ *or* $A = \mathbb{D}^2_1$ *or* $A = \mathbb{D}^1_k$. *Then all natural transformations* $T^AT^* \to T^*T^A$ *are of the form* (7).

Proof. Consider first $A = \mathbb{D}^1_k$ and denote by $(x^i, y^i_\alpha, r_i, s^\alpha_i, \alpha = 1, \dots, k)$ the canonical coordinates on $T^1_kT^*M$. By [6], the coordinate form of all natural transformations $T^1_kT^* \to T^*T^1_k$ is $y^i_\alpha = A^\beta_\alpha x^i_\beta$, $s^\alpha_i = B^\alpha p_i$ and $r_i = A^\beta_\alpha B^\alpha p_{i,\beta} + Cp_i$. Clearly, this is the coordinate form of t described in item VII. For $A = \mathbb{D}$ the assertion follows from [11], for $A = \mathbb{D} \otimes \mathbb{D}$ from [4] and finally for $A = \mathbb{D}^2_1$ from [3]. ■

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