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Reduction theorem for general connections

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Abstract. We prove the (first) reduction theorem for general and classical connections, i.e. we prove that any natural operator of a general connection Γ on a fibered manifold and a classical connection Λ on the base manifold can be expressed as a zero order operator of the curvature tensors of Γ and Λ and their appropriate derivatives.

1. Introduction. A *classical connection* on a manifold M is assumed to be a linear symmetric connection Λ on TM. Among the most important results concerning classical connections on manifolds are the replacement and the (first) reduction theorems which are closely related to the so called normal coordinates associated with a classical connection Λ. Let us recall that Λ-*normal coordinates* centered at $x_0 \in M$ (see [V1, VT]) are local coordinates (x^λ), $\lambda = 1, \dots, \dim M$, such that

$$(1.1)\qquad \Lambda_\mu{}^\lambda{}_\nu(x) = \sum_{i=1} \frac{1}{i!} N^\lambda_{\rho_1\dots\rho_i\mu\nu}(x_0) x^{\rho_1}\dots x^{\rho_i}, \qquad \Lambda_\mu{}^\lambda{}_\nu x^\mu x^\nu = 0,$$

where $N_i = (N^\lambda_{\rho_1\cdots\rho_i\mu\nu})$ are the *normal tensors* satisfying the following identities:

$$(1.2)\qquad N^\lambda_{\rho_{\sigma(1)}\cdots\rho_{\sigma(i)}\mu\nu} = N^\lambda_{\rho_1\dots\rho_i\mu\nu}$$

for any permutation σ of i indices,

$$(1.3)\qquad N^\lambda_{\rho_1\dots\rho_i\mu\nu} = N^\lambda_{\rho_1\dots\rho_i\nu\mu}$$

and

$$(1.4)\qquad N^\lambda_{(\rho_1\dots\rho_i\mu\nu)} = 0,$$

where $(\dots)$ denotes symmetrization. The independence of a natural differential operator from given local coordinates (the main property of natural operators) implies that any differential operator of order r of Λ is a zero

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order natural operator of the normal tensors N_i, $i \leq r$, i.e., if D is a natural operator with values in a natural bundle, then $D(j^r \Lambda) = \widetilde{D}(0, N_1, \ldots, N_r)$. This result is known as the *replacement theorem* (see [Th, TM]). The first reduction theorem now follows from the fact that each N_i can be expressed as a linear combination, with real coefficients, of covariant derivatives of order $i-1$ of the curvature tensor $R[\Lambda]$ of Λ and a tensor field constructed by tensor products and contractions from covariant derivatives of $R[\Lambda]$ of orders $\leq i-2$ [S, p. 162], [V2, p. 91].

So, the normal tensors and the covariant derivatives of the curvature tensor of Λ form two bases of natural operators of Λ. Of course, we can find many other bases of natural operators of Λ which differ in the right hand side of the Bianchi–Ricci identities. For instance there is a base satisfying the so called ideal Bianchi–Ricci identities with vanishing right hand side (see [JM]).

The replacement theorem was generalized by Horndeski [H] to principal connections Γ of a principal G-bundle $p : P \to M$ and classical connections Λ on M. *Γ-normal coordinates* (over the Λ-normal coordinates of the base) are given by *normal tensorial gauge concomitants* B_k. So any natural operator of order r in Γ and of order $r-2$ in Λ (the minimal order with respect to Λ we have to use) is of the type $D(j^r\Gamma, j^{r-2}\Lambda) = \tilde{D}(0, B_1, \ldots, B_r, 0, N_1, \ldots N_{r-2})$. Normal tensorial gauge concomitants are given by covariant derivatives of $R[\Gamma]$ with respect to Γ and Λ (such covariant derivatives are also called *double covariant derivatives* in the literature) and covariant derivatives of the curvature tensor of Λ. This leads to the corresponding reduction theorem (higher order Utiyama theorem) for principal connections. The reduction theorem for principal connections was proved in [J3] and, for the case of general linear connections on a vector bundle considered as principal connections on the corresponding principal frame bundle, in [J2]. Another approach to the reduction theorem for principal connections can be found in [DM]. In all versions of reduction theorems a key role is played by covariant derivatives of the curvature tensors.

This paper was inspired by the paper by Mikulski [M] who introduced the so called "special" fibered coordinates associated with a general connection Γ on a fibered manifold $p : Y \to M$ and a classical connection Λ on M. As a consequence, any natural operator of Γ and Λ with values in a natural bundle of a certain order can be expressed via "normal" fields which are sections of a natural bundle over Y whose fibers are $(G^1_m \times G^r_n)$-manifolds, $m = \dim M$, $m + n = \dim Y$.

The aim of this paper is to prove the (first) reduction theorem for general connections, i.e. to show that any natural operator of Γ and Λ can be expressed by the curvature tensor of Γ and its "general covariant" and vertical derivatives and by the curvature tensor of Λ and its standard covariant derivatives. As a consequence, Mikulski's normal fields can be expressed as zero order operators of the curvature tensors of Γ and Λ and their appropriate derivatives.

All manifolds and mappings considered are assumed to be smooth.

2. Preliminaries. In this paper we use the terminology and properties of natural bundles and natural operators in the sense of [KMS, KJ, N, T].

Let us denote by $\mathcal{M}f$ the category of all smooth manifolds and smooth morphisms, by $\mathcal{FM}$ the subcategory of all fibered manifolds and fibered manifold morphisms, and by $\mathcal{FM}_{m,n}$ the subcategory of fibered manifolds with m-dimensional bases and n-dimensional fibers and fibered diffeomorphisms over diffeomorphisms of bases. In this paper we mainly consider natural bundle functors on the subcategory $\mathcal{FM}_{m,n}$. Standard fibers of such natural bundle functors are left $G^r_{m,n}$-manifolds, where the group $G^r_{m,n}$ is a subgroup of the rth order differential group $G^r_{m+n} = \operatorname{inv} J^r_0(\mathbb{R}^{m+n}, \mathbb{R}^{m+n})_0$. The elements of $G^r_{m,n}$ are r-jets $j^r_{(0,0)}\varphi$, where $\varphi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ is a diffeomorphism such that $\varphi(0,0) = (0,0)$ and φ is projectable onto an origin preserving diffeomorphism $\underline{\varphi} : \mathbb{R}^m \to \mathbb{R}^m$.

We have the group epimorphism $\pi^r_s : G^r_{m,n} \to G^s_{m,n}$, $r > s$, given by $\pi^r_s(j^r_{(0,0)}\varphi) = j^s_{(0,0)}\varphi$. Moreover, we have the group epimorphism $p^r_1 : G^r_{m,n} \to G^r_m$ given by $p^r_1(j^r_{(0,0)}\varphi) = j^r_0\underline{\varphi}$ and the group epimorphism $p^r_2 : G^r_{m,n} \to G^r_n$ given by $p^r_2(j^r_{(0,0)}\varphi) = j^r_0(\mathrm{pr}_2 \circ \varphi \circ \iota)$, where $\iota : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ is the canonical inclusion and $\mathrm{pr}_2 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection on the second component. On the other hand G^r_m and G^r_n can be viewed as subgroups in $G^r_{m,n}$ by extending origin preserving diffeomorphisms of $\mathbb{R}^m$ and $\mathbb{R}^n$ to diffeomorphisms of $\mathbb{R}^m \times \mathbb{R}^n$ via the identity on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If we denote by $\pi^r_s(m) : G^r_m \to G^s_m$ and $\pi^r_s(n) : G^r_n \to G^s_n$ the canonical group epimorphisms and by $B^r_s(m)$ and $B^r_s(n)$ the corresponding kernels, we obtain the group epimorphisms $q^r_s(m) : G^r_{m,n} \to G^s_{m,n} \times B^r_s(m)$ and $q^r_s(n) : G^r_{m,n} \to G^s_{m,n} \times B^r_s(n)$.

In particular $G^1_{m,n}$ is given by matrices of the type

$$A = \begin{pmatrix} a^\lambda_\mu & 0 \\ a^I_\mu & a^I_J \end{pmatrix}, \qquad |a^\lambda_\mu| \neq 0,\ |a^I_J| \neq 0,\ \lambda, \mu = 1, \dots, m,\ I, J = 1, \dots, n.$$

In what follows, a tilde indicates inverse, so we have the following identities for $G^1_{m,n}$:

$$a^\lambda_\rho \tilde{a}^\rho_\mu = \delta^\lambda_\mu, \qquad a^I_\rho \tilde{a}^\rho_\mu + a^I_P\, \tilde{a}^P_\mu = 0, \qquad a^I_P \tilde{a}^P_J = \delta^I_J,$$

which gives, by formal differentiation, identities on $G^r_{m,n}$.

We say that a natural bundle functor on the category $\mathcal{FM}_{m,n}$ is of *order* $(1,k)$, $k \geq 1$, if its standard fiber is a left $(G^1_m \times G^k_n)$-manifold. A typical example of a $(1,1)$-order bundle functor is $V \otimes \bigotimes^s T_B \otimes \bigotimes^r T^*_B$ where V is the vertical tangent bundle functor and T_B and T^*_B are the tangent and cotangent functors applied to base manifolds. A section $\Phi : Y \to VY \otimes_Y (\bigotimes^s TM \otimes \bigotimes^r T^*M)$ is a $(1,1)$-order field and its vertical derivative $V\Phi$ can be considered as a section $V\Phi : Y \to V(VY \otimes_Y (\bigotimes^s TM \otimes \bigotimes^r T^*M)) \otimes_{VY\otimes_Y(\bigotimes^s TM \otimes \bigotimes^r T^*M)} V^*Y)$, i.e. a field of order $(1,2)$, and, by iteration, $V^k\Phi$ is a field of order $(1,k)$.

Let us recall that classification of natural operators between natural bundle functors is equivalent to classification of equivariant maps between standard fibers. An important tool in classifications of equivariant maps is the *orbit reduction theorem* [KMS, KJ]. Let $p : G \to H$ be a Lie group epimorphism with kernel K, M be a left G-space, Q be a left H-space and $\pi : M \to Q$ be a p-equivariant surjective submersion, i.e. $\pi(gx) = p(g)\pi(x)$ for all $x \in M$, $g \in G$. Having p, we can consider every left H-space N as a left G-space by $gy = p(g)y$, $g \in G$, $y \in N$.

THEOREM 2.1 ([KMS, p. 233]). *If each $\pi^{-1}(q)$, $q \in Q$, is a K-orbit in M, then there is a bijection between the G-maps $f : M \to N$ and the H-maps $\varphi : Q \to N$ given by $f = \varphi \circ \pi$.* ∎

3. General connections on fibered manifolds. Let $p : Y \to M$ be in the category $\mathcal{FM}_{m,n}$. A *general connection* on Y is defined to be a section $\Gamma : Y \to J^1Y$, or equivalently a tangent-valued 1-form $\Gamma : Y \to TY \otimes T^*M$, over the identity of TM. The corresponding horizontal lift will be denoted by $h^\Gamma : Y \times_M TM \to TY$.

We assume (x^λ, y^I) is a fibered coordinate chart on Y, $(x^\lambda, y^I, y^I_\lambda)$ is the induced fibered coordinate chart on J^1Y, and $(\partial_\lambda, \partial_I)$ and (d^λ, d^I) are the associated local bases of vector fields and 1-forms, respectively. Then Γ is given by

$$(x^\lambda, y^I, y^I_\lambda) \circ \Gamma = (x^\lambda, y^I, \Gamma^I{}_\lambda(x,y))$$

or, when considered as a tangent valued 1-form, by

$$\Gamma = d^\lambda \otimes (\partial_\lambda + \Gamma^I{}_\lambda\, \partial_I).$$

Moreover, if we identify Γ with its coefficients $\Gamma^I{}_\lambda(x,y)$, then Γ can be considered as a section of the natural bundle $\operatorname{Gen} Y \to Y$, where Gen is a first order natural bundle functor from $\mathcal{FM}_{m,n}$ to $\mathcal{FM}$. The standard fiber of Gen is $S_{\mathrm{Gen}} = \mathbb{R}^n \otimes \mathbb{R}^{m*}$ with coordinates $(\Gamma^I{}_\lambda)$ and an action of the subgroup $G^1_{m,n} \subset G^1_{m+n}$. The coordinate expression of the action of $G^1_{m,n}$ on S_{Gen} is

$$\bar{\Gamma}^I{}_\lambda = (a^I_P\, \Gamma^P{}_\rho + a^I_\rho)\tilde{a}^\rho_\lambda. \tag{3.1}$$

Let us note that the standard fiber of $J^r\,\mathrm{Gen}$ is the space of $(m+n,r)$-velocities $T^r_{m+n}S_{\mathrm{Gen}} = J^r_0(\mathbb{R}^{m+n}, S_{\mathrm{Gen}})$ with the action of $G^{r+1}_{m,n}$ given by composition of jets. For instance, for $r=1$, we have the induced action of $G^2_{m,n}$ on $T^1_{m+n}S_{\mathrm{Gen}}$ given by (3.1) and

$$\begin{aligned}(3.2)\qquad \bar{\Gamma}^I{}_{\lambda,\mu} &= (a^I_{PQ}\,\tilde{a}^Q_\mu + a^I_{P\rho}\,\tilde{a}^\rho_\mu)\Gamma^P{}_\rho\,\tilde{a}^\rho_\lambda + a^I_P\,(\Gamma^P{}_{\rho,Q}\,\tilde{a}^Q_\mu + \Gamma^P{}_{\rho,\sigma}\,\tilde{a}^\sigma_\mu)\tilde{a}^\rho_\lambda \\ &\quad + a^I_P\,\Gamma^P{}_\rho\,\tilde{a}^\rho_{\lambda\mu} + (a^I_{\rho P}\,\tilde{a}^P_\mu + a^I_{\rho\sigma}\,\tilde{a}^\sigma_\mu)\tilde{a}^\rho_\lambda + a^I_\rho\,\tilde{a}^\rho_{\lambda\mu},\end{aligned}$$

$$(3.3)\qquad \bar{\Gamma}^I{}_{\lambda,J} = (a^I_{PQ}\,\Gamma^P{}_\rho + a^I_P\,\Gamma^P{}_{\rho,Q} + a^I_{\rho Q})\,\tilde{a}^\rho_\lambda\,\tilde{a}^Q_J.$$

REMARK 3.1. A *classical connection* is a linear section $\Lambda : TM \to J^1TM$ with the corresponding horizontal lift $\dot{d}^\lambda = \Lambda_\mu{}^\lambda{}_\nu(x)\dot{x}^\mu d^\nu$, where $\Lambda_\mu{}^\lambda{}_\nu = \Lambda_\nu{}^\lambda{}_\mu$ and we denote by "˙" the induced coordinates on the tangent bundle. Λ can be considered to be a section of the second order natural bundle $\mathrm{Cla}\,M \to M$, where the standard fiber of Cla is $S_{\mathrm{Cla}} = \mathbb{R}^{m*}\otimes\mathbb{R}^m\otimes\mathbb{R}^{m*}$ with the action of G^2_m given by

$$(3.4)\qquad \bar{\Lambda}_\mu{}^\lambda{}_\nu = (a^\lambda_\rho\,\Lambda_\sigma{}^\rho{}_\tau + a^\lambda_{\sigma\tau})\tilde{a}^\sigma_\mu\tilde{a}^\tau_\nu.$$

If we consider Γ as a tangent-valued 1-form $\Gamma : Y \to TY\otimes T^*M$ over the identity of TM, the curvature of Γ can be defined as the section

$$R[\Gamma] : Y \to VY \otimes \textstyle\bigwedge^2 T^*M$$

given by $R[\Gamma] = -[\Gamma,\Gamma]$, where $[\,,]$ is the Frölicher–Nijenhuis bracket. We have the coordinate expression

$$R[\Gamma] = -2(\partial_\lambda\Gamma^I{}_\mu + \Gamma^P{}_\lambda\partial_P\Gamma^I{}_\mu)\partial_I\otimes d^\lambda\wedge d^\mu.$$

Now, let us consider the standard fiber $\mathcal{U}_0 = \mathbb{R}^n\otimes\bigwedge^2\mathbb{R}^{m*}$ of the natural bundle functor $V\otimes\bigwedge^2 T^*_B$ with the induced coordinates $(u^I{}_{\lambda\mu})$ and the tensorial action of the subgroup $G^1_m\times G^1_n \subset G^1_{m,n}$. Then it is easy to see that the curvature operator is a natural first order operator $\mathrm{Gen}\to V\otimes\bigwedge^2 T^*_B$ with the corresponding $G^2_{m,n}$-equivariant mapping $\mathcal{R}_G : T^1_{m+n}S_{\mathrm{Gen}} \to \mathcal{U}_0$ given by

$$(3.5)\qquad u^I{}_{\lambda\mu}\circ\mathcal{R}_G = \Gamma^I{}_{\lambda,\mu} - \Gamma^I{}_{\mu,\lambda} + \Gamma^P{}_\mu\Gamma^I{}_{\lambda,P} - \Gamma^P{}_\lambda\Gamma^I{}_{\mu,P}.$$

4. General covariant derivatives. A key role in reduction theorems for (general) linear and principal connections is played by covariant derivatives of curvature tensor fields. But in the case of a general connection we cannot define the standard covariant derivatives of $R[\Gamma]$. We have to consider a more general concept of general covariant derivatives. In this section we recall the definition and basic properties of general covariant derivatives (see [J4]) of vertical-valued tensor fields with respect to general and classical connections, which allows us to define also general covariant derivatives of the curvature tensor $R[\Gamma]$.

The *general covariant derivative* is a first order natural operator transforming sections $\Phi : Y \to VY \otimes_Y (\bigotimes^s TM \otimes \bigotimes^r T^*M)$ (in order one), general connections Γ (in order one) and classical connections Λ (in order zero) to sections of $VY \otimes_Y (\bigotimes^s TM \otimes \bigotimes^{r+1} T^*M)$. The coordinate expression of a general covariant derivative is

$$\tag{4.1} \begin{aligned} \nabla^{\Gamma,\Lambda}\Phi = \Big(&\partial_\mu \Phi^{I\underline{\nu}}{}_{\underline{\lambda}} + \Gamma^P{}_\mu \partial_P \Phi^{I\underline{\nu}}{}_{\underline{\lambda}} - \Phi^{P\underline{\nu}}{}_{\underline{\lambda}} \partial_P \Gamma^I{}_\mu \\ &- \sum_{k=1}^{s} \Phi^{I\nu_1\dots\rho\dots\nu_s}{}_{\underline{\lambda}} \Lambda_\mu{}^{\nu_k}{}_\rho + \sum_{j=1}^{r} \Phi^{I\underline{\nu}}{}_{\lambda_1\dots\rho\dots\lambda_r} \Lambda_\mu{}^\rho{}_{\lambda_j}\Big) \partial_I \otimes \partial_{\underline{\nu}} \otimes d^{\underline{\lambda}} \otimes d^\mu, \end{aligned}$$

where we have used multiindices $\underline{\nu} = (\nu_1 \dots \nu_s)$, $\underline{\lambda} = (\lambda_1 \dots \lambda_r)$ and we have set $\partial_{\underline{\nu}} = \partial_{\nu_1} \otimes \dots \otimes \partial_{\nu_s}$ and $d^{\underline{\lambda}} = d^{\lambda_1} \otimes \dots \otimes d^{\lambda_r}$. If we consider the corresponding mappings of standard fibers $T^1_{m+n} S_{\mathrm{Gen}} \times S_{\mathrm{Cla}} \times T^1_{m+n}(\mathbb{R}^n \otimes \bigotimes^s \mathbb{R}^m \otimes \bigotimes^r \mathbb{R}^{m*}) \to \mathbb{R}^n \otimes \bigotimes^s \mathbb{R}^m \otimes \bigotimes^{r+1} \mathbb{R}^{m*}$, and we denote by ";" the *formal general covariant derivative* and by "," the *formal partial derivative*, then

$$\tag{4.2} \begin{aligned} \Phi^{I\underline{\nu}}{}_{\underline{\lambda};\mu} = \Phi^{I\underline{\nu}}{}_{\underline{\lambda},\mu} &+ \Gamma^P{}_\mu \Phi^{I\underline{\nu}}{}_{\underline{\lambda},P} - \Phi^{P\underline{\nu}}{}_{\underline{\lambda}} \Gamma^I{}_{\mu,P} \\ &- \sum_{k=1}^{s} \Phi^{I\nu_1\dots\rho\dots\nu_s}{}_{\underline{\lambda}} \Lambda_\mu{}^{\nu_k}{}_\rho + \sum_{j=1}^{r} \Phi^{I\underline{\nu}}{}_{\lambda_1\dots\rho\dots\lambda_r} \Lambda_\mu{}^\rho{}_{\lambda_j}. \end{aligned}$$

By iteration we get the kth order general covariant derivative $(\nabla^{\Gamma,\Lambda})^k\Phi : Y \to VY \otimes_Y (\bigotimes^s TM \otimes \bigotimes^{r+k} T^*M)$ which is a natural operator of order k with respect to Φ and Γ and of order $k-1$ with respect to Λ.

In what follows we shall write simply ∇ instead of $\nabla^{\Gamma,\Lambda}$ and we shall set $\nabla^{(k)} = (\mathrm{id} = \nabla^0, \nabla, \dots, \nabla^k)$.

Remark 4.1. Let $p : E \to M$ be a vector bundle with linear fibered coordinates, K be a (general) linear connection on E, and Φ be a linear vertical-valued tensor field. Then the general covariant derivative $\nabla^{K,\Lambda}\Phi$ is a linear vertical-valued tensor field which is given by the standard covariant derivative of Φ with respect to the pair (K, Λ) (see [J1]). So, the general covariant derivative generalizes the standard covariant derivatives with respect to linear connections.

Remark 4.2. Let us recall that for a section $s : M \to Y$ we can define the *covariant derivative* $\nabla^\Gamma s : M \to VY \otimes T^*M$ with respect to a general connection Γ by

$$\nabla^\Gamma s = j^1 s - \Gamma \circ s.$$

In coordinates, if $(x^\lambda, y^I) \circ s = (x^\lambda, s^I(x))$, then

$$(x^\lambda, y^I, u^I{}_\lambda) \circ \nabla^\Gamma s = (x^\lambda, s^I(x), \partial_\lambda s^I(x) - \Gamma^I{}_\lambda(x, s(x))),$$

i.e.

$$\nabla^{\Gamma} s = (\partial_\lambda s^I - \Gamma^I{}_\lambda)\partial_I \otimes d^\lambda.$$

In [CK] the above covariant derivative is called the *absolute differential* of s and, by using an auxiliary classical connection on M, the second order absolute differential of s is defined. The second order absolute differential of s coincides with the general covariant derivative of $\nabla^{\Gamma} s$ and we can consider general covariant derivatives of s of any order.

The vertical prolongation $V\Phi : VY \to V(VY \otimes_Y (\bigotimes^s TM \otimes \bigotimes^r T^*M))$ can be considered as the section

$$V\Phi : Y \to V(VY \otimes_Y (\bigotimes^s TM \otimes \bigotimes^r T^*M)) \otimes_{VY \otimes_Y (\bigotimes^s TM \otimes \bigotimes^r T^*M)} V^*Y$$

given in coordinates by

$$V\Phi = \partial_J \Phi^{I\boldsymbol{\nu}}{}_{\underline{\boldsymbol{\lambda}}}\, \partial_I{}^{\underline{\boldsymbol{\lambda}}}{}_{\boldsymbol{\nu}} \otimes d^J,$$

where we have put $\partial_I{}^{\boldsymbol{\lambda}}{}_{\underline{\boldsymbol{\nu}}} = \partial/\partial u^{I\boldsymbol{\nu}}{}_{\underline{\boldsymbol{\lambda}}}$. The functor $(V \otimes \bigotimes^s T_B \otimes \bigotimes^r T^*_B) \oplus (V(V \otimes \bigotimes^s T_B \otimes \bigotimes^r T^*_B) \otimes V^*)$ on the category $\mathcal{FM}_{m,n}$ is of order $(1,2)$ and the action of the subgroup $G^1_m \times G^2_n \subset G^2_{m,n}$ on its standard fiber is given by

$$\bar{u}^{I\boldsymbol{\nu}}{}_{\underline{\boldsymbol{\lambda}}} = a^I_P\, a^{\boldsymbol{\nu}}_{\underline{\boldsymbol{\sigma}}}\, u^{P\boldsymbol{\sigma}}{}_{\underline{\boldsymbol{\rho}}}\, \tilde{a}^{\boldsymbol{\rho}}_{\underline{\boldsymbol{\lambda}}}, \tag{4.3}$$

$$\bar{u}^{I\boldsymbol{\nu}}{}_{\underline{\boldsymbol{\lambda}}J} = a^{\boldsymbol{\nu}}_{\underline{\boldsymbol{\sigma}}}\, (a^I_{PQ}\, u^{P\boldsymbol{\sigma}}{}_{\underline{\boldsymbol{\rho}}} + a^I_P\, u^{P\boldsymbol{\sigma}}{}_{\underline{\boldsymbol{\rho}}Q})\, \tilde{a}^{\boldsymbol{\rho}}_{\underline{\boldsymbol{\lambda}}}\, \tilde{a}^Q_J, \tag{4.4}$$

where we have set $a^{\boldsymbol{\nu}}_{\underline{\boldsymbol{\sigma}}} = a^{\nu_1}_{\sigma_1} \dots a^{\nu_s}_{\sigma_s}$ and $\tilde{a}^{\boldsymbol{\rho}}_{\underline{\boldsymbol{\lambda}}} = \tilde{a}^{\rho_1}_{\lambda_1} \dots \tilde{a}^{\rho_r}_{\lambda_r}$. By iteration we can deduce that the functor

$$\bigoplus_{i=1}^{k} \underbrace{V(\dots(V(V}_{i\,\text{times}} \otimes \bigotimes^s T_B \otimes \bigotimes^r T^*_B) \otimes \underbrace{V^*)\dots) \otimes V^*}_{i-1\,\text{times}}$$

is of order $(1,k)$ and on its standard fiber we have the action of the group $G^1_m \times G^k_n$ given by the formal vertical derivatives of (4.4). The sequence of operators $V^{(k)}\Phi = (\Phi, V\Phi, \dots, V^k\Phi)$ has values in sections of

$$\bigoplus_{i=1}^{k} \underbrace{V(\dots(V(V}_{i\,\text{times}} Y \otimes \bigotimes^s T_B \otimes \bigotimes^r T^*_B) \otimes \underbrace{V^*Y)\dots) \otimes V^*Y}_{(i-1)\,\text{times}}$$

and defines a kth order operator which depends on vertical derivatives of Φ only.

Let us consider the general covariant derivatives of the curvature tensor $R[\Gamma]$. Then we obtain the *general Bianchi identity* [J4].

LEMMA 4.3. *We have*

$$\nabla R[\Gamma](\xi,\eta,\zeta) + \nabla R[\Gamma](\eta,\zeta,\xi) + \nabla R[\Gamma](\zeta,\xi,\eta) = 0, \tag{4.5}$$

i.e. in coordinates

$$R^I{}_{\lambda\mu;\nu} + R^I{}_{\mu\nu;\lambda} + R^I{}_{\nu\lambda;\mu} = 0. \tag{4.6}$$

For the second order general covariant derivative we have the following *general Ricci identity* [J4].

LEMMA 4.4. *The antisymmetrization of the second order general covariant derivative is a section* $\operatorname{Alt}\nabla^2\Phi : Y \to VY \otimes_Y (\bigotimes^s TM \otimes \bigotimes^r T^*M \otimes \bigwedge^2 T^*M)$ *given by*

$$\begin{aligned}2\operatorname{Alt}\nabla^2\Phi(\alpha_1,\dots,\xi_r,\eta,\zeta) &= \nabla^2\Phi(\alpha_1,\dots,\xi_r,\eta,\zeta) - \nabla^2\Phi(\alpha_1,\dots,\xi_r,\zeta,\eta)\\ &= V(\Phi(\alpha_1,\dots,\xi_r))(R[\Gamma](\eta,\zeta)) - V(R[\Gamma](\eta,\zeta))(\Phi(\alpha_1,\dots,\xi_r))\\ &\quad - \sum_{k=1}^{s}\Phi(\alpha_1,\dots,R[\Lambda](\alpha_k,\eta,\zeta),\dots,\alpha_s,\xi_1,\dots,\xi_r)\\ &\quad + \sum_{j=1}^{r}\Phi(\alpha_1,\dots,\alpha_s,\xi_1,\dots,R[\Lambda](\xi_j,\eta,\zeta),\dots,\xi_r)\end{aligned}$$

for any 1-forms α_k, $k = 1,\dots,s$, *and vector fields* ξ_j, η, ζ, $j = 1,\dots,r$, *on* M. *In coordinates*

$$\begin{aligned}2\Phi^{I\underline{\nu}}{}_{\underline{\lambda};[\mu;\kappa]} &= \Phi^{I\underline{\nu}}{}_{\underline{\lambda};\mu;\kappa} - \Phi^{I\underline{\nu}}{}_{\underline{\lambda};\kappa;\mu} = \Phi^{I\underline{\nu}}{}_{\underline{\lambda},P}\, R^P{}_{\mu\kappa} - R^I{}_{\mu\kappa,P}\,\Phi^{P\underline{\nu}}{}_{\underline{\lambda}}\\ &\quad - \sum_{k=1}^{s}\Phi^{I\nu_1\dots\rho\dots\nu_s}{}_{\underline{\lambda}}\, R_\rho{}^{\nu_k}{}_{\mu\kappa} + \sum_{j=1}^{r}\Phi^{I\underline{\nu}}{}_{\lambda_1\dots\rho\dots\lambda_r}\, R_{\lambda_j}{}^{\rho}{}_{\mu\kappa}.\end{aligned} \tag{4.7}$$

We can write

$$\operatorname{Alt}\nabla^2\Phi = \operatorname{pol}(\Phi, V\Phi, R[\Gamma], VR[\Gamma], R[\Lambda]),$$

where pol is a polynomial (zero order) operator on the indicated fields. But $\operatorname{Alt}\nabla^2\Phi$ is the section of $VY \otimes_Y (\bigotimes^s TM \otimes \bigotimes^{r+2} T^*M)$, i.e. a section of a $(1,1)$-order bundle and we can apply general covariant and vertical derivatives of higher orders. Then we have [J4]

THEOREM 4.5. *We have*

$$\begin{aligned}\nabla^{r-2}(\operatorname{Alt}\nabla^2\Phi) = \operatorname{pol}\big(&\nabla^{(r-2)}\Phi, V(\nabla^{(r-2)}\Phi),\\ &\nabla^{(r-2)}R[\Gamma], V(\nabla^{(r-2)}R[\Gamma]), \nabla^{(r-2)}R[\Lambda]\big),\end{aligned} \tag{4.8}$$

$$V^{r-2}(\operatorname{Alt}\nabla^2\Phi) = \operatorname{pol}(V^{(r-1)}\Phi, V^{(r-1)}R[\Gamma], R[\Lambda]). \tag{4.9}$$

If we apply Theorem 4.5 to the curvature tensor of Γ we get

COROLLARY 4.6.

$$\nabla^{r-2}(\operatorname{Alt}\nabla^2 R[\Gamma]) = \operatorname{pol}(\nabla^{(r-2)}R[\Gamma], V(\nabla^{(r-2)}R[\Gamma]), \nabla^{(r-2)}R[\Lambda]), \tag{4.10}$$

$$V^{r-2}(\operatorname{Alt}\nabla^2 R[\Gamma]) = \operatorname{pol}(V^{(r-1)}R[\Gamma], R[\Lambda]). \tag{4.11}$$

REMARK 4.7. Let us remark that the general Ricci identity differs from the Ricci identities for (general) linear or principal connections in the order of fields on the right hand side [J2, J3]. Indeed, in the cases of general linear and principal connections we have the field Φ and the curvature tensors $R[\Gamma]$ and $R[\Lambda]$ on the right hand side, i.e. the right hand side is of order zero with respect to Φ and of order one with respect to connections, so antisymmetrization decreases the degree with respect to Φ by 2. In the case of general connections on fibered manifolds the right hand side of the general Ricci identity is given by the vertical derivatives of Φ and $R[\Gamma]$, i.e. it is of order one with respect to Φ and of order two with respect to Γ, so antisymmetrization decreases the order with respect to Φ by one, and if we apply the general Ricci identity for the covariant derivatives of the curvature tensor, antisymmetrization decreases the order by one.

5. Replacement and reduction theorems. By Mikulski [M] there are fibered coordinates on Y, over Λ-normal coordinates on M, such that in a neighborhood of $y_0 = (0,0)$ the coordinate expression of Γ is

$$\Gamma^I{}_\lambda = \sum_{i\geq 1, j\geq 0} \frac{1}{i!}\frac{1}{j!} B^I_{\mu_1\ldots\mu_i P_1\ldots P_j\lambda}\, x^{\mu_1}\ldots x^{\mu_i}\, y^{P_1}\ldots y^{P_j}, \qquad \Gamma^I{}_\lambda\, x^\lambda = 0. \tag{5.1}$$

Here $B_r = \{B_{(i,j)} = (B^I_{\mu_1\ldots\mu_i P_1\ldots P_j\lambda}),\ i+j=r\}$ are the so called *normal fields* which are sections of a natural bundle over Y on whose fibers the subgroup $G^1_m \times G^r_n \subset G^r_{m,n}$ acts, i.e. normal fields are sections of a natural bundle of order $(1,r)$. Moreover, the following identities are satisfied:

$$B^I_{(\mu_1\ldots\mu_i P_1\ldots P_j\lambda)} = 0, \tag{5.2}$$

$$B^I_{\mu_1\ldots\mu_i P_1\ldots P_j\lambda} = B^I_{\mu_{\sigma(1)}\ldots\mu_{\sigma(i)} P_{\rho(1)}\ldots P_{\rho(j)}\lambda}, \tag{5.3}$$

where $(\ldots)$ denotes symmetrization with respect to the Greek indices only, while σ and ρ are permutations of i and j indices, respectively.

In what follows we shall consider natural operators of order r in general connections Γ and of order s, $r-2 \leq s$, with respect to Λ. The reason is that general covariant derivatives of $R[\Gamma]$ are natural operators such that the order with respect to Λ is the order with respect to Γ minus 2.

Now we can prove the *replacement theorem for general connections.*

THEOREM 5.1. *Any natural operator of order $r \geq 1$ with respect to Γ and of order s, $r-2 \leq s$, with respect to Λ with values in a natural bundle factorizes through normal fields of Γ and Λ up to orders r and s, respectively, i.e.*

$$D(j^r\Gamma, j^s\Lambda) = \widetilde{D}(0, B_1, \ldots, B_r, 0, N_1, \ldots, N_s). \tag{5.4}$$

Proof. Natural operators are independent of the choice of a local fibered coordinate chart. In Γ-normal fibered coordinates centered at $y_0 \in Y$ over Λ-normal coordinates on the base centered at $x_0 = p(y_0)$, r-jets of Γ are given by normal fields B, and s-jets of Λ are given by normal tensors N. ∎

By the above replacement theorem, the Γ-normal and Λ-normal fields form a basis of natural operators of Γ and Λ. In the rest of the paper we shall prove that there is another basis of natural operators of Γ and Λ formed by the general covariant and vertical derivatives of the curvature tensor of Γ and by the standard covariant derivatives of the curvature tensor of Λ. This result is the (first) reduction theorem for general connections.

Remark 5.2. For $r = 2$, $s = 0$ we can prove the reduction theorem very easily. Let us put

$$B^I_{\mu\lambda} = R^I{}_{\lambda\mu}, \qquad B^I_{\mu_1\mu_2\lambda} = R^I{}_{\lambda(\mu_1;\mu_2)}, \qquad B^I_{\mu P\lambda} = R^I{}_{\lambda\mu,P}.$$

Then it is easy to see that the Bianchi identity (4.6) implies the identity (5.2). On the other hand if we have normal fields (B_1, B_2) then

$$R^I{}_{\lambda\mu} = B^I_{\mu\lambda}, \qquad R^I{}_{\lambda\mu_1;\mu_2} = \frac{2}{3}(B^I_{\mu_1\mu_2\lambda} - B^I_{\lambda\mu_2\mu_1}), \qquad R^I{}_{\lambda\mu,P} = B^I_{\mu P\lambda}$$

is the inverse mapping and the identity (5.2) implies the Bianchi identity. Then

$$\widetilde{D}(0, B_1, B_2) = \widetilde{\widetilde{D}}(R[\Gamma], \nabla R[\Gamma], VR[\Gamma]).$$

So, the normal fields (B_1, B_2) and the fields $(R[\Gamma], \nabla R[\Gamma], VR[\Gamma])$ form two different bases of second order natural operators of Γ.

Remark 5.3. If $r = 1$ then $B^I_{\mu\lambda} = R^I{}_{\lambda\mu}$ and we see that Utiyama's theorem [U] is also true for general connections.

Now, we can formulate the reduction theorem for general connections.

Theorem 5.4. *Any natural operator of order r with respect to Γ and of order s, $r-2 \le s$, with respect to Λ with values in a $(1, r+1)$-order natural bundle factorizes through general covariant and vertical derivatives of $R[\Gamma]$ up to order $r-1$ and through covariant derivatives of $R[\Lambda]$ up to order $s-1$, i.e.*

$$D(j^r\Gamma, j^s\Lambda) = \widetilde{D}(V^{(j}(\nabla^{i)}R[\Gamma]), \nabla^{(s-1)}R[\Lambda]), \quad j+i = 0, 1, \ldots, r-1. \tag{5.5}$$

Remark 5.5. The difference between the reduction theorem and the replacement theorem is in the target natural bundle. In the replacement theorem the order of the target natural bundle is arbitrary but in the reduction theorem the target bundle has to be a natural bundle of order $(1, r+1)$. But in the last section we shall prove that Γ-normal fields can be expressed via general covariant and vertical derivatives of $R[\Gamma]$ and covariant derivatives of $R[\Lambda]$. So natural differential operators with values in a natural bundle of any order can be factorized through curvature tensors and their appropriate covariant derivatives.

6. Curvature bundles. To prove the above reduction Theorem 5.4 we have to prove first some technical results.

Let us introduce the following notation.

Let $\mathcal{W}M = \mathcal{W}_0M := T^*M \otimes TM \otimes \bigwedge^2 T^*M$, $\mathcal{W}_iM = \mathcal{W}M \otimes \bigotimes^i T^*M$, $i \geq 0$. Let us put $\mathcal{W}^{(r)}M = \mathcal{W}_0M \times_M \cdots \times_M \mathcal{W}_rM$. Then $\mathcal{W}_iM$ and $\mathcal{W}^{(r)}M$ are natural bundles of order one on the category $\mathcal{M}f$, and the corresponding standard fibers will be denoted by $\mathcal{W}_i$ and $\mathcal{W}^{(r)}$, where $\mathcal{W}_0 := \mathcal{W} = \mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes \bigwedge^2 \mathbb{R}^{m*}$, $\mathcal{W}_i = \mathcal{W}_0 \otimes \bigotimes^i \mathbb{R}^{m*}$, $i \geq 0$, and $\mathcal{W}^{(r)} = \mathcal{W}_0 \times \cdots \times \mathcal{W}_r$. We denote by $(w_\mu{}^\lambda{}_{\nu\kappa\rho_1\dots\rho_i})$ the canonical coordinates on $\mathcal{W}_i$.

We denote by

$$\mathcal{R}_{C,i} : T^{i+1}_m S_{\mathrm{Cla}} \to \mathcal{W}_i$$

the G^{i+3}_m-equivariant map associated with the standard ith covariant derivative of curvature tensors of classical connections

$$\nabla^i R[\Lambda] : C^\infty(\mathrm{Cla}\, M) \to C^\infty(\mathcal{W}_iM).$$

The map $\mathcal{R}_{C,i}$ is said to be the *formal curvature map of order* i of classical connections.

Let $C_{C,i} \subset \mathcal{W}_i$ be the subset given by the identities of the ith covariant derivatives of the curvature tensors of classical connections (see [J3, KMS]). Recall that these identities are the first and second Bianchi identities of the curvature tensor of classical connections and their covariant derivatives, and the Ricci identity and its covariant derivatives. Let us put $C^{(r)}_C = C_{C,0} \times \cdots \times C_{C,r}$ and

$$\mathcal{R}^{(r)}_C := (\mathcal{R}_{C,0}, \dots, \mathcal{R}_{C,r}) : T^{r+1}_m S_{\mathrm{Cla}} \to \mathcal{W}^{(r)}, \tag{6.1}$$

which has values in $C^{(r)}_C$. In [KMS] it was proved that $C^{(r)}_C$ is a submanifold in $\mathcal{W}^{(r)}$ and the restriction of (6.1) to $C^{(r)}_C$ is a surjective submersion. The corresponding first order natural bundle $C^{(r)}_C M$ is called the rth order *curvature natural bundle* for classical connections on M.

In order to describe the *curvature bundle for general connections* we have to recall that the general Ricci identity applied to the second order general covariant derivatives of $R[\Gamma]$ (or $\nabla^i R[\Gamma]$) depends also on the vertical derivatives of $R[\Gamma]$ (or $\nabla^i R[\Gamma]$). So we have to include into the curvature bundle for general connections also the vertical prolongation of the bundle $VY \otimes_Y \bigwedge^2 T^*M \otimes \bigotimes^i T^*M$.

Let

$$\mathcal{U}Y = \mathcal{U}_{(0,0)}Y := VY \otimes_Y \bigwedge^2 T^*M,$$

$$\mathcal{U}_{(i,j)}Y = \underbrace{V(\dots(V}_{j\text{ times}}(VY \otimes_Y (\bigwedge^2 T^*M \otimes \bigotimes^i T^*M)) \otimes \underbrace{V^*Y)\dots) \otimes V^*Y}_{j\text{ times (symmetric)}},$$

$$i \geq 0,\ j \geq 0,$$

$$\mathcal{U}_k Y = \bigtimes_{i+j=k} \mathcal{U}_{(i,j)}Y, \qquad \mathcal{U}^{(r)}Y = \bigtimes_{k=0}^{r} \mathcal{U}_k Y.$$

Then $\mathcal{U}^{(r)}Y$ is a $(1, r+1)$-order natural graded vector bundle on the category $\mathcal{FM}_{m,n}$ on whose standard fiber the subgroup $G^1_m \times G^{r+1}_n \subset G^{r+1}_{m,n}$ acts. The corresponding standard fibers will be denoted by $\mathcal{U}_{(i,j)}$, $\mathcal{U}_k$ and $\mathcal{U}^{(r)}$, respectively, where $\mathcal{U}_{(0,0)} := \mathcal{U} = \mathbb{R}^n \otimes \bigwedge^2 \mathbb{R}^{m*}$, $\mathcal{U}_{(i,j)} = \mathbb{R}^n \otimes \bigwedge^2 \mathbb{R}^{m*} \otimes \bigotimes^i \mathbb{R}^{m*} \otimes S^j \mathbb{R}^{n*}$, $i, j \geq 0$, $\mathcal{U}_k = \bigtimes_{i+j=k} \mathcal{U}_{(i,j)}$ and $\mathcal{U}^{(r)} = \bigtimes_{k=0}^{r} \mathcal{U}_k$. We denote by $(u^I{}_{\lambda\mu\nu_1\dots\nu_i K_1\dots K_j})$ the canonical coordinates on $\mathcal{U}_{(i,j)}$. The action of the subgroup $G^1_m \times G^{r+1}_n \subset G^{r+1}_{m,n}$ on $\mathcal{U}^{(r)}$ can be deduced from iterated formal vertical derivatives of the tensor action of $G^1_m \times G^1_n$ on $\mathbb{R}^n \otimes \bigwedge^2 \mathbb{R}^{m*} \otimes \bigotimes^i \mathbb{R}^{m*}$. For instance, for $r = 1$, we have the coordinates on the standard fiber $\mathcal{U}^{(1)}$ given by $(u^I{}_{\lambda\mu}, u^I{}_{\lambda\mu\nu}, u^I{}_{\lambda\mu J})$ with the action of the subgroup $G^1_m \times G^2_n \subset G^2_{m,n}$ given by

$$\bar{u}^I{}_{\lambda\mu} = a^I_P\, u^P{}_{\rho\sigma}\, \tilde{a}^\rho_\lambda\, \tilde{a}^\sigma_\mu, \qquad \bar{u}^I{}_{\lambda\mu\nu} = a^I_P\, u^P{}_{\rho\sigma\tau}\, \tilde{a}^\rho_\lambda\, \tilde{a}^\sigma_\mu\, \tilde{a}^\tau_\nu,$$
$$\bar{u}^I{}_{\lambda\mu J} = (a^I_P\, u^P_{\rho\sigma Q} + a^I_{PQ}\, u^P{}_{\rho\sigma})\, \tilde{a}^\rho_\lambda\, \tilde{a}^\sigma_\mu\, \tilde{a}^Q_J.$$

We denote by

$$\mathcal{R}_{G,(i,j)} : T^{i+j+1}_{m+n} S_{\mathrm{Gen}} \times T^{i+j-1}_m S_{\mathrm{Cla}} \to \mathcal{U}_{(i,j)} \tag{6.2}$$

the $G^{i+j+2}_{m,n}$-equivariant map associated with the ith general covariant and the jth vertical derivative of curvature tensors of general connections

$$V^j(\nabla^i R[\Gamma]) : C^\infty(\mathrm{Gen}\, Y \times_Y \mathrm{Cla}\, M) \to C^\infty(\mathcal{U}_{(i,j)}Y).$$

The map $\mathcal{R}_{G,(i,j)}$ is said to be the *formal curvature map of order* (i,j) of general connections. We set

$$\mathcal{R}_{G,k} = \bigtimes_{i+j=k} \mathcal{R}_{G,(i,j)} : T^{k+1}_{m+n} S_{\mathrm{Gen}} \times T^{k-1}_m S_{\mathrm{Cla}} \to \mathcal{U}_k.$$

Let $C_{G,(i,j)} \subset \mathcal{U}_{(i,j)}$ be the subset given by the identities of the (i,j)th general covariant and vertical derivatives of the curvature tensors of general connections which are obtained by taking the general covariant and vertical derivatives of the general Bianchi and general Ricci identities (see Lemmas 4.3 and 4.4 or [J4]). Let us put $C_{G,k} = \bigtimes_{i+j=k} C_{G,(i,j)}$ and $C^{(r)}_G =$

$\times^{r}_{k=0} C_{G,k}$ and

$$(6.3)\qquad \mathcal{R}^{(r)}_G := \mathop{\times}_{k=0}^{r} \mathcal{R}_{G,k} : T^{r+1}_{m+n} S_{\text{Gen}} \times T^{r-1}_m S_{\text{Cla}} \to \mathcal{U}^{(r)},$$

which has values in $C^{(r)}_G$.

Now we shall prove

Lemma 6.1. *If $s \geq r-2$, $r \geq 0$, then $C^{(r)}_G \times C^{(s)}_C$ is a submanifold of $\mathcal{U}^{(r)} \times \mathcal{W}^{(s)}$. Then $C^{(r)}_G \times C^{(s)}_C = (\mathcal{R}^{(r)}_G, \mathcal{R}^{(s)}_C)(T^{r+1}_{m+n} S_{\text{Gen}} \times T^{s+1}_m S_{\text{Cla}})$ and the restricted map*

$$(\mathcal{R}^{(r)}_G, \mathcal{R}^{(s)}_C) : T^{r+1}_{m+n} S_{\text{Gen}} \times T^{s+1}_m S_{\text{Cla}} \to C^{(r)}_G \times C^{(s)}_C$$

is a surjective submersion.

Proof. First, let us recall that $C^{(s)}_C$ is a submanifold of $\mathcal{W}^{(s)}$ and $\mathcal{R}^{(s)}_C : T^{s+1}_m S_{\text{Cla}} \to C^{(s)}_C$ is a surjective submersion [KMS, p. 236].

Next, we prove by induction that $C^{(r)}_G$ is a submanifold of $\mathcal{U}^{(r)}$.

For $r = 0$, $C^{(0)}_G = \mathcal{U}^{(0)} = \mathcal{U}$. For $r = 1$ we have $C_{G,1} = C_{G,(1,0)} \times C_{G,(0,1)}$ a linear subspace of $\mathcal{U}_1 = \mathcal{U}_{(1,0)} \times \mathcal{U}_{(0,1)}$ given by the solution of the formal general Bianchi identity (4.6). Hence $C^{(1)}_G$ is a vector subbundle of $C^{(0)}_G \times \mathcal{U}_1$.

Assume $C^{(r-1)}_G \subset \mathcal{U}^{(r-1)}$ is a submanifold and consider the product bundle $C^{(r-1)}_G \times \mathcal{U}_r$. The equations defining $C_{G,r}$ consist of the formal general covariant and vertical derivatives of (4.6) and of formal expressions of alternations of the second order general covariant derivatives of curvature tensors and their general covariant and vertical derivatives. By Lemmas 4.3, 4.4 and Corollary 4.6 we have the following two systems of equations:

$$(6.4)\qquad u^I{}_{(\lambda\mu\nu_1)\nu_2\dots\nu_i K_1\dots K_j} = 0, \qquad i \geq 1,$$

$$(6.5)\qquad u^I{}_{\lambda\mu\nu_1\dots[\nu_{s-1}\nu_s]\dots\nu_i K_1\dots K_j} + \operatorname{pol}(C^{(r-1)}_G \times C^{(r-2)}_C) = 0, \qquad i \geq 2,$$

$i + j = r$, where $\operatorname{pol}(C^{(r-1)}_G \times C^{(r-2)}_C)$ are some polynomials in elements of $C^{(r-1)}_G \times C^{(r-2)}_C$. The map defined by the left-hand sides of (6.4) and (6.5) represents an affine bundle morphism $C^{(r-1)}_G \times C^{(s)}_C \times \mathcal{U}_r \to C^{(r-1)}_G \times C^{(s)}_C \times \mathbb{R}^N$ of constant rank, $N =$ the number of equations (6.4) and (6.5). Its kernel $C^{(r)}_G \times C^{(s)}_C$ is a subbundle of $C^{(r-1)}_G \times C^{(s)}_C \times \mathcal{U}_r$.

To prove $(\mathcal{R}^{(r)}_G, \mathcal{R}^{(s)}_C)$ is a surjective submersion it is sufficient to prove that the map $\mathcal{R}^{(r)}_G(-, j^{s+1}_0 \lambda) : T^{r+1}_{m+n} S_{\text{Gen}} \to C^{(r)}_G$ is a surjective submersion for any $j^{s+1}_0 \lambda \in T^{s+1}_m S_{\text{Cla}}$. We shall prove this by induction. For $r = 0$ we have $\mathcal{R}^{(0)}_G(-, j^{s+1}_0 \lambda) \equiv \mathcal{R}_G$ and we shall prove $\mathcal{R}_G(T^1_{m+n} S_{\text{Gen}}) = C^{(0)}_G = \mathcal{U}$. The coordinate expression of $\mathcal{R}_G$ is (3.5). This is an affine bundle morphism of the affine bundle $\pi^1_0 : T^1_{m+n}S_{\text{Gen}} \to S_{\text{Gen}}$ into $\mathcal{U}$ of constant rank. We know that the values of $\mathcal{R}_G$ lie in $\mathcal{U}$, so that it suffices to prove that the image is the whole $\mathcal{U}$ at one point $0 \in S_{\text{Gen}}$. Consider the restriction $\bar{\mathcal{R}}_G$ of $\mathcal{R}_G$ to the fiber $(\pi^1_0)^{-1}(0)$, $0 \in S_{\text{Gen}}$. Then

$$u^I{}_{\lambda\mu} \circ \bar{\mathcal{R}}_G = \Gamma^I{}_{\lambda,\mu} - \Gamma^I{}_{\mu,\lambda}.$$

$\bar{\mathcal{R}}_G$ is a surjection if and only if

$$\dim \mathcal{U} = \dim (\pi^1_0)^{-1}(0) - \dim \operatorname{Ker} \bar{\mathcal{R}}_P.$$

But $\dim \mathcal{U} = \frac{1}{2}nm(m-1)$, $\dim (\pi^1_0)^{-1}(0) = nm^2 + n^2m$ and $\dim \operatorname{Ker} \bar{\mathcal{R}}_G = \frac{1}{2}nm(m+1) + n^2m$. This implies that $\bar{\mathcal{R}}_G$ is surjective and hence $\mathcal{R}_G$ is a surjective submersion.

Assume by induction

$$\mathcal{R}^{(r-1)}_G(-, j^{s+1}_0\lambda) : T^r_{m+n}S_{\text{Gen}} \to C^{(r-1)}_G$$

is a surjective submersion. So we have the commutative diagram

$$\begin{array}{ccc}
T^{r+1}_{m+n}S_{\text{Gen}} & \xrightarrow{\mathcal{R}^{(r)}_G(-,j^{s+1}_0\lambda)} & C^{(r)}_G \\
{\scriptstyle \pi^{r+1}_r}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}^r_{r-1}} \\
T^r_{m+n}S_{\text{Gen}} & \xrightarrow{\mathcal{R}^{(r-1)}_G(-,j^{s+1}_0\lambda)} & C^{(r-1)}_G
\end{array}$$

where the bottom arrow is by assumption a surjective submersion. The mapping $\mathcal{R}^{(r)}_G(-, j^{s+1}_0\lambda)$ is affine (with respect to affine structures of both projections) and of constant rank on all fibers. Hence if the fibers of the projection $\pi^{r+1}_r : T^{r+1}_{m+n}S_{\text{Gen}} \to T^r_{m+n}S_{\text{Gen}}$ are mapped surjectively onto fibers of the projection $\mathrm{pr}^r_{r-1} : C^{(r)}_G \to C^{(r-1)}_G$, the top arrow is a surjective submersion. Let us denote by $\bar{\mathcal{R}}^{(r)}_G(-, j^{s+1}_0\lambda)$ the restriction of $\mathcal{R}^{(r)}_G(-, j^{s+1}_0\lambda)$ to the fiber over $0 \in T^r_{m+n}S_{\text{Gen}}$. Then $\bar{\mathcal{R}}^{(r)}_G(-, j^{s+1}_0\lambda) = \bar{\mathcal{R}}_{G,r}(-, j^{s+1}_0\lambda) : (\pi^{r+1}_r)^{-1}(0) \to (C_{G,r})_0$, where $(C_{G,r})_0$ is the fiber of the projection $\mathrm{pr}^r_{r-1} : C^{(r)}_G \to C^{(r-1)}_G$ over the zero in $C^{(r-1)}_G$.

We shall prove $\bar{\mathcal{R}}_{G,r}(-, j^{s+1}_0\lambda) : (\pi^{r+1}_r)^{-1}(0) \to (C_{G,r})_0$ is surjective. In coordinates

$$(u^I{}_{\lambda\mu\nu_1\dots\nu_i K_1\dots K_j}) \circ \bar{\mathcal{R}}_{G,(i,j)}(-, j^{s+1}_0\lambda) = \Gamma^I{}_{\lambda,\mu\nu_1\dots\nu_i K_1\dots K_j} - \Gamma^I{}_{\mu,\lambda\nu_1\dots\nu_i K_1\dots K_j}.$$

The fiber $(C_{G,(i,j)})_0$ equals

$$(C_{G,(0,0)} \otimes S^i\mathbb{R}^{m*} \otimes S^j\mathbb{R}^{n*}) \cap (C_{G,(1,0)} \otimes S^{i-1}\mathbb{R}^{m*} \otimes S^j\mathbb{R}^{n*}),$$

which follows from (6.4) and (6.5).

Let us note that $(\pi^{r+1}_r)^{-1}(0) = \times_{i+j=r+1} \mathbb{R}^n \otimes \mathbb{R}^{m*} \otimes S^i\mathbb{R}^{m*} \otimes S^j\mathbb{R}^{n*}$.

By induction assumption the map

$$(\bar{\mathcal{R}}_{G,(1,0)} \times \bar{\mathcal{R}}_{G,(0,1)})(-, j_0^{s+1}\lambda) : \\ (\mathbb{R}^n \otimes \mathbb{R}^{m*} \otimes S^2\mathbb{R}^{m*}) \times (\mathbb{R}^n \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{n*}) \to (C_{G,(1,0)} \times C_{G,(0,1)})_0$$

is a surjection which splits into two surjective mappings

$$\bar{\mathcal{R}}_{G,(1,0)}(-, j_0^{s+1}\lambda) : \mathbb{R}^n \otimes \mathbb{R}^{m*} \otimes S^2\mathbb{R}^{m*} \to (C_{G,(1,0)})_0,$$
$$\bar{\mathcal{R}}_{G,(0,1)}(-, j_0^{s+1}\lambda) : \mathbb{R}^n \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{n*} \to (C_{G,(0,1)})_0,$$

with the coordinate expressions

$$u^I{}_{\lambda\mu\nu} = \Gamma^I{}_{\lambda,\mu\nu} - \Gamma^I{}_{\mu,\lambda\nu}, \qquad u^I{}_{\lambda\mu J} = \Gamma^I{}_{\lambda,\mu J} - \Gamma^I{}_{\mu,\lambda J}.$$

Hence

$$\bar{\mathcal{R}}_{G,(1,0)}(-, j_0^{s+1}\lambda) \otimes \mathrm{id}_{S^{i-1}\mathbb{R}^{m*}\otimes S^j\mathbb{R}^{n*}} : \\ (\mathbb{R}^n \otimes \mathbb{R}^{m*} \otimes S^2\mathbb{R}^{m*}) \otimes S^{i-1}\mathbb{R}^{m*} \otimes S^j\mathbb{R}^{n*} \to (C_{G,(1,0)})_0 \otimes S^{i-1}\mathbb{R}^{m*} \otimes S^j\mathbb{R}^{n*}$$

is also a surjection.

Consider an element $X \in (C_{G,(i,j)})_0$, $i + j = r$. Then there exists

$$Z \in (\mathbb{R}^n \otimes \mathbb{R}^{m*} \otimes S^2\mathbb{R}^{m*}) \otimes S^{i-1}\mathbb{R}^{m*} \otimes S^j\mathbb{R}^{n*}$$

such that

$$(\bar{\mathcal{R}}_{G,(1,0)}(-, j_0^{s+1}\lambda) \otimes \mathrm{id}_{S^{i-1}\mathbb{R}^{m*}\otimes S^j\mathbb{R}^{n*}})(Z) = X,$$

i.e. in coordinates

$$X^I{}_{\lambda\mu\nu_1\dots\nu_i K_1\dots K_j} = Z^I{}_{\lambda\mu\nu_1\dots\nu_i K_1\dots K_j} - Z^I{}_{\mu\lambda\nu_1\dots\nu_i K_1\dots K_j}.$$

Symmetrization gives

$$\bar{Z}^I{}_{\lambda\mu\nu_1\nu_2\nu_3\dots\nu_i K_1\dots K_j} = Z^I{}_{\lambda\mu(\nu_1\nu_2)\nu_3\dots\nu_i K_1\dots K_j} \\ \in \mathbb{R}^n \otimes \mathbb{R}^{m*} \otimes S^{i+1}\mathbb{R}^{m*} \otimes S^j\mathbb{R}^{n*} = (\pi^{r+1}_r)^{-1}(0)$$

such that $\bar{\mathcal{R}}_{G,(i,j)}(\bar{Z}, j_0^{s+1}\lambda) = X$ and hence $\bar{\mathcal{R}}_{G,(i,j)}(-, j_0^{s+1}\lambda)$ is a surjection. This implies that the top arrow in the above diagram is a surjective submersion. ∎

In the above Lemma 6.1 we have proved that $C^{(r)}_G \times C^{(s)}_C$ is a submanifold of $\mathcal{U}^{(r)} \times \mathcal{W}^{(s)}$. It is easy to see that $C^{(r)}_G \times C^{(s)}_C$ is closed with respect to the action of the group $G^1_m \times G^{r+2}_n$ and, according to the general theory of natural bundles, we have the $(1, r+2)$-order natural vector subbundle $C^{(r)}_G Y \times_Y C^{(s)}_C M$ of the $(1, r+2)$-order natural vector bundle $\mathcal{U}^{(r)}Y \times_Y \mathcal{W}^{(s)}M$ called the *(r, s)-order curvature bundle of general and classical connections.*

7. Orbit reduction theorem for general connections. Let F be a natural bundle functor of order $(1, r+1)$, i.e. S_F is a left $(G^1_m \times G^{r+1}_n)$-manifold. From the fact that $G^1_m \times G^{r+1}_n$ is a subgroup of $G^{r+1}_{m,n}$ we see that S_F can be considered as a left $G^{r+1}_{m,n}$-manifold. Similarly, as G^s_m is a subgroup of $G^s_{m,n}$, we can consider $T^{s-2}_m S_{\mathrm{Cla}}$ as a left $G^s_{m,n}$-manifold.

THEOREM 7.1. *Let $s \geq r-2$, $r \geq 0$, $t = \max(r+1, s+2)$. For every $G^t_{m,n}$-equivariant map*

$$f : T^r_{m+n} S_{\mathrm{Gen}} \times T^s_m S_{\mathrm{Cla}} \to S_F$$

there exists a unique $(G^1_m \times G^t_n)$-equivariant map $g : C^{(r-1)}_G \times C^{(s-1)}_C \to S_F$ satisfying

$$f = g \circ (\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(s-1)}_C).$$

Proof. Let us consider the spaces

$$S_{C,s} := \mathbb{R}^m \otimes S^s \mathbb{R}^{m*} \quad \text{and} \quad S_{G,(i+1,j)} := \mathbb{R}^n \otimes S^{i+1}\mathbb{R}^{m*} \otimes S^j \mathbb{R}^{n*}$$

with coordinates $(s^\lambda{}_{\mu_1\mu_2\dots\mu_s})$ and $(s^I{}_{\mu_1\dots\mu_{i+1}K_1\dots K_j})$, respectively. Let us consider the action of G^s_m on $S_{C,s}$ and the action of $G^{i+j+1}_{m,n}$ on $S_{G,(i+1,j)}$ such that the symmetrization maps

$$\sigma_{C,s} : T^s_m S_{\mathrm{Cla}} \to S_{C,s+2}, \qquad \sigma_{G,r} : T^r_{m+n} S_{\mathrm{Gen}} \to S_{G,r+1} := \bigtimes_{i+j=r} S_{G,(i+1,j)}$$

given by

$$(s^\lambda{}_{\mu_1\mu_2\dots\mu_{s+2}}) \circ \sigma_{C,s} = \Lambda_{(\mu_1}{}^\lambda{}_{\mu_2,\mu_3\dots\mu_{s+2})},$$
$$(s^I{}_{\mu_1\dots\mu_{i+1}K_1\dots K_j}) \circ \sigma_{G,r} = \Gamma^I{}_{(\mu_1,\mu_2\dots\mu_{i+1})K_1\dots K_j}, \qquad i+j=r,$$

are G^{s+2}_m- and $G^{r+1}_{m,n}$-equivariant, respectively.

We have the G^{s+2}_m-equivariant map

$$\varphi_{C,s} := (\sigma_{C,s}, \pi^s_{s-1}, \mathcal{R}_{C,s-1}) : T^s_m S_{\mathrm{Cla}} \to S_{C,s+2} \times T^{s-1}_m S_{\mathrm{Cla}} \times \mathcal{W}_{s-1}.$$

On the other hand we can define the G^{s+2}_m-equivariant map

$$\psi_{C,s} : S_{C,s+2} \times T^{s-1}_m S_{\mathrm{Cla}} \times \mathcal{W}_{s-1} \to T^s_m S_{\mathrm{Cla}}$$

over the identity of $T^{s-1}_m S_{\mathrm{Cla}}$ such that

$$\psi_{C,s} \circ \varphi_{C,s} = \mathrm{id}_{T^s_m S_{\mathrm{Cla}}}.$$

This map is given by the coordinate expression

$$\Lambda_{\mu_1}{}^\lambda{}_{\mu_2,\mu_3\dots\mu_{s+2}} = s^\lambda{}_{\mu_1\dots\mu_{s+2}} + \sum_\sigma A_\sigma w_{\mu_{\sigma(1)}}{}^\lambda{}_{\mu_{\sigma(2)}\dots\mu_{\sigma(s+2)}} - \mathrm{pol}(T^{s-1}_m S_{\mathrm{Cla}}), \tag{7.1}$$

where A_σ are real coefficients and σ is a permutation of $s+2$ indices (for details see [J3, KMS]).

Similarly we have the $G^{r+1}_{m,n}$-equivariant map

$$\varphi_{G,r} := (\sigma_{G,r}, \pi^r_{r-1} \times \mathrm{id}_{T^{r-2}_m S_{\mathrm{Cla}}}, \mathcal{R}_{G,r-1}) : T^r_{m+n} S_{\mathrm{Gen}} \times T^{r-2}_m S_{\mathrm{Cla}} \\ \to S_{G,r+1} \times T^{r-1}_{m+n} S_{\mathrm{Gen}} \times T^{r-2}_m S_{\mathrm{Cla}} \times \mathcal{U}_{r-1}$$

where we have used the notation $\mathcal{R}_{G,r-1} = \times_{i+j=r} \mathcal{R}_{G,(i-1,j)}$ and $\mathcal{U}_{r-1} = \times_{i+j=r} \mathcal{U}_{(i-1,j)}$, and we define the $G^{r+1}_{m,n}$-equivariant map

$$\psi_{G,r} : S_{G,r+1} \times T^{r-1}_{m+n} S_{\mathrm{Gen}} \times T^{r-2}_m S_{\mathrm{Cla}} \times \mathcal{U}_{r-1} \to T^r_{m+n} S_{\mathrm{Gen}} \times T^{r-2}_m S_{\mathrm{Cla}}$$

over the identity of $T^{r-1}_{m+n} S_{\mathrm{Gen}} \times T^{r-2}_m S_{\mathrm{Cla}}$ by the coordinate expression

$$\Gamma^I{}_{\mu_1,\mu_2\dots\mu_{i+1}K_1\dots K_j} = s^I{}_{\mu_1\dots\mu_{i+1}K_1\dots K_j} + \sum_\sigma A_\sigma u^I{}_{\mu_{\sigma(1)}\dots\mu_{\sigma(i+1)}K_1\dots K_j} \\ - \mathrm{pol}\Big(\bigtimes_{l\geq i} S_{G,(l,r-l+1)} \times T^{r-1}_{m+n} S_{\mathrm{Gen}} \times T^{i-2}_m S_{\mathrm{Cla}} \times \bigtimes_{l \geq i} \mathcal{U}_{(l-2,r-l+1)} \Big), \tag{7.2}$$

$i+j = r$, where σ is a permutation of $i+1$ indices and A_σ are real coefficients. $\psi_{G,r}$ can be constructed in the following way. First,

$$\Gamma^I{}_{\mu,K_1\dots K_r} = s^I{}_{\mu K_1\dots K_r}.$$

Second, by taking the $(r-1)$-order formal vertical derivative of (3.5) we get

$$\begin{aligned} u^I{}_{\mu_1\mu_2K_1\dots K_{r-1}} &= 2\Gamma^I{}_{[\mu_1,\mu_2]K_1\dots K_{r-1}} \\ &\quad + \Gamma^I{}_{\mu_1,PK_1\dots K_{r-1}}\,\Gamma^P{}_{\mu_2} - \Gamma^I{}_{\mu_2,PK_1\dots K_{r-1}}\,\Gamma^P{}_{\mu_1} + \mathrm{pol}(T^{r-1}_{m+n} S_{\mathrm{Gen}}) \\ &= 2\Gamma^I{}_{[\mu_1,\mu_2]K_1\dots K_{r-1}} \\ &\quad + s^I_{\mu_1PK_1\dots K_{r-1}}\,\Gamma^P{}_{\mu_2} - s^I_{\mu_2PK_1\dots K_{r-1}}\,\Gamma^P{}_{\mu_1} + \mathrm{pol}(T^{r-1}_{m+n} S_{\mathrm{Gen}}) \\ &= 2\Gamma^I{}_{[\mu_1,\mu_2]K_1\dots K_{r-1}} + \mathrm{pol}(S_{G,(1,r)} \times T^{r-1}_{m+n} S_{\mathrm{Gen}}), \end{aligned}$$

where $[\dots]$ denotes antisymmetrization. We can write

$$\Gamma^I{}_{\mu_1,\mu_2K_1\dots K_{r-1}} = s^I{}_{\mu_1\mu_2K_1\dots K_{r-1}} + (\Gamma^I{}_{\mu_1,\mu_2K_1\dots K_{r-1}} - \Gamma^I{}_{(\mu_1,\mu_2)K_1\dots K_{r-1}}).$$

Then the term in brackets can be written as

$$\Gamma^I{}_{[\mu_1,\mu_2]K_1\dots K_{r-1}} = \tfrac{1}{2} u^I{}_{\mu_1\mu_2K_1\dots K_{r-1}} - \mathrm{pol}(S_{G,(1,r)} \times T^{r-1}_{m+n} S_{\mathrm{Gen}}),$$

and we get

$$\Gamma^I{}_{\mu_1,\mu_2K_1\dots K_{r-1}} = s^I{}_{\mu_1\mu_2K_1\dots K_{r-1}} \\ + \tfrac{1}{2} u^I{}_{\mu_1\mu_2K_1\dots K_{r-1}} - \mathrm{pol}(S_{G,(1,r)} \times T^{r-1}_{m+n} S_{\mathrm{Gen}}). \tag{7.3}$$

Next, by taking the formal general covariant and the $(r-2)$-order formal vertical derivative of (3.5) we get

$$u^I{}_{\mu_1\mu_2\mu_3K_1\dots K_{r-2}} = 2\Gamma^I{}_{[\mu_1,\mu_2]\mu_3K_1\dots K_{r-2}} + \Gamma^I{}_{\mu_1,\mu_3PK_1\dots K_{r-2}}\Gamma^P{}_{\mu_2} \\ - \Gamma^I{}_{\mu_2,\mu_3PK_1\dots K_{r-2}}\Gamma^P{}_{\mu_1} + \mathrm{pol}(T^{r-1}_{m+n} S_{\mathrm{Gen}} \times S_{\mathrm{Cla}})$$

and from (7.3) we get

$$u^I{}_{\mu_1\mu_2\mu_3K_1\dots K_{r-2}} = 2\Gamma^I{}_{[\mu_1,\mu_2]\mu_3K_1\dots K_{r-2}} + \mathrm{pol}(S_{G,(2,r-1)} \times S_{G,(1,r)} \times T^{r-1}_{m+n}S_{\mathrm{Gen}} \times S_{\mathrm{Cla}} \times \mathcal{U}_{G,(0,r-1)}).$$

We can write

$$\Gamma^I{}_{\mu_1,\mu_2\mu_3K_1\dots K_{r-2}} = s^I{}_{\mu_1\mu_2\mu_3K_1\dots K_{r-2}} + (\Gamma^I{}_{\mu_1,\mu_2\mu_3K_1\dots K_{r-2}} - \Gamma^I{}_{(\mu_1,\mu_2\mu_3)K_1\dots K_{r-2}}).$$

The term in brackets can be written as

$$\begin{aligned}
&\tfrac{2}{3}\Gamma^I{}_{[\mu_1,\mu_2]\mu_3K_1\dots K_{r-2}} + \tfrac{2}{3}\Gamma^I{}_{[\mu_1,\mu_3]\mu_2K_1\dots K_{r-2}} \\
&\quad = \tfrac{1}{3}(u^I{}_{\mu_1\mu_2\mu_3K_1\dots K_{r-2}} \\
&\qquad - \mathrm{pol}(S_{G,(2,r-1)} \times S_{G,(1,r)} \times T^{r-1}_{m+n}S_{\mathrm{Gen}} \times S_{\mathrm{Cla}} \times \mathcal{U}_{G,(0,r-1)})) \\
&\qquad + \tfrac{1}{3}\,(u^I{}_{\mu_1\mu_3\mu_2K_1\dots K_{r-2}} \\
&\qquad - \mathrm{pol}(S_{G,(2,r-1)} \times S_{G,(1,r)} \times T^{r-1}_{m+n}S_{\mathrm{Gen}} \times S_{\mathrm{Cla}} \times \mathcal{U}_{G,(0,r-1)})),
\end{aligned}$$

and we get

$$\begin{aligned}
(7.4)\qquad \Gamma^I{}_{\mu_1,\mu_2\mu_3K_1\dots K_{r-2}} &= s^I{}_{\mu_1\mu_2\mu_3K_1\dots K_{r-2}} \\
&\quad + \tfrac{1}{3}\,(u^I{}_{\mu_1\mu_2\mu_3K_1\dots K_{r-2}} + u^I{}_{\mu_1\mu_3\mu_2K_1\dots K_{r-2}}) \\
&\quad - \mathrm{pol}(S_{G,(2,r-1)} \times S_{G,(1,r)} \times T^{r-1}_{m+n}S_{\mathrm{Gen}} \times S_{\mathrm{Cla}} \times \mathcal{U}_{G,(0,r-1)}).
\end{aligned}$$

Continuing in this way by increasing the order of general covariant derivatives we get (7.2).

Moreover, it is easy to see that

$$\psi_{G,r} \circ \varphi_{G,r} = \mathrm{id}_{T^r_{m+n}S_{\mathrm{Gen}} \times T^{r-2}_m S_{\mathrm{Cla}}}\,.$$

Now we have to distinguish three possibilities.

(A) Let $s = r-1$. The groups G^{r+1}_m and $G^{r+1}_{m,n}$ acting on $T^{r-1}_m S_{\mathrm{Cla}}$ and $T^r_m S_{\mathrm{Gen}}$, respectively, are of the same order. Moreover, we consider G^{r+1}_m as a subgroup in $G^{r+1}_{m,n}$.

Let us set

$$A^r := T^{r-1}_{m+n}S_{\mathrm{Gen}} \times T^{r-2}_m S_{\mathrm{Cla}} \times \mathcal{U}_{r-1} \times \mathcal{W}_{r-2}.$$

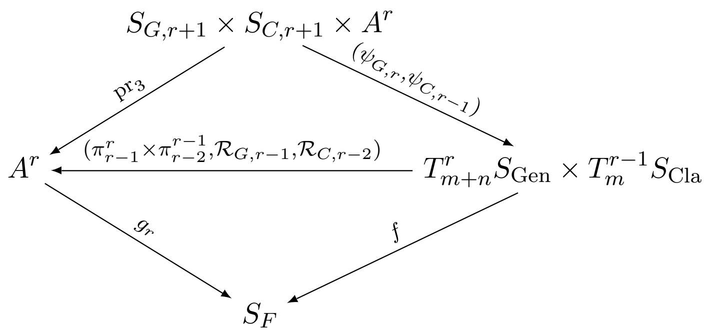
Then the map $f\circ(\psi_{G,r},\psi_{C,r-1}) : S_{G,r+1}\times S_{C,r+1}\times A^r \to S_F$ satisfies the conditions of the orbit reduction (see Theorem 2.1) for the group epimorphism $\pi^{r+1}_r \times q^{r+1}_r(n) : G^{r+1}_{m,n} \to G^r_{m,n} \times B^{r+1}_r(n)$ and the surjective submersion $\mathrm{pr}_3 : S_{G,r+1} \times S_{C,r+1} \times A^r \to A^r$. Indeed, if we denote by K^{r+1}_r the kernel of $\pi^{r+1}_r \times q^{r+1}_r(n)$, then $S_{G,r+1} \times S_{C,r+1}$ is a K^{r+1}_r-orbit. Indeed, it is easy to see that we have the coordinates

$$(a^\lambda_{\mu_1\dots\mu_{r+1}}, a^I_{\mu_1\dots\mu_{i+1}K_1\dots K_j}), \quad i+j=r,$$

on K_r^{r+1} and the restriction of the action of $G_{m,n}^{r+1}$ on $S_{G,r+1} \times S_{C,r+1}$ to K_r^{r+1} has the coordinate expression

$$\bar{s}^{\lambda}_{\mu_1\ldots\mu_{r+1}} = s^{\lambda}_{\mu_1\ldots\mu_{r+1}} + a^{\lambda}_{\mu_1\ldots\mu_{r+1}},$$
$$\bar{s}^{I}_{\mu_1\ldots\mu_{i+1}K_1\ldots K_j} = s^{i}_{\mu_1\ldots\mu_{i+1}K_1\ldots K_j} + a^{I}_{\mu_1\ldots\mu_{i+1}K_1\ldots K_j}.$$

Hence this action is simply transitive and there exists a unique $(G^r_{m,n} \times B_r^{r+1}(n))$-equivariant map $g_r : A^r \to S_F$ such that the diagram



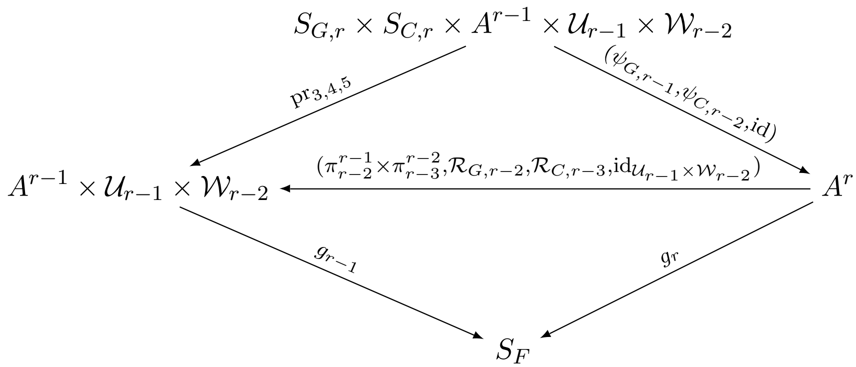
commutes. So $f \circ (\psi_{G,r}, \psi_{C,r-1}) = g_r \circ \mathrm{pr}_3$ and if we compose both sides with $(\varphi_{G,r}, \varphi_{C,r-1})$, by considering

$$\mathrm{pr}_3 \circ (\varphi_{G,r}, \varphi_{C,r-1}) = (\pi^r_{r-1} \times \pi^{r-1}_{r-2}, \mathcal{R}_{G,r-1}, \mathcal{R}_{C,r-2}),$$

we obtain

$$f = g_r \circ (\pi^r_{r-1} \times \pi^{r-1}_{r-2}, \mathcal{R}_{G,r-1}, \mathcal{R}_{C,r-2}).$$

In the second step we consider the same construction for the map g_r and obtain the commutative diagram



So there exists a unique $(G^{r-1}_{m,n} \times B^{r+1}_{r-1}(n))$-equivariant map $g_{r-1} : A^{r-1} \times \mathcal{U}_{r-1} \times \mathcal{W}_{r-2} \to S_F$ such that

$$g_r = g_{r-1} \circ (\pi^{r-1}_{r-2} \times \pi^{r-2}_{r-3}, \mathcal{R}_{G,r-2}, \mathcal{R}_{C,r-3}, \mathrm{id}_{\mathcal{U}_{r-1} \times \mathcal{W}_{r-2}}),$$

i.e.

$$f = g_{r-1} \circ (\pi^r_{r-2} \times \pi^{r-1}_{r-3}, \mathcal{R}_{G,r-2}, \mathcal{R}_{G,r-1}, \mathcal{R}_{C,r-3}, \mathcal{R}_{C,r-2}).$$

Proceeding in this way we get in the last step a unique $(G^1_{m,n}\times B^{r+1}_1(n))$-equivariant map $g_1 : S_{\mathrm{Gen}}\times \mathcal{U}^{(r-1)}\times \mathcal{W}^{(r-2)}\to S_F$ such that

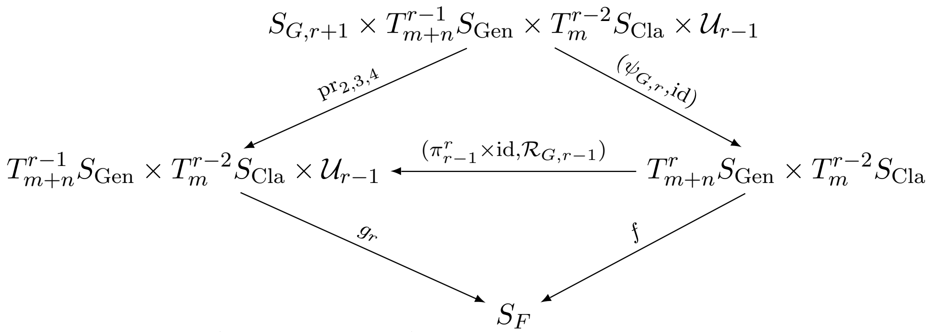
$$f = g_1\circ(\pi^r_0\circ \mathrm{pr}_1, \mathcal{R}^{(r-1)}_G, \mathcal{R}^{(r-2)}_C).$$

Finally, g_1 satisfies the orbit reduction theorem for the group epimorphism $G^1_{m,n}\times B^{r+1}_1(n)\to G^1_m\times G^{r+1}_n$, given by the epimorphism $p^1_1\times p^1_2 : G^1_{m,n}\to G^1_m\times G^1_n$ and the surjective submersion $\mathrm{pr}_{2,3} : S_{\mathrm{Gen}}\times\mathcal{U}^{(r-1)}\times\mathcal{W}^{(r-2)}\to\mathcal{U}^{(r-1)}\times\mathcal{W}^{(r-2)}$. Indeed, S_{Gen} is an orbit with respect to the action of the kernel of the group epimorphism $G^1_{m,n}\times B^{r+1}_1(n)\to G^1_m\times G^{r+1}_n$ (this follows from $\bar{\Gamma}^I{}_\lambda = \Gamma^I{}_\lambda + a^I_\lambda$, where a^I_λ are the coordinates on the kernel of $G^1_{m,n}\times B^{r+1}_1(n)\to G^1_m\times G^{r+1}_n$). So there is a unique $(G^1_m\times G^{r+1}_n)$-equivariant map $g : \mathcal{U}^{(r-1)}\times\mathcal{W}^{(r-2)}\to S_F$ such that $g_1 = g\circ\mathrm{pr}_{2,3}$ and hence

$$f = g\circ(\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(r-2)}_C).$$

(B) Let $s = r-2$. We have the action of the group G^r_m on $T^{r-2}_m S_{\mathrm{Cla}}$ and the action of the group $G^{r+1}_{m,n}$ on $T^r_{m+n}S_{\mathrm{Gen}}$. We consider G^r_m as a subgroup in $G^{r+1}_{m,n}$.

Then the map $f\circ(\psi_{G,r}\,\mathrm{id}_{T^{r-2}_m S_{\mathrm{Cla}}}) : S_{G,r+1}\times T^{r-1}_{m+n}S_{\mathrm{Gen}}\times T^{r-2}_m S_{\mathrm{Cla}}\times\mathcal{U}_{r-1}\to S_F$ satisfies the conditions of the orbit reduction (Theorem 2.1) for the group epimorphism $\pi^{r+1}_r\times q^{r+1}_r(n) : G^{r+1}_{m,n}\to G^r_{m,n}\times B^{r+1}_r(n)$ and the surjective submersion $\mathrm{pr}_{2,3,4} : S_{G,r+1}\times T^{r-1}_{m+n}S_{\mathrm{Gen}}\times T^{r-2}_m S_{\mathrm{Cla}}\times\mathcal{U}_{r-1}\to T^{r-1}_{m+n}S_{\mathrm{Gen}}\times T^{r-2}_m S_{\mathrm{Cla}}\times\mathcal{U}_{r-1}$. Indeed, the space $S_{G,r+1} = \bigtimes_{i+j=r} S_{G,(i+1,j)}$ is a K^{r+1}_r-orbit. Let us note that the action of B^{r+1}_r on $\bigtimes S_{G,r+1}$ is transitive, but not simply transitive. Hence there exists a unique $(G^r_{m,n}\times B^{r+1}_r(n))$-equivariant map $g_r : T^{r-1}_{m+n}S_{\mathrm{Gen}}\times T^{r-2}_m S_{\mathrm{Cla}}\times\mathcal{U}_{r-1}\to S_F$ such that the diagram



commutes. So $f\circ(\psi_{P,r}, \mathrm{id}_{T^{r-2}_m S_{\mathrm{Cla}}}) = g_r\circ\mathrm{pr}_{2,3,4}$ and if we compose both sides with the mapping $(\varphi_{G,r}, \mathrm{id}_{T^{r-2}_m S_{\mathrm{Cla}}})$, by considering $\mathrm{pr}_{2,3,4}\circ(\varphi_{G,r}, \mathrm{id}_{T^{r-2}_m S_{\mathrm{Cla}}}) = (\pi^r_{r-1}\times\mathrm{id}_{T^{r-2}_m S_{\mathrm{Cla}}}, \mathcal{R}_{G,r-1})$, we obtain

$$f = g_r\circ(\pi^r_{r-1}\times\mathrm{id}_{T^{r-2}_m S_{\mathrm{Cla}}}, \mathcal{R}_{G,r-1}).$$

Further we proceed as in the second step in (A) to get a unique $(G^1_m \times G^{r+1}_n)$-equivariant map $g : \mathcal{W}^{(r-3)} \times \mathcal{U}^{(r-1)} \to S_F$ such that

$$f = g \circ (\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(r-3)}_C).$$

(C) Let $s > r-1$. We have the action of the group G^{s+2}_m on $T^s_m S_{\mathrm{Cla}}$ and $G^{r+1}_{m,n}$ on $T^r_m S_{\mathrm{Gen}}$. We consider G^{s+2}_m as a subgroup of $G^{s+2}_{m,n}$ and both $T^s_m S_{\mathrm{Cla}}$ and $T^r_m S_{\mathrm{Gen}}$ as left $G^{s+2}_{m,n}$-manifolds.

By [J2] there exists a $G^{r+1}_{m,n}$-equivariant mapping

$$g_{r+1} : T^r_m S_{\mathrm{Gen}} \times T^{r-1}_m S_{\mathrm{Cla}} \times \mathcal{W}_{r-1} \times \dots \times \mathcal{W}_{s-1} \to S_F$$

such that

$$f = g_{r+1} \circ (\mathrm{id}_{T^r_m S_{\mathrm{Gen}}} \times \pi^s_{r-1}, \mathcal{R}_{C,r-1}, \dots, \mathcal{R}_{C,s-1}).$$

g_{r+1} then satisfies condition (A) and we obtain a unique $(G^1_m \times G^{r+1}_n)$-equivariant morphism

$$g : \mathcal{U}^{(r-1)} \times \mathcal{W}^{(r-2)} \to S_F$$

such that $g_{r+1} = g \circ (\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(r-2)}_C, \mathrm{id}_{\mathcal{W}_{r-1} \times \dots \times \mathcal{W}_{s-1}})$, i.e.

$$f = g \circ (\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(s-1)}_C).$$

Summarizing all cases we have

$$f = g \circ (\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(s-1)}_C)$$

for any $s \geq r-2$ and the restriction of g to $C^{(r-1)}_G \times C^{(r-2)}_C$ is the uniquely determined map we wished to find. ∎

Let us denote by K^t the kernel of the group epimorphism $G^t_{m,n} \to G^1_m \times G^{r+1}_n$. In the above Theorem 7.1 we have found a map g which factorizes f, but we have not proved that $(\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(s-1)}_C) : T^r_{m+n} S_{\mathrm{Gen}} \times T^s_m S_{\mathrm{Cla}} \to C^{(r-1)}_G \times C^{(s-1)}_C$ satisfies the orbit condition, that is,

$$(\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(s-1)}_C)^{-1}(r^{(r-1)}_G, r^{(s-1)}_C)$$

is a K^t-orbit for any $(r^{(r-1)}_G, r^{(s-1)}_C) \in C^{(r-1)}_G \times C^{(s-1)}_C$. Now we shall prove it.

Lemma 7.2. *If* $(j^r_0\gamma, j^s_0\lambda), (j^r_0\acute{\gamma}, j^s_0\acute{\lambda}) \in T^r_{m+n} S_{\mathrm{Gen}} \times T^s_m S_{\mathrm{Cla}}$ *satisfy*

$$(\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(s-1)}_C)(j^r_0\gamma, j^s_0\lambda) = (\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(s-1)}_C)(j^r_0\acute{\gamma}, j^s_0\acute{\lambda}),$$

then there is an element $h \in K^t$ *such that* $h(j^r_0\acute{\gamma}, j^s_0\acute{\lambda}) = (j^r_0\gamma, j^s_0\lambda)$.

Proof. Consider the orbit set $(T^r_{m+n} S_{\mathrm{Gen}} \times T^s_m S_{\mathrm{Cla}})/K^t$. It is a $(G^1_m \times G^t_n)$-set. Clearly the factor projection

$$p : T^r_{m+n} S_{\mathrm{Gen}} \times T^s_m S_{\mathrm{Cla}} \to (T^r_{m+n} S_{\mathrm{Gen}} \times T^s_m S_{\mathrm{Cla}})/K^t$$

is a $G^t_{m,n}$-map. By Theorem 7.1 there is a $(G^1_m \times G^{r+1}_n)$-equivariant map

$$g : C^{(r-1)}_G \times C^{(s-1)}_C \to (T^r_{m+n} S_{\mathrm{Gen}} \times T^s_m S_{\mathrm{Cla}})/K^t$$

satisfying $p = g \circ (\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(s-1)}_C)$. If

$$(\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(s-1)}_C)(j^r_0\gamma, j^s_0\lambda) = (\mathcal{R}^{(r-1)}_G, \mathcal{R}^{(s-1)}_C)(j^r_0\acute{\gamma}, j^s_0\acute{\lambda}) = (r^{(r-1)}_G, r^{(s-1)}_C),$$

then $p(j^r_0\gamma, j^s_0\lambda) = p(j^r_0\acute{\gamma}, j^s_0\acute{\lambda}) = g(r^{(r-1)}_P, r^{(s-1)}_C)$. This proves Lemma 7.2. ∎

Proof of Theorem 5.4. From Theorem 7.1 and Lemma 7.2 it follows that the equivariant mappings associated with natural operators on general connections on Y and classical connections on M with values in $(1, r+1)$-order natural bundles factorize through formal curvature mappings. So, the operators factorize through curvature operators and their covariant, general covariant and vertical derivatives. ∎

8. Examples. We give applications of the reduction Theorem 5.4 for general connections on concrete examples. The first example is prolongation of Γ to $J^1Y \to M$.

By [KMS] there are two canonical prolongations of Γ, by means of a classical symmetric connection Λ on M, to general connections on $p^1 : J^1Y \to M$. Let Γ be given by its horizontal lift

$$d^I = \Gamma^I{}_\lambda(x, y) d^\lambda. \tag{8.1}$$

Then the flow prolongation $\mathcal{J}^1(\Gamma, \Lambda)$ is given by (8.1) and

$$dy^I_\mu = \Big(\frac{\partial \Gamma^I{}_\lambda}{\partial x^\mu} + \frac{\partial \Gamma^I{}_\lambda}{\partial y^P} y^P_\mu + \Lambda_\lambda{}^\rho{}_\mu(\Gamma^I{}_\rho - y^I_\rho)\Big) d^\lambda. \tag{8.2}$$

The second prolongation $P(\Gamma, \Lambda)$ is given by (8.1) and

$$dy^I_\mu = \Big(\frac{\partial \Gamma^I{}_\mu}{\partial x^\lambda} + \frac{\partial \Gamma^I{}_\lambda}{\partial y^P}(y^P_\mu - \Gamma^P{}_\mu) + \frac{\partial \Gamma^I{}_\mu}{\partial y^p}\Gamma^P{}_\lambda - \Lambda_\lambda{}^\rho{}_\mu(y^I_\rho - \Gamma^I{}_\rho)\Big) d^\lambda. \tag{8.3}$$

The difference $\Sigma = \mathcal{J}^1(\Gamma, \Lambda) - P(\Gamma, \Lambda)$ is a section

$$\Sigma : Y \to V_M J^1Y \otimes_Y T^*M.$$

In coordinates this section is given by

$$(x^\lambda, y^I, y^I_\lambda, u^I{}_\lambda, u^I{}_{\lambda\mu}) \circ \Sigma = (x^\lambda, y^I, y^I_\lambda, 0, R[\Gamma]^I{}_{\lambda\mu}),$$

i.e. Σ is in the kernel of the projection $V_M J^1Y \otimes_Y T^*M \to VY \otimes_Y T^*M$. But this kernel is identified with $VY \otimes_Y (T^*M \otimes T^*M)$ and $\Sigma = \mathcal{J}^1(\Gamma, \Lambda) - P(\Gamma, \Lambda) = R[\Gamma]$.

Theorem 5.4 can be applied to find operators with values in a $(G^1_m \times G^r_n)$-bundle. For instance there is only the zero operator with values in $VY \otimes_Y T^*M$ (this follows from the homogeneous function theorem [KMS, p. 213]). As a consequence there is no other natural connection on Y given by Γ

and Λ. This implies that any prolongations of Γ, by means of Λ, to general connections on $p^1 : J^1Y \to M$ are over Γ. So the difference of any two such prolonged connections is a section of $VY \otimes (T^*M \otimes T^*M)$. From Theorem 5.4 and the homogeneous function theorem we can easily see that all operators with values in $VY \otimes (T^*M \otimes T^*M)$ are scalar multiples of $R[\Gamma]$ and we conclude that any connection on $p^1 : J^1Y \to M$ is one of the canonical prolongations, for instance the flow prolongation, plus scalar multiples of $R[\Gamma]$. So we have

$$D(\Gamma, \Lambda) = \mathcal{J}^1(\Gamma, \Lambda) + tR[\Gamma], \qquad t \in \mathbb{R}.$$

If we put $R[\Gamma] = \mathcal{J}^1(\Gamma, \Lambda) - P(\Gamma, \Lambda)$ we find that all prolonged connections are affine combinations of the canonical prolongations $\mathcal{J}^1(\Gamma, \Lambda)$ and $P(\Gamma, \Lambda)$, which is exactly Proposition 45.8 of [KMS].

As the second example we express normal fields by Mikulski [M] via derivatives of curvature tensors. Consider the normal field $B_{(i,j)}$ which is a field of order $(1, j+1)$. By Theorem 5.4 and the homogeneous function theorem [KMS, p. 213] we get

$$B^I{}_{\mu_1,\dots,\mu_i K_1\dots K_j \lambda} = R^I{}_{\lambda(\mu_1;\dots;\mu_i),K_1\dots K_j} + \text{pol},$$

where pol is a polynomial constructed from the general covariant and vertical derivatives of $R[\Gamma]$ of total orders less than $r-1 = i+j-1$ and from the covariant derivatives of $R[\Lambda]$ of orders less than $r-1$. Moreover this polynomial has to satisfy the identities (5.2) and (5.3). For $r = 1, 2$ this relation is given in Remark 5.2. For $r = 3$ we get $B_3 = (B^I_{\mu_1\mu_2\mu_3\lambda}, B^I_{\mu_1\mu_2K\lambda}, B^I_{\mu K_1K_2\lambda})$ given by

$$B^I_{\mu_1\mu_2\mu_3\lambda} = R^I{}_{\lambda(\mu_1;\mu_2;\mu_3)} + A\, R^I{}_{\rho(\mu_1}\, R_{\mu_2}{}^\rho{}_{\mu_3)\lambda} + B\, R^I{}_{\lambda(\mu_1}\, R_{\mu_2}{}^\rho{}_{\mu_3)\rho},$$
$$B^I_{\mu_1\mu_2K\lambda} = R^I{}_{\lambda(\mu_1;\mu_2),K}, \qquad B^I_{\mu K_1K_2\lambda} = R^I{}_{\lambda\mu,K_1K_2},$$

where $A, B \in \mathbb{R}$.

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