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Relaxed hyperelastic curves

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Abstract. We define relaxed hyperelastic curve, which is a generalization of relaxed elastic lines, on an oriented surface in three-dimensional Euclidean space E^3, and we derive the intrinsic equations for a relaxed hyperelastic curve on a surface. Then, by examining relaxed hyperelastic curves in a plane, on a sphere and on a cylinder, we show that geodesics are relaxed hyperelastic curves in a plane and on a sphere. But on a cylinder, they are relaxed hyperelastic curves only in special cases.

1. Introduction. Consider smooth curves in a surface. A "relaxed elastic line", as defined by Manning in [6] and characterized in [7], minimizes the total squared curvature $\int_0^\ell \kappa^2(s)\,ds$ among curves with fixed initial point, direction, and length ℓ. Here we generalize this notion by considering κ^r, with $r \geq 2$, and introduce corresponding hyperelastic curves. A differential condition is derived together with boundary conditions that must be satisfied for any hyperelastic curve on an oriented surface. These conditions are subsequently examined in the context of geodesic curves on spheres and cylinders. The relationship between geodesics and Euler elasticity ($r = 2$) continues to be an active area of research, especially in the case of curves subject to periodic boundary conditions on two-dimensional surfaces; see for instance Linnér [4] and Linnér and Renka [5].

2. Preliminaries. We consider a curve α on a connected oriented surface M in three-dimensional Euclidean space E^3, parametrized by arc length s, $0 \leq s \leq \ell$. At a point $\alpha(s)$ of α, let $T = \alpha'(s)$ denote the unit tangent vector to α, let $N(s)$ denote the unit normal to M and let $Q(s) = N(s) \times T(s)$. Then $\{T, N, Q\}$ gives an orthonormal basis for all vectors at $\alpha(s)$ on M and it is known as the Darboux frame. The differential equations of the Darboux frame $\{T, N, Q\}$ are

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$$
\begin{pmatrix} T' \\ Q' \\ N' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ Q \\ N \end{pmatrix}, \tag{2.1}
$$

where κ_g, κ_n and τ_g are the geodesic curvature, normal curvature, and geodesic torsion of α, respectively [3].

The curvature of α is obtained as

$$
\kappa = \sqrt{\langle T', T' \rangle} = \sqrt{\kappa_g^2 + \kappa_n^2}. \tag{2.2}
$$

3. Intrinsic equations for a relaxed hyperelastic curve on an oriented surface. Our problem is to minimize the energy K,

$$
K = \int_0^\ell \kappa^r(s)\, ds, \qquad r \geq 2, \tag{3.1}
$$

among curves with trajectories $x(u(s), v(s))$ of fixed length ℓ and arc length s, $0 \leq s \leq \ell$, contained in a surface M, where x is the three-vector from the origin of three-space to a point on the surface with coordinates (u, v), and r is a natural number.

Suppose that

$$
\alpha(s) = x(u(s), v(s)), \qquad 0 \leq s \leq \ell,
$$

is a trajectory on the surface that minimizes K, and that α lies in a coordinate patch $(u, v) \to x(u, v)$ of M. Since $x_u = \partial x/\partial u$ and $x_v = \partial x/\partial v$ span the plane tangent to the surface at $x(u, v)$, both T and Q may be represented as linear combinations of these vectors. By the chain rule, we get

$$
T(s) = \frac{d\alpha}{ds}(s) = \frac{du}{ds} x_u + \frac{dv}{ds} x_v.
$$

We can write

$$
Q(s) = p(s) x_u + q(s) x_v
$$

for suitable scalar functions $p(s)$ and $q(s)$.

Next we must define variational fields for our problem. In order to obtain variational arcs of length ℓ, it is generally necessary to extend α to a curve $\alpha^*(s)$ defined for $0 \leq s \leq \ell^*$ with $\ell^* > \ell$ but sufficiently close to ℓ so that α^* lies in the coordinate patch. Let $\mu(s)$, $0 \leq s \leq \ell^*$, be a scalar function of class C^2, not vanishing identically. Define $\eta(s) = \mu(s) p^*(s)$, $\zeta(s) = \mu(s) q^*(s)$. Then

$$
\eta(s) x_u + \zeta(s) x_v = \mu(s) Q(s) \tag{3.2}
$$

along α. Assume also that

$$
\mu(0) = 0, \qquad \mu'(0) = 0. \tag{3.3}
$$

No further restrictions will be placed on μ. Now, we define

$$\beta(\sigma;t) = x(u(\sigma), v(\sigma)) + t(\eta(\sigma), \zeta(\sigma)) \tag{3.4}$$

for $0 \le \sigma \le \ell^*$. For $|t| < \varepsilon_1$ (where $\varepsilon_1 > 0$ depends upon the choice of α^* and μ), the point $\beta(\sigma;t)$ lies in the coordinate patch. For fixed t, $\beta(\sigma;t)$ gives an arc with the same initial point and initial direction as α, because of (3.3). For $t = 0$, $\beta(\sigma;0)$ is the same as α^*, and σ is arc length. For $t \neq 0$, the parameter σ is not arc length in general.

For fixed t, $|t| < \varepsilon_1$, let $L^*(t)$ denote the length of the arc $\beta(\sigma;t)$, $0 \le \sigma \le \ell^*$. Then

$$L^*(t) = \int_0^{\ell^*} \sqrt{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle}\, d\sigma \tag{3.5}$$

with

$$L^*(0) = \ell^* > \ell. \tag{3.6}$$

It is clear from (3.5) and (3.4) that $L^*(t)$ is continuous (even differentiable) in t. In particular, it follows from (3.6) that

$$L^*(t) > \frac{\ell + \ell^*}{2} > \ell \quad \text{for } |t| < \varepsilon \tag{3.7}$$

for a suitable ε satisfying $0 < \varepsilon \le \varepsilon_1$. Because of (3.7) we can restrict $\beta(\sigma;t)$, $0 \le |t| < \varepsilon$, to an arc of length ℓ by restricting the parameter σ to an interval $0 \le \sigma \le \lambda(t) \le \ell^*$ by requiring

$$\int_0^{\lambda(t)} \sqrt{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle}\, d\sigma = \ell. \tag{3.8}$$

Note that $\lambda(0) = \ell$. The function $\lambda(t)$ need not be determined explicitly, but we shall need its derivative, given in Lemma 3.1 below.

The following calculation aims at proving that lemma (and other results).

From (2.1) and (3.4), we obtain

$$\left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0} = T, \tag{3.9}$$

$$\left.\frac{\partial^2\beta}{\partial\sigma^2}\right|_{t=0} = T' = \kappa_g Q + \kappa_n N. \tag{3.10}$$

Also, we get

$$\left.\frac{\partial\beta}{\partial t}\right|_{t=0} = \mu Q \tag{3.11}$$

from (3.2). By using (2.1), we have

$$\left.\frac{\partial^2\beta}{\partial\sigma\partial t}\right|_{t=0} = \left.\frac{\partial^2\beta}{\partial t\partial\sigma}\right|_{t=0} = -\mu k_g T + \mu' Q + \mu\tau_g N. \tag{3.12}$$

Then

$$\left.\frac{\partial^3\beta}{\partial t\partial\sigma^2}\right|_{t=0} = (-2\mu' k_g - \mu k_g' - \mu k_n\tau_g)T + (-\mu k_g^2 + \mu'' - \mu\tau_g^2)Q \\ + (-\mu k_g k_n + 2\mu'\tau_g + \mu\tau_g')N. \tag{3.13}$$

LEMMA 3.1.

$$\left.\frac{d\lambda}{dt}\right|_{t=0} = \int_0^\ell \mu k_g\, ds. \tag{3.14}$$

Proof. Differentiating (3.8) with respect to t yields

$$\frac{d\lambda}{dt}\sqrt{\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle}\Bigg|_{\sigma=\lambda(t)} + \int_0^{\lambda(t)}\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial^2\beta}{\partial\sigma\partial t}\right\rangle\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-1/2} d\sigma = 0.$$

The proof is completed by using (3.9) and (3.12) with $\lambda(0) = \ell$ at $t = 0$. ■

Let $K(t)$ denote the relaxed hyperelastic functional of the arc $\beta(\sigma;t)$, $0 \le \sigma \le \lambda(t)$, $|t| < \varepsilon$. Because σ is not generally the arc-length parameter for $t \neq 0$, the functional (3.1) is calculated as

$$\begin{aligned} K(t) &= \int_0^{\lambda(t)} \langle T', T'\rangle^{r/2}\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-1/2} d\sigma \qquad (r \ge 2) \\ &= \int_0^{\lambda(t)} \Bigg\{\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-1} \\ &\quad - \left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-2}\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial\beta}{\partial\sigma}\right\rangle^{2}\Bigg\}^{r/2}\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-1/2} d\sigma \\ &= \int_0^{\lambda(t)} \Bigg\{\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-(r+1)/r}\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle \\ &\quad - \left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-(2r+1)/r}\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial\beta}{\partial\sigma}\right\rangle^{2}\Bigg\}^{r/2} d\sigma. \end{aligned} \tag{3.15}$$

The required conditions for α to be an extremal are $\left.\frac{dK}{dt}\right|_{t=0} = 0$ (see [8]) for arbitrary μ satisfying (3.3). By (3.15),

$$\begin{aligned} \frac{dK}{dt} &= \frac{d\lambda(t)}{dt}\left\{\frac{\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle}{\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{(r+1)/r}} - \frac{\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial\beta}{\partial\sigma}\right\rangle^2}{\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{(2r+1)/r}}\right\}^{r/2}_{\sigma=\lambda(t)} \\ &\quad + \frac{r}{2}\int_0^{\lambda(t)}\left\{\frac{\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle}{\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{(r+1)/r}} - \frac{\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial\beta}{\partial\sigma}\right\rangle^2}{\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{(2r+1)/r}}\right\}^{(r-2)/2} \end{aligned}$$

$$\times \left\{ 2 \frac{\left\langle \frac{\partial^3\beta}{\partial\sigma^2\partial t}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle}{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle^{(r+1)/r}} - \frac{2(r+1)}{r} \frac{\left\langle \frac{\partial^2\beta}{\partial\sigma\partial t}, \frac{\partial\beta}{\partial\sigma} \right\rangle \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle}{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle^{(2r+1)/r}} \right.$$
$$- 2 \frac{\left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial\beta}{\partial\sigma} \right\rangle \left(\left\langle \frac{\partial^3\beta}{\partial\sigma^2\partial t}, \frac{\partial\beta}{\partial\sigma} \right\rangle + \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma\partial t} \right\rangle \right)}{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle^{(2r+1)/r}}$$
$$\left. + \frac{2(2r+1)}{r} \frac{\left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial\beta}{\partial\sigma} \right\rangle^2 \left\langle \frac{\partial^2\beta}{\partial\sigma\partial t}, \frac{\partial\beta}{\partial\sigma} \right\rangle}{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle^{(3r+1)/r}} \right\} d\sigma.$$

We omitted terms with a factor $\left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial\beta}{\partial\sigma} \right\rangle$, which vanishes at $t = 0$, since $\langle T, T' \rangle = 0$, where $\frac{\partial\beta}{\partial\sigma}\big|_{t=0} = T$, $\frac{\partial^2\beta}{\partial\sigma^2}\big|_{t=0} = T'$. Thus,

$$\frac{dK}{dt}\bigg|_{t=0} = \frac{d\lambda(t)}{dt}\bigg|_{t=0} \left\{ \frac{\left\langle \frac{\partial^2\beta}{\partial\sigma^2}\big|_{t=0}, \frac{\partial^2\beta}{\partial\sigma^2}\big|_{t=0} \right\rangle}{\left\langle \frac{\partial\beta}{\partial\sigma}\big|_{t=0}, \frac{\partial\beta}{\partial\sigma}\big|_{t=0} \right\rangle^{\frac{(r+1)}{r}}} \right\}^{r/2}_{\sigma=\lambda(0)=\ell}$$
$$+ \frac{r}{2} \int_0^{\lambda(0)=\ell} \left\{ \frac{\left\langle \frac{\partial^2\beta}{\partial\sigma^2}\big|_{t=0}, \frac{\partial^2\beta}{\partial\sigma^2}\big|_{t=0} \right\rangle}{\left\langle \frac{\partial\beta}{\partial\sigma}\big|_{t=0}, \frac{\partial\beta}{\partial\sigma}\big|_{t=0} \right\rangle^{(r+1)/r}} \right\}^{(r-2)/2}$$
$$\times \left\{ 2 \frac{\left\langle \frac{\partial^3\beta}{\partial\sigma^2\partial t}\big|_{t=0}, \frac{\partial^2\beta}{\partial\sigma^2}\big|_{t=0} \right\rangle}{\left\langle \frac{\partial\beta}{\partial\sigma}\big|_{t=0}, \frac{\partial\beta}{\partial\sigma}\big|_{t=0} \right\rangle^{(r+1)/r}} \right.$$
$$\left. - \frac{2(r+1)}{r} \frac{\left\langle \frac{\partial^2\beta}{\partial\sigma\partial t}\big|_{t=0}, \frac{\partial\beta}{\partial\sigma}\big|_{t=0} \right\rangle \left\langle \frac{\partial^2\beta}{\partial\sigma^2}\big|_{t=0}, \frac{\partial^2\beta}{\partial\sigma^2}\big|_{t=0} \right\rangle}{\left\langle \frac{\partial\beta}{\partial\sigma}\big|_{t=0}, \frac{\partial\beta}{\partial\sigma}\big|_{t=0} \right\rangle^{(2r+1)/r}} \right\} ds.$$

Using (3.14), (2.2), (3.10), (3.12) and (3.13), we obtain

$$\frac{dK}{dt}\bigg|_{t=0} = \Big(\int_0^\ell \mu\kappa_g \, ds \Big) (\kappa_g^2(\ell) + \kappa_n^2(\ell))^{r/2}$$
$$+ \frac{r}{2} \int_0^\ell (\kappa_g^2 + \kappa_n^2)^{(r-2)/2} \Big\{ 2\kappa_g(\mu'' - \mu\tau_g^2 - \mu\kappa_g^2)$$
$$+ 2\kappa_n(2\mu'\tau_g + \mu\tau_g' - \mu\kappa_g\kappa_n) + \frac{2(r+1)}{r}\mu\kappa_g(\kappa_g^2 + \kappa_n^2) \Big\} ds.$$

By integration by parts and (3.3), we get

$$\int_0^\ell \mu_g''\kappa_g \, ds = \mu'(\ell)\kappa_g(\ell) - \mu(\ell)\kappa_g''(\ell) + \int_0^\ell \mu\kappa_g'' \, ds, \tag{3.16}$$

$$\int_0^\ell \mu'\kappa_n\tau_g \, ds = \mu(\ell)\kappa_n(\ell)\tau_g(\ell) - \int_0^\ell \mu(\kappa_n'\tau_g + \kappa_n\tau_g') \, ds. \tag{3.17}$$

Thus,

$$
\begin{aligned}
\left.\frac{dK}{dt}\right|_{t=0} = \int_0^{\ell} \mu\{ & \kappa_g(\kappa_g^2(\ell)+\kappa_n^2(\ell))^{r/2} + r\kappa_g[(\kappa_g^2+\kappa_n^2)^{(r-2)/2}]'' \\
& + 2r(\kappa_g' - \kappa_n\tau_g)[(\kappa_g^2+\kappa_n^2)^{(r-2)/2}]' \\
& + (r\kappa_g'' + \kappa_g^3 - r\kappa_g\tau_g^2 - 2r\kappa_n'\tau_g \\
& - r\kappa_n\tau_g' + \kappa_g\kappa_n^2)(\kappa_g^2+\kappa_n^2)^{(r-2)/2}\}\, ds \\
& + \mu(\ell)(2r\kappa_n(\ell)\tau_g(\ell) - r\kappa_g'(\ell))(\kappa_g^2(\ell)+\kappa_n^2(\ell))^{(r-2)/2} \\
& + r\mu'(\ell)\kappa_g(\ell)(\kappa_g^2(\ell)+\kappa_n^2(\ell))^{(r-2)/2} \\
& - r(r-2)\mu(\ell)\kappa_g(\ell)(\kappa_g(\ell)\kappa_g'(\ell) \\
& + \kappa_n(\ell)\kappa_n'(\ell))(\kappa_g^2(\ell)+\kappa_n^2(\ell))^{(r-4)/2}.
\end{aligned}
$$

In order that $\frac{dK}{dt}|_{t=0} = 0$ for all functions μ satisfying (3.3) with arbitrary values of $\mu(\ell)$ and $\mu'(\ell)$, the given arc α must satisfy two boundary conditions and a differential equation. Thus, we obtain the following theorem.

THEOREM 3.2. *The intrinsic equations for a relaxed hyperelastic curve of length ℓ on a connected oriented surface in three-dimensional Euclidean space E^3 are given by the differential equation*

$$
\begin{aligned}
(3.18)\qquad & \kappa_g(\kappa_g^2(\ell)+\kappa_n^2(\ell))^{r/2} + r\kappa_g[(\kappa_g^2+\kappa_n^2)^{(r-2)/2}]'' \\
& + 2r(\kappa_g' - \kappa_n\tau_g)[(\kappa_g^2+\kappa_n^2)^{(r-2)/2}]' \\
& + (r\kappa_g'' + \kappa_g^3 - r\kappa_g\tau_g^2 - 2r\kappa_n'\tau_g - r\kappa_n\tau_g' + \kappa_g\kappa_n^2)(\kappa_g^2+\kappa_n^2)^{(r-2)/2} = 0
\end{aligned}
$$

together with the boundary conditions

$$
(3.19)\qquad \kappa_g(\ell)(\kappa_g^2(\ell)+\kappa_n^2(\ell))^{(r-2)/2} = 0
$$

and

$$
(3.20)\qquad (2\kappa_n(\ell)\tau_g(\ell) - \kappa_g'(\ell))(\kappa_g^2(\ell)+\kappa_n^2(\ell))^{(r-2)/2} = 0
$$

at the free end. Here κ_g, κ_n and τ_g are the geodesic curvature, the normal curvature and the geodesic torsion as functions of arc length along the line.

The following corollary gives a condition under which geodesics are relaxed hyperelastic curves.

COROLLARY 3.3. *On a general surface, an arc of a geodesic is a relaxed hyperelastic curve if and only if*

$$
(3.21)\qquad \kappa_n^{2(r-1)}\tau_g = 0.
$$

Proof. The geodesic curvature κ_g vanishes for a geodesic on a general surface. Thus, by using (3.18), we get

$$\begin{aligned} -2r\kappa_n\tau_g(\kappa_n^{r-2})' + (-2r\kappa_n'\tau_g - r\kappa_n\tau_g')(\kappa_n^{r-2}) &= 0, \\ 2(r-2)\kappa_n^{r-2}\kappa_n'\tau_g + 2\kappa_n^{r-2}\kappa_n'\tau_g + \kappa_n^{r-1}\tau_g' &= 0, \\ 2(r-1)\kappa_n^{r-2}\kappa_n'\tau_g + \kappa_n^{r-1}\tau_g' &= 0. \end{aligned}$$

If we multiply the last equation by κ_n^{r-1}, we have

$$2(r-1)\kappa_n^{2r-3}\kappa_n'\tau_g + \kappa_n^{2r-2}\tau_g' = 0.$$

Integrating, we obtain

$$\kappa_n^{2(r-1)}\tau_g = \text{const},$$

and the constant must vanish because of (3.20). The boundary condition (3.19) is clearly satisfied. ∎

Example 3.4 (plane and sphere). For all curves in a plane or on a sphere, the geodesic torsion τ_g vanishes and $\kappa_n^2 = c^2$, where c^2 is zero for the plane and $1/R^2$ for a sphere of radius R. Thus, a geodesic in a plane or on a sphere is a relaxed hyperelastic curve.

Example 3.5 (cylinder). Consider a cylinder parametrized by

$$x(u,v) = \left(R\cos\frac{u}{R}, R\sin\frac{u}{R}, v\right)$$

where R is the radius of the cylinder. For an arbitrary arc $\alpha(s)$ on the cylinder, we have

$$\kappa_g = \frac{d\theta}{ds}, \qquad \kappa_n = -\frac{1}{R}\cos^2\theta, \qquad \tau_g = \frac{1}{2R}\sin 2\theta,$$

where $\theta = \theta(s)$ is the angle between the u-coordinate curve through $\alpha(s)$ and the curve α. The geodesics on the cylinder are characterized by $\theta \equiv$ const and from (3.21) if $\theta = 0$, $\theta = \pm\pi/2$ and $\theta = \pm\pi$, they are relaxed hyperelastic curves.

Conclusion 3.6. In this work, we generalize the notion of “relaxed elastic line” defined, given in [1] and [2] and define the notion “relaxed hyperelastic curve”. Then we obtain the formulation determining a relaxed hyperelastic curve on an oriented surface. We apply this formulation to give results about relaxed hyperelastic curves on various surfaces. Not surprisingly, a geodesic (straight line and great circle, respectively) in a plane and on a sphere is always a relaxed hyperelastic curve. However, geodesics on a cylinder are relaxed hyperelastic curves only in special cases.

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