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Distribution of zeros and shared values of difference operators

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Abstract. We investigate the distribution of zeros and shared values of the difference operator on meromorphic functions. In particular, we show that if f is a transcendental meromorphic function of finite order with a small number of poles, c is a non-zero complex constant such that $\Delta_c^k f \not\equiv 0$ for $n \geq 2$, and a is a small function with respect to f, then $f^n \Delta_c^k f$ equals a ($\not\equiv 0, \infty$) at infinitely many points. Uniqueness of difference polynomials with the same 1-points or fixed points is also proved.

1. Introduction and results. We apply the standard notation of value distribution theory: $m(r, f)$, $N(r, f)$, $T(r, f)$, $N(r, f)$ and $S(r, f)$. Let $f(z)$ be meromorphic in the plane and let c be a non-zero complex number. Define the differences of $f(z)$ by

$$\Delta_c f(z) = f(z+c) - f(z),\ \Delta_c^{k+1} f(z) = \Delta_c(\Delta_c^k f(z)),\ k = 1, 2, \ldots. \tag{1.1}$$

Value distribution of difference operators (1.1) on meromorphic functions has become a subject of great interest recently. Bergweiler and Langley [BL] investigated the distribution of zeros of $\Delta_c^k f(z)$. Halburd and Korhonen [HK1] established a difference analogue of Nevanlinna'a second main theorem. For other results in this field, the reader is referred to [Z] and [ZK]. An analogue of the logarithmic derivative lemma, developed by Chiang and Feng [CF] and Halburd and Korhonen [HK2] independently, reads as follows.

Theorem 1.1. *Let f be a meromorphic function of finite order and let c be a non-zero complex constant. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f),$$

where $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure.

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Noting that

$$\Delta_c^k f(z) = \sum_{j=0}^{k} C_k^j (-1)^{k-j} f(z+jc),$$

it is easy to get

$$m\left(r, \frac{\Delta_c^k f}{f}\right) = S(r,f) \tag{1.2}$$

from Theorem 1.1 (see also [HK1, Lemma 2.3]).

Let f be a transcendental meromorphic function and let n be a positive integer. Hayman [H67] conjectured that $f^n f'$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often. For results concerning this conjecture, see [BE, H59, M]. Laine and Yang [LY] proved that if $f(z)$ is a transcendental entire function of finite order and $n \geq 2$, then $f(z)^n f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often. Liu and Yang [LiY] considered the case of $f(z)^n \Delta_c f(z)$. They proved

Theorem 1.2. *Let f be a transcendental entire function of finite order and let c be non-zero complex constant with $\Delta_c f(z) \not\equiv 0$. Then for $n \geq 2$ and any polynomial $p(z) \not\equiv 0$, $f(z)^n \Delta_c f(z) - p(z)$ has infinitely many zeros.*

We intend to study the distribution of zeros of $f(z)^n \Delta_c^k f(z) - a(z)$, where $a(z)$ $(\not\equiv 0, \infty)$ is a small function with respect to $f(z)$, i.e., $T(r,a) = S(r,f)$. The following theorem is an extension of Theorem 1.2.

Theorem 1.3. *Let $f(z)$ be a transcendental meromorphic function of finite order ρ, and let n, k be positive integers. Suppose that c is a non-zero complex number such that $\Delta_c^k f(z) \not\equiv 0$ and $a(z)$ $(\not\equiv 0, \infty)$ is a small function with respect to $f(z)$. If $n \geq 2$ and the exponent of convergence of poles of f satisfies*

$$\lambda(1/f) := \limsup_{r\to\infty} \frac{\log N(r,f)}{\log r} < \rho,$$

then $f(z)^n \Delta_c^k f(z) - a(z)$ has infinitely many zeros, and

$$\overline{N}\left(r, \frac{1}{f^n \Delta_c^k f - a}\right) \geq T(r,f) + S(r,f).$$

The following two counterexamples from [LiY] show that Theorem 1.3 is not true if $n = 1$ or the restriction on the order of f is removed.

Example 1.4. Let $f(z) = z + e^z$, $c = 2k\pi i$. Then

$$f(z)\Delta_c f - cz = ce^z \neq 0, \qquad \forall z \in \mathbb{C}.$$

Example 1.5. Let $f(z) = ze^{e^z}$, $e^c = -n$. Then f is an entire function of infinite order and

$$f(z)^n \Delta_c f - z^n(z+c) = -z^{n+1} e^{(n+1)e^z}$$

has a finite number of zeros for any $n \in \mathbb{N}$.

We have the following result for meromorphic functions in general.

THEOREM 1.6. *Let $f(z)$ be a transcendental meromorphic function of finite order. Suppose that n, k, c and $a(z)$ are as in Theorem 1.3. If $n \geq 3k+5$, then $f(z)^n \Delta_c^k f(z) - a(z)$ has infinitely many zeros, and*

$$\overline{N}\left(r, \frac{1}{f^n \Delta_c^k f - a}\right) \geq T(r,f) + S(r,f).$$

We now introduce the definition of sharing a common value: we say that functions f and g *share a value a CM* (counting multiplicities) if $f-a$ and $g-a$ have the same zeros with the same multiplicities (see, e.g., [H64, YY]). We say that f and g *share z* CM if $f-z$ and $g-z$ share 0 CM. A finite value z_0 is called a *fixed point* of f if $f(z_0) = z_0$. It is easy to see that a polynomial P with degree $n \geq 2$ has n fixed points (counting multiplicities). A transcendental function may not have fixed points: for example, $f(z) = e^z + z$. Obviously, if f and g share z CM, then f and g have the same fixed points with the same multiplicities.

Yang and Hua [YH] studied differential polynomials sharing a common value and obtained the following uniqueness theorem corresponding to Hayman's conjecture.

THEOREM 1.7. *Let f and g be nonconstant entire functions, and let $n \geq 6$ be an integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

Fang and Qiu [FQ] obtained the following result concerning fixed points.

THEOREM 1.8. *Let f and g be nonconstant entire functions, and let $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share z CM, then either $f(z) = c_1 e^{cz^2}$ and $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

In the present paper, we get analogous results on difference operators.

THEOREM 1.9. *Let f and g be nonconstant entire functions of finite order, and let $n \geq 5$ be an integer. Suppose that c is a non-zero complex constant such that $\Delta_c f(z) \not\equiv 0$ and $\Delta_c g(z) \not\equiv 0$. If $f^n \Delta_c f$ and $g^n \Delta_c g$ share 1 CM, and $g(z+c)$ and $g(z)$ share 0 CM, then*

(1) *$f(z) = c_1 e^{az}$ and $g(z) = c_2 e^{-az}$, where c_1, c_2 and a are constants satisfying $(c_1c_2)^{n+1}(e^{ac} + e^{-ac} - 2) = -1$; or*
(2) *$f = tg$, where t is a constant satisfying $t^{n+1} = 1$.*

Theorem 1.10. *Let f and g be nonconstant entire functions of finite order, and let $n \geq 5$ be an integer. Suppose that c is a non-zero complex constant such that $\Delta_c f(z) \not\equiv 0$ and $\Delta_c g(z) \not\equiv 0$. If $f^n \Delta_c f$ and $g^n \Delta_c g$ share z CM, and $g(z+c)$ and $g(z)$ share 0 CM, then $f = tg$, where t is a constant satisfying $t^{n+1} = 1$.*

2. Some preliminary results. In this section, we give some results needed in this paper. Chiang and Feng [CF] estimated the characteristic function and the counting function of $f(z+c)$, and got the following two results.

Lemma 2.1 ([CF, Theorem 2.1]). *Let f be a meromorphic function of finite order ρ and let c be a non-zero complex constant. Then, for each $\varepsilon > 0$,*

$$T(r, f(z+c)) = T(r, f) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

It is evident that $S(r, f(z+c)) = S(r, f)$ from Lemma 2.1.

Recall that $\lambda(1/f)$ denotes the exponent of convergence of poles of a meromorphic function f, defined in Theorem 1.3.

Lemma 2.2 ([CF, Theorem 2.2]). *Let f be a meromorphic function with $\lambda(1/f)$ finite and let c be a non-zero complex constant. Then, for each $\varepsilon > 0$,*

$$N(r, f(z+c)) = N(r, f) + O(r^{\lambda(1/f)-1+\varepsilon}) + O(\log r).$$

The following result bases on Lemmas 2.1 and 2.2.

Lemma 2.3. *Let $f(z)$ be a transcendental meromorphic function of finite order, and let n, k be positive integers. Suppose that c is a non-zero complex number such that $\Delta_c^k f(z) \not\equiv 0$ and $a(z)$ ($\not\equiv 0, \infty$) is a small function with respect to $f(z)$. Denote $F(z) = f(z)^n \Delta_c^k f(z)$. Then*

$$(2.1)\qquad \begin{aligned}(n+1)T(r,f) \leq{}& \overline{N}(r,1/f) + N(r, \Delta_c^k f/f) \\ &+ \overline{N}(r, 1/(F-a)) - N(r,F) \\ &+ N(r, f^{n+1}) + \overline{N}(r,F) + S(r,f).\end{aligned}$$

Proof. We deduce from the second main theorem that

$$(2.2)\qquad \begin{aligned}(n+1)T(r,f) &= T(r, f^{n+1}) = m(r, f^{n+1}) + N(r, f^{n+1}) \\ &\leq m(r, f^{n+1}/F) + m(r,F) + N(r, f^{n+1}) \\ &= m(r, f/\Delta_c^k f) + T(r,F) - N(r,F) + N(r, f^{n+1}) \\ &\leq m(r, f/\Delta_c^k f) + \overline{N}(r, 1/F) + \overline{N}(r, 1/(F-a)) \\ &\quad + \overline{N}(r,F) - N(r,F) + N(r, f^{n+1}).\end{aligned}$$

Noting that

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{f}{f^{n+1}\Delta_c^k f}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{f}{\Delta_c^k f}\right),$$

we see from (1.2) that

$$m\left(r,\frac{f}{\Delta_c^k f}\right)+\overline{N}\left(r,\frac{1}{F}\right)\le m\left(r,\frac{f}{\Delta_c^k f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{f}{\Delta_c^k f}\right) \tag{2.3}$$
$$=T\left(r,\frac{f}{\Delta_c^k f}\right)+\overline{N}\left(r,\frac{1}{f}\right)$$
$$=T\left(r,\frac{\Delta_c^k f}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+O(1)$$
$$=N\left(r,\frac{\Delta_c^k f}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f).$$

Applying (2.3) and (2.2), we obtain the assertion. ■

When nonconstant meromorphic functions F, G share at least one finite value CM, the following lemma plays a key role.

Lemma 2.4 ([YH, Lemma 3]). *Let F and G be nonconstant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:*

(1) $\max\{T(r,F),T(r,G)\}\le N_2(r,1/F)+N_2(r,1/G)+N_2(r,F)+N_2(r,G)+S(r,F)+S(r,G)$;
(2) $FG=1$;
(3) $F=G$,

where $N_2(r,1/F)$ denotes the counting function of zeros of F such that simple zeros are counted once and multiple zeros twice.

3. Proofs

Proof of Theorem 1.3. For each $\varepsilon>0$ we have $N(r,f)=O(r^{\lambda(1/f)+\varepsilon})$. Since f is transcendental, it follows from Lemma 2.2 that

$$N(r,f(z+c))=N(r,f)+O(r^{\lambda(1/f)-1+\varepsilon})+O(\log r)=O(r^{\lambda(1/f)+\varepsilon}). \tag{3.1}$$

Thus $N(r,\Delta_c^k f)=O(r^{\lambda(1/f)+\varepsilon})$. Denote $F(z)=f(z)^n\Delta_c^k f(z)$. Lemma 2.3 gives

$$(n+1)T(r,f)\le\overline{N}(r,1/f)+N(r,\Delta_c^k f/f)+\overline{N}(r,1/(F-a)) \tag{3.2}$$
$$\quad -N(r,F)+N(r,f^{n+1})+\overline{N}(r,F)+S(r,f)$$
$$\le 2N(r,1/f)+N(r,\Delta_c^k f)+\overline{N}(r,1/(F-a))$$
$$\quad +N(r,f^{n+1})+S(r,f)$$
$$\le 2T(r,f)+\overline{N}(r,1/(F-a))+O(r^{\lambda(1/f)+\varepsilon})+S(r,f).$$

Since $\lambda(1/f) < \rho$, we can choose ε small enough such that $\lambda(1/f) + 2\varepsilon < \rho$. Hence $O(r^{\lambda(1/f)+\varepsilon}) = S(r,f)$, and (3.2) yields

$$(n-1)T(r,f) \le \overline{N}(r, 1/(F-a)) + S(r,f). \ \blacksquare$$

Proof of Theorem 1.6. Define $F(z)$ as before. Using (1.2), we have

$$\begin{aligned} N(r, f^{n+1}) - N(r,F) &= N(r, Ff/\Delta_c^k f) - N(r,F) \le N(r, f/\Delta_c^k f) \\ &\le T(r, \Delta_c^k f/f) + O(1) \le N(r, \Delta_c^k f/f) + S(r,f). \end{aligned}$$

From this and (2.1), we deduce using Lemma 2.2 that

$$\begin{aligned} nT(r,f) &\le N(r, \Delta_c^k f/f) + \overline{N}(r, 1/(F-a)) - N(r,F) \\ &\quad + N(r, f^{n+1}) + \overline{N}(r,F) + S(r,f) \\ &\le 2N(r, \Delta_c^k f/f) + \overline{N}(r,F) + \overline{N}(r, 1/(F-a)) + S(r,f) \\ &\le 3N(r, \Delta_c^k f/f) + \overline{N}(r,f) + \overline{N}(r, 1/(F-a)) + S(r,f) \\ &\le 3(N(r,1/f) + kN(r,f)) + \overline{N}(r,f) + \overline{N}(r, 1/(F-a)) + S(r,f) \\ &\le (3k+4)T(r,f) + \overline{N}(r, 1/(F-a)) + S(r,f), \end{aligned}$$

which is

$$(n-3k-4)T(r,f) \le \overline{N}(r, 1/(F-a)) + S(r,f).$$

The assertion follows as $n \ge 3k+5$. $\blacksquare$

Proof of Theorem 1.10. Denote

$$F(z) = \frac{f(z)^n \Delta_c f(z)}{z}, \quad G(z) = \frac{g(z)^n \Delta_c g(z)}{z}. \tag{3.3}$$

Then F and G share 1 CM. Since f is a transcendental entire function, we deduce from the definition of F that $N_2(r,F) = O(\log r) = S(r,f)$ and

$$\begin{aligned} N_2(r, 1/F) &\le N_2(r, 1/(f^n \Delta_c f)) \le N_2(r, 1/f^{n+1}) + N_2(r, f/\Delta_c f) \\ &\le 2\overline{N}(r,1/f) + T(r, \Delta_c f/f) + O(1) \\ &\le 2\overline{N}(r,1/f) + N(r, \Delta_c f/f) + S(r,f) \\ &\le 2\overline{N}(r,1/f) + N(r,1/f) + S(r,f) \le 3T(r,f) + S(r,f). \end{aligned}$$

Thus

$$N_2(r,1/F) + N_2(r,F) \le 3T(r,f) + S(r,f). \tag{3.4}$$

Similarly, we get $N_2(r,G) = S(r,g)$ and

$$N_2(r,1/G) \le 2\overline{N}(r,1/g) + N(r, \Delta_c g/g) + S(r,g).$$

Noting that $g(z+c)$ and $g(z)$ share 0 CM, we obtain $N(r, \Delta_c g/g) = 0$ and the last inequality gives

$$N_2(r,1/G) \le 2T(r,g) + S(r,g).$$

Thus

$$N_2(r,1/G) + N_2(r,G) \le 2T(r,g) + S(r,g). \tag{3.5}$$

Assume that case (1) of Lemma 2.4 holds. Using (3.4) and (3.5), we have

$$\max\{T(r,F),T(r,G)\} \le 3T(r,f)+2T(r,g)+S(r,f)+S(r,g). \tag{3.6}$$

On the other hand, it follows from Theorem 1.1 that

$$\begin{aligned}(n+1)T(r,f) = T(r,f^{n+1}) &= m(r,f^{n+1}) \le m(r,f^{n+1}/F)+m(r,F)\\ &= m(r,zf/\Delta_c f)+T(r,F)\\ &\le m(r,f/\Delta_c f)+T(r,F)+O(\log r)\\ &\le N(r,\Delta_c f/f)+T(r,F)+S(r,f)\\ &\le T(r,f)+T(r,F)+S(r,f),\end{aligned}$$

which means

$$T(r,F) \ge nT(r,f)+S(r,f). \tag{3.7}$$

By the same reasoning, it follows from $N(r,\Delta_c g/g)=0$ that

$$T(r,G) \ge (n+1)T(r,g)+S(r,g). \tag{3.8}$$

Combining (3.7), (3.8) with (3.6), we conclude that

$$(n-3)T(r,f) \le 2T(r,g)+S(r,f)+S(r,g)$$

and

$$(n-1)T(r,g) \le 3T(r,f)+S(r,f)+S(r,g).$$

The last two inequalities yield an immediate contradiction, as $n \ge 5$. Hence $F(z)\cdot G(z) \equiv 1$ or $F(z) \equiv G(z)$ by Lemma 2.4. We discuss the two cases separately.

Case 1. Suppose that $F(z)\cdot G(z) \equiv 1$. Then

$$f^n(z)(f(z+c)-f(z))g^n(z)(g(z+c)-g(z)) = z^2. \tag{3.9}$$

Notice, that for $n \ge 5$, zero is a Picard exceptional value of both f and g from (3.9). Then $f(z)=e^{Q(z)}$ and $g(z)=e^{P(z)}$, where $Q(z)$ and $P(z)$ are polynomials. It follows from (3.9) that

$$(e^{Q(z+c)-Q(z)}-1)(e^{P(z+c)-P(z)}-1) = z^2e^{-(n+1)[Q(z)+P(z)]}.$$

Denote $\phi(z)=e^{Q(z+c)-Q(z)}$. Then $\phi(z)\ne 0,\infty$ for any $z\in\mathbb{C}$. If $\phi(z)\not\equiv$ const, then it is a transcendental entire function. We infer from the above equation and the second main theorem that

$$\begin{aligned}T(r,\phi) &\le \overline{N}(r,\phi)+\overline{N}(r,1/\phi)+\overline{N}(r,1/(\phi-1))+S(r,\phi)\\ &= \overline{N}(r,1/z^2)+S(r,\phi) = O(\log r)+S(r,\phi)=S(r,\phi),\end{aligned}$$

which is a contradiction. Therefore, $\phi(z)\equiv$ const. Similarly, $e^{P(z+c)-P(z)}\equiv$ const. It follows that $z^2e^{-(n+1)[Q(z)+P(z)]}\equiv$ const, which is impossible.

Case 2. Suppose that $F(z)\equiv G(z)$. Then

$$f^n(z)(f(z+c)-f(z)) = g^n(z)(g(z+c)-g(z)).$$

Let $h(z) = f(z)/g(z)$. We deduce that

$$(h^{n+1}(z) - 1)g(z) = (h^n(z)h(z+c) - 1)g(z+c). \tag{3.10}$$

If $h(z+c) \equiv h(z)$, then $(h^{n+1} - 1)\Delta_c g = 0$ and $h(z)^{n+1} \equiv 1$. Thus $h(z)$ is a constant, say t, satisfying $t^{n+1} = 1$.

Assume now that $h(z+c) \not\equiv h(z)$. Suppose that there exists a point z_0 such that $h(z_0)^{n+1} = 1$. Then $h(z_0)^n h(z_0 + c) = 1$ from (3.10) since $g(z)$ and $g(z+c)$ share 0 CM. Hence $h(z_0) = h(z_0 + c)$ and

$$\overline{N}(r, 1/(h^{n+1} - 1)) \le \overline{N}(r, 1/(h(z+c) - h)) \le 2T(r,h) + S(r,h)$$

by Lemma 2.1. In the above inequality, we apply the second main theorem to h^{n+1}, resulting in

$$\begin{aligned} T(r, h^{n+1}) &\le \overline{N}(r, h^{n+1}) + \overline{N}(r, 1/h^{n+1}) + \overline{N}(r, 1/(h^{n+1} - 1)) + S(r,h) \\ &\le 4T(r,h) + S(r,h). \end{aligned}$$

Then

$$(n+1)T(r,h) \le 4T(r,h) + S(r,h),$$

which means h is a constant, because $n \ge 5$. Let $h = t$. Making use of (3.10) again, we obtain $t^{n+1} = 1$. ■

Proof of Theorem 1.9. Denote

$$F(z) = f(z)^n \Delta_c f(z), \qquad G(z) = g(z)^n \Delta_c g(z).$$

Then F and G share 1 CM. By the same arguments as in the proof of Theorem 1.10, we have $F(z) \cdot G(z) \equiv 1$ or $F(z) \equiv G(z)$ by Lemma 2.4. If $F(z) \equiv G(z)$, we obtain the second assertion by the same reasoning as in Case 2 in the proof of Theorem 1.10.

It thus remains to consider the case $F(z) \cdot G(z) \equiv 1$. Then

$$f^n(z)(f(z+c) - f(z))g^n(z)(g(z+c) - g(z)) = 1. \tag{3.11}$$

Case 1 in the proof of Theorem 1.10 gives us $f(z) = e^{Q(z)}$ and $g(z) = e^{P(z)}$, where $Q(z)$ and $P(z)$ are polynomials. We conclude from (3.11) that $e^{Q(z+c)-Q(z)} - 1$ is never zero. Denote $H(z) = e^{Q(z+c)-Q(z)}$. Then $H(z) \ne 0, 1, \infty$ for any $z \in \mathbb{C}$. By Picard's theorem, H is a constant, so $\deg Q = 1$. Similarly, $\deg P = 1$. Assume now that

$$f(z) = c_1 e^{az}, \quad g(z) = c_2 e^{bz},$$

where a, b, c_1 and c_2 are non-zero constants. Applying (3.11) again, we get $a = -b$ and $(c_1 c_2)^{n+1}(e^{ac} + e^{-ac} - 2) = -1$. The first assertion follows. This completes the proof of Theorem 1.9. ■

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