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Unicity of meromorphic mappings sharing few hyperplanes

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Abstract. We prove some theorems on uniqueness of meromorphic mappings into complex projective space $\mathbb{P}^n(\mathbb{C})$, which share $2n+3$ or $2n+2$ hyperplanes with truncated multiplicities.

1. Introduction. In 1926, R. Nevanlinna showed that two distinct nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$ cannot have the same inverse images for five distinct values, and that g is a special type of linear fractional transformation of f if they have the same inverse images counted with multiplicities for four distinct values [N].

In 1975, H. Fujimoto [Fu1] generalized Nevalinna's results to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. He considered two distinct meromorphic maps f and g of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ satisfying the condition that $\nu_{(f,H_j)} = \nu_{(g,H_j)}$ for q hyperplanes $H_1, \dots, H_q$ of $\mathbb{P}^n(\mathbb{C})$ in general position, where $\nu_{(f,H_j)}$ is the map of $\mathbb{C}^m$ into $\mathbb{Z}$ whose value $\nu_{(f,H_j)}(a)$ ($a \in \mathbb{C}^m$) is the intersection multiplicity of the images of f and H_j at $f(a)$. He proved the following

THEOREM A ([Fu1]). *Let H_i, $1 \le i \le 3n+2$, be $3n+2$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position, and let f and g be nonconstant meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with $f(\mathbb{C}^m) \not\subseteq H_i$ and $g(\mathbb{C}^m) \not\subseteq H_i$ such that $\nu_{(f,H_i)} = \nu_{(g,H_i)}$ for $1 \le i \le 3n+2$. Assume that either f or g is linearly nondegenerate over $\mathbb{C}$, that is, the image is not included in any hyperplane in $\mathbb{P}^n(\mathbb{C})$. Then $f \equiv g$.*

Since that time, the unicity problem without truncated mutiplicities has been studied intensively by many authors, including M. Ru, Y. Aihara, D. D. Thai–S. D. Quang, G. Dethloff–T. V. Tan, Z. Chen–Q. Yan and others.

We state here the recent result of Z. Chen and Q. Yan which is the best result available at present.

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Take a meromorphic mapping f of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ which is linearly nondegenerate over $\mathbb{C}$, a positive integer d, and q hyperplanes $H_1, \dots, H_q$ in $\mathbb{P}^n(\mathbb{C})$ in general position with

$$\dim f^{-1}(H_i \cap H_j) \leq m-2 \quad (1 \leq i < j \leq q),$$

and consider the set $\mathcal{F}(f, \{H_i\}_{i=1}^q, d)$ of all linearly nondegenerate (over $\mathbb{C}$) meromorphic maps $g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ satisfying the conditions:

(a) $\min(\nu_{(f,H_j)}, d) = \min(\nu_{(g,H_j)}, d)$ $(1 \leq j \leq q)$,
(b) $f(z) = g(z)$ on $\bigcup_{j=1}^q f^{-1}(H_j)$.

Denote by $\sharp S$ the cardinality of the set S.

Theorem B (Z. Chen–Q. Yan [ChY]). $\sharp\mathcal{F}(f, \{H_i\}_{i=1}^{2n+3}, 1) = 1$.

We emphasize that the proof of Theorem B was complicated.

Our first purpose is to prove a more general and slightly stronger form of the result of Z. Chen and Q. Yan. Moreover, we simplify its proof. First of all, let us recall the following.

Let f be a nonconstant meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ and let k be a positive integer. For every $z \in \mathbb{C}^m$, we set

$$\nu_{(f,H),\leq k}(z) = \begin{cases} 0 & \text{if } \nu_{(f,H)}(z) > k, \\ \nu_{(f,H)}(z) & \text{if } \nu_{(f,H)}(z) \leq k, \end{cases}$$

$$\nu_{(f,H),>k}(z) = \begin{cases} \nu_{(f,H)}(z) & \text{if } \nu_{(f,H)}(z) > k, \\ 0 & \text{if } \nu_{(f,H)}(z) \leq k. \end{cases}$$

We now take a meromorphic mapping f of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ which is linearly nondegenerate over $\mathbb{C}$, positive integers k, d, and q hyperplanes $H_1, \dots, H_q$ of $\mathbb{P}^n(\mathbb{C})$ in general position with

$$\dim\{z \in \mathbb{C}^m : \nu_{(f,H_i),\leq k}(z) > 0 \text{ and } \nu_{(f,H_j),\leq k}(z) > 0\} \leq m-2$$

$(1 \leq i < j \leq q)$, and consider the set $\mathcal{F}(f, \{H_j\}_{j=1}^q, k, d)$ of all linearly nondegenerate meromorphic maps $g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ satisfying the conditions:

(a) $\min(\nu_{(f,H_j),\leq k}, d) = \min(\nu_{(g,H_j),\leq k}, d)$ $(1 \leq j \leq q)$,
(b) $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z \in \mathbb{C}^m : \nu_{(f,H_j),\leq k}(z) > 0\}$.

Then we see that

$$\mathcal{F}(f, \{H_j\}_{j=1}^q, d) = \mathcal{F}(f, \{H_j\}_{j=1}^q, \infty, d) \subset \mathcal{F}(f, \{H_j\}_{j=1}^q, k, d).$$

We will improve Theorem B to the following.

Theorem 1. $\sharp\mathcal{F}(f, \{H_i\}_{i=1}^{2n+3}, k, 1) = 1$ *for* $k > \dfrac{n(4n^2+11n+4)}{3n+2} - 1.$

Our second main aim is to show a unicity theorem for meromorphic mappings sharing $2n+2$ hyperplanes with truncated multiplicities to level 1. Namely, we will prove the following.

THEOREM 2. *Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and let $H_1, \dots, H_{2n+2}$ be $2n+2$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position with*

$$\dim f^{-1}(H_i \cap H_j) \leq m-2 \quad (1 \leq i < j \leq q).$$

Let g be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ satisfying:

(a) $\min\{\nu_{(f,H_j),\leq n}, 1\} = \min\{\nu_{(g,H_j),\leq n}, 1\}$,
$\min\{\nu_{(f,H_j),\geq n}, 1\} = \min\{\nu_{(g,H_j),\geq n}, 1\}$ $(1 \leq j \leq q)$,

(b) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{2n+2} f^{-1}(H_j)$.

If $n \geq 2$ then $f \equiv g$.

2. Basic notions in Nevanlinna theory

2.1. We set $\|z\| = (|z_1|^2 + \dots + |z_m|^2)^{1/2}$ for $z = (z_1, \dots, z_m) \in \mathbb{C}^m$ and define

$$B(r) := \{z \in \mathbb{C}^m : \|z\| < r\}, \quad S(r) := \{z \in \mathbb{C}^m : \|z\| = r\} \quad (0 < r < \infty).$$

Set

$$\sigma(z) := (dd^c\|z\|^2)^{m-1},$$
$$\eta(z) := d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{m-1} \quad \text{on } \mathbb{C}^m \setminus \{0\}.$$

2.2. Let F be a nonzero holomorphic function on a domain Ω in $\mathbb{C}^m$. For a set $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\mathcal{D}^\alpha F = \partial^{|\alpha|} F / \partial^{\alpha_1} z_1 \cdots \partial^{\alpha_m} z_m$. We define the map $\nu_F : \Omega \to \mathbb{Z}$ by

$$\nu_F(z) := \max\{l : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < l\} \quad (z \in \Omega).$$

A *divisor* on a domain Ω in $\mathbb{C}^m$ is a map $\nu : \Omega \to \mathbb{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighborhood $U \subset \Omega$ of a such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq m-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq m-2$. For a divisor ν on Ω we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(m-1)$-dimensional analytic subset of Ω or an empty set.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbb{C}^m$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighborhood $U \subset \Omega$ such that $\varphi = F/G$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq m-2$, and we define the divisors ν_φ, ν_φ^∞ by $\nu_\varphi := \nu_F$, $\nu_\varphi^\infty := \nu_G$, which are independent of the choices of F and G and so globally well-defined on Ω.

2.3. For a divisor ν on $\mathbb{C}^m$ and for positive integers k, M or $M = \infty$, we define the counting function of ν by

$$\nu^{(M)}(z) = \min\{M, \nu(z)\},$$

$$\nu^{(M)}_{\le k}(z) = \begin{cases} 0 & \text{if } \nu(z) > k, \\ \nu^{(M)}(z) & \text{if } \nu(z) \le k, \end{cases} \qquad \nu^{(M)}_{\ge k}(z) = \begin{cases} \nu^{(M)}(z) & \text{if } \nu(z) \ge k, \\ 0 & \text{if } \nu(z) < k, \end{cases}$$

$$n(t) = \begin{cases} \int\limits_{B(t)} \nu(z)\sigma & \text{if } m \ge 2, \\ \sum\limits_{|z| \le t} \nu(z) & \text{if } m = 1. \end{cases}$$

Similarly, we define $n^{(M)}(t)$, $n^{(M)}_{\le k}(t)$, $n^{(M)}_{\ge k}(t)$. Set

$$N(r, \nu) = \int\limits_1^r \frac{n(t)}{t^{2m-1}}\, dt \qquad (1 < r < \infty).$$

Similarly, we define $N(r, \nu^{(M)})$, $N(r, \nu^{(M)}_{\le k})$, $N(r, \nu^{(M)}_{\ge k})$ and denote them by $N^{(M)}(r, \nu)$, $N^{(M)}_{\le k}(r, \nu)$, $N^{(M)}_{\ge k}(r, \nu)$ respectively.

Let $\varphi : \mathbb{C}^m \to \mathbb{C}$ be a meromorphic function. Define

$$N_\varphi(r) = N(r, \nu_\varphi), \qquad N^{(M)}_\varphi(r) = N^{(M)}(r, \nu_\varphi),$$
$$N^{(M)}_{\varphi, \le k}(r) = N^{(M)}_{\le k}(r, \nu_\varphi), \qquad N^{(M)}_{\varphi, \ge k}(r) = N^{(M)}_{\ge k}(r, \nu_\varphi).$$

For brevity we will omit the superscript $^{(M)}$ if $M = \infty$.

2.4. Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. For fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbb{P}^n(\mathbb{C})$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $I(f) = \{f_0 = \cdots = f_n = 0\}$ of codimension ≥ 2. Set $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$.

The *characteristic function* of f is defined by

$$T_f(r) = \int\limits_{S(r)} \log \|f\|\, \eta - \int\limits_{S(1)} \log \|f\|\, \eta.$$

Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ given by $H = \{a_0\omega_0 + \cdots + a_n\omega_n = 0\}$, where $a := (a_0, \ldots, a_n) \ne (0, \ldots, 0)$. We set $(f, H) = \sum_{i=0}^n a_i f_i$. We define the corresponding divisor f^*H by $f^*H(z) = \nu_{(f,H)}(z)$ $(z \in \mathbb{C}^m)$, which is independent of the choice of the reduced representation of f. From now on, we will write $\nu_{(f,H)}$ for f^*H if there is no confusion. Moreover, we define the *proximity function* of f with respect to H by

$$m_{f,H}(r) = \int\limits_{S(r)} \log \frac{\|f\| \cdot \|H\|}{|(f, H)|} \eta - \int\limits_{S(1)} \log \frac{\|f\| \cdot \|H\|}{|(f, H)|} \eta,$$

where $\|H\| = (\sum_{i=0}^n |a_i|^2)^{1/2}$.

2.5. Let φ be a nonzero meromorphic function on $\mathbb{C}^m$, which is occasionally regarded as a meromorphic map into $\mathbb{P}^1(\mathbb{C})$. The *proximity function* of φ is defined by

$$m(r,\varphi) := \int_{S(r)} \log^+ |\varphi|\, \eta,$$

where $\log^+ t = \max\{0, \log t\}$ for $t > 0$. The *Nevanlinna characteristic function* of φ is defined by

$$T(r,\varphi) = N_{1/\varphi}(r) + m(r,\varphi).$$

Then

$$T_\varphi(r) = T(r,\varphi) + O(1).$$

The meromorphic function φ is said to be *small* with respect to f if $\| \; T(r,\varphi) = o(T_f(r))$

2.6. As usual, the notation $\| \; P$ means the assertion P holds for all $r \in [0,\infty)$ excluding a Borel subset E of $[0,\infty)$ with $\int_E dr < \infty$.

The following statements are essential in Nevanlinna theory (see [NO]).

2.7. THE FIRST MAIN THEOREM. *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping and let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ such that $f(\mathbb{C}^m) \not\subset H$. Then*

$$N_{(f,H)}(r) + m_{f,H}(r) = T_f(r) \quad (r > 1).$$

2.8. THE SECOND MAIN THEOREM. *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and $H_1, \ldots, H_q$ be q hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then*

$$\Big\| \; (q-n-1)T_f(r) \le \sum_{i=1}^{q} N^{(n)}_{(f,H_i)}(r) + o(T_f(r)).$$

2.9. LEMMA ON LOGARITHMIC DERIVATIVE. *Let f be a nonzero meromorphic function on $\mathbb{C}^m$. Then*

$$\Big\| \; m\left(r, \frac{\mathcal{D}^\alpha(f)}{f}\right) = O(\log^+ T(r,f)) \quad (\alpha \in \mathbb{Z}^m_+).$$

2.10. Denote by $\mathcal{M}^*_m$ the abelian multiplicative group of all nonzero meromorphic functions on $\mathbb{C}^m$. Denote by $\mathcal{R}^*_f$ the group of all nonzero meromorphic functions on $\mathbb{C}^m$ which are small with respect to f. Then $\mathcal{R}^*_f$ is a subgroup of $\mathcal{M}^*_m$ and the multiplicative group $\mathcal{M}^*_m/\mathcal{R}^*_f$ is a torsion free abelian group.

Let G be a torsion free abelian group and let $A = (a_1, \ldots, a_q)$ be a q-tuple of elements of G. Let $q \ge r > s > 1$. We say that the q-tuple A has the *property* $(P_{r,s})$ if any r elements $a_{l(1)}, \ldots, a_{l(r)}$ in A satisfy the condition that for any given $i_1, \ldots, i_s$ $(1 \le i_1 < \cdots < i_s \le r)$, there exist

$j_1, \dots, j_s$ $(1 \le j_1 < \cdots < j_s \le r)$ with $\{i_1, \dots, i_s\} \neq \{j_1, \dots, j_s\}$ such that $a_{l(i_1)} \dots a_{l(i_s)} = a_{l(j_1)} \dots a_{l(j_s)}$.

2.11. PROPOSITION (H. Fujimoto [Fu1]). *Let G be a torsion free abelian group and $A = (a_1, \dots, a_q)$ a q-tuple in G. If A has the property $(P_{r,s})$ for some r, s with $q \ge r > s > 1$, then there exist $i_1, \dots, i_{q-r+2}$ with $1 \le i_1 < \cdots < i_{q-r+2} \le q$ such that $a_{i_1} = \cdots = a_{i_{q-r+2}}$.*

3. Proofs of Theorems 1 and 2

3.1. LEMMA. *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and let $H_1, \dots, H_q$ be q hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position and let k be a positive integer. Assume that $q \ge n+2$ and $k \ge nq/(q-n-1)$. Then*

$$\Big\| \; T_f(r) \le \frac{k+1-n}{(k+1)(q-n-1)-nq} \sum_{i=1}^{q} N^{(n)}_{(f,H_i),\le k}(r) + o(T_f(r)).$$

Proof. By the Second Main Theorem, we have

$$\begin{aligned}
\Big\| \; (q-n-1)T_f(r) &\le \sum_{i=1}^{q} N^{(n)}_{(f,H_i)}(r) + o(T_f(r)) \\
&= \sum_{i=1}^{q} N^{(n)}_{(f,H_i),\le k}(r) + \sum_{i=1}^{q} N^{(n)}_{(f,H_i),\ge k+1}(r) + o(T_f(r)) \\
&\le \sum_{i=1}^{q} N^{(n)}_{(f,H_i),\le k}(r) + \frac{n}{k+1} \sum_{i=1}^{q} N_{(f,H_i),\ge k+1}(r) + o(T_f(r)) \\
&\le \left(1 - \frac{n}{k+1}\right) \sum_{i=1}^{q} N^{(n)}_{(f,H_i),\le k}(r) \\
&\quad + \frac{n}{k+1} \sum_{i=1}^{q} (N_{(f,H_i),\ge k+1}(r) + N_{(f,H_i),\le k}(r)) + o(T_f(r)) \\
&= \left(1 - \frac{n}{k+1}\right) \sum_{i=1}^{q} N^{(n)}_{(f,H_i),\le k}(r) + \frac{n}{k+1} \sum_{i=1}^{q} N_{(f,H_i)}(r) + o(T_f(r)) \\
&\le \left(1 - \frac{n}{k+1}\right) \sum_{i=1}^{q} N^{(n)}_{(f,H_i),\le k}(r) + \frac{nq}{k+1} T_f(r) + o(T_f(r)).
\end{aligned}$$

Hence

$$\Big\| \; T_f(r) \le \frac{k+1-n}{(k+1)(q-n-1)-nq} \sum_{i=1}^{q} N^{(n)}_{(f,H_i),\le k}(r) + o(T_f(r)). \; \blacksquare$$

3.2. Lemma. *Suppose $k \geq 2n+1$ and $q \geq 2n+2$. Then*
$$\| \; T_g(r) = O(T_f(r)) \quad \textit{and} \quad \| \; T_f(r) = O(T_g(r))$$
for each $g \in \mathcal{F}(f, \{H_i\}_{i=1}^q, k, 1)$.

Proof. By the Second Main Theorem, we have
$$\begin{aligned}\Big\| \; (q-n-1)T_g(r) &\leq \sum_{i=1}^q N^{(n)}_{(g,H_i)}(r) + o(T_g(r)) \\ &\leq \sum_{i=1}^q nN^{(1)}_{(g,H_i)}(r) + o(T_g(r)) \\ &\leq \sum_{i=1}^q nN^{(1)}_{(f,H_i),\leq k}(r) + \sum_{i=1}^q \frac{n}{k+1} N^{(1)}_{(g,H_i),\geq k+1}(r) \\ &\quad + o(T_g(r)) \\ &\leq qnT_f(r) + \frac{qn}{k+1}T_g(r) + o(T_g(r)).\end{aligned}$$
Thus
$$\Big\| \; \left(\frac{q(k+1-n)}{k+1} - n - 1\right)T_g(r) \leq qnT_f(r) + o(T_g(r)).$$
Hence $\| \; T_g(r) = O(T_f(r))$. Similarly, we get $\| \; T_f(r) = O(T_g(r))$. ∎

3.3. *Proof of Theorem 1.* Suppose that there exist two distinct maps $f, g \in \mathcal{F}(f, \{H_i\}_{i=1}^{2n+3}, k, 1)$.

By changing indices if necessary, we may assume that
$$\underbrace{\frac{(f,H_1)}{(g,H_1)} \equiv \cdots \equiv \frac{(f,H_{k_1})}{(g,H_{k_1})}}_{\text{group 1}} \not\equiv \underbrace{\frac{(f,H_{k_1+1})}{(g,H_{k_1+1})} \equiv \cdots \equiv \frac{(f,H_{k_2})}{(g,H_{k_2})}}_{\text{group 2}}$$
$$\not\equiv \underbrace{\frac{(f,H_{k_2+1})}{(g,H_{k_2+1})} \equiv \cdots \equiv \frac{(f,H_{k_3})}{(g,H_{k_3})}}_{\text{group 3}} \not\equiv \cdots \not\equiv \underbrace{\frac{(f,H_{k_{s-1}+1})}{(g,H_{k_{s-1}+1})} \equiv \cdots \equiv \frac{(f,H_{k_s})}{(g,H_{k_s})}}_{\text{group } s},$$
where $k_s = 2n+3$.

For each $1 \leq i \leq 2n+3$, we set
$$\sigma(i) = \begin{cases} i+n & \text{if } i+n \leq 2n+3, \\ i-n-3 & \text{if } i+n > 2n+3. \end{cases}$$
and
$$P_i = (f,H_i)(g,H_{\sigma(i)}) - (g,H_i)(f,H_{\sigma(i)}).$$

Since $f \not\equiv g$, the number of elements of each group is at most n. Hence $(f,H_i)/(g,H_i)$ and $(f,H_{\sigma(i)})/(g,H_{\sigma(i)})$ belong to distinct groups. This means that $P_i \not\equiv 0$ $(1 \leq i \leq 2n+3)$.

Fix an index i with $1 \le i \le 2n+3$. For $z \notin I(f) \cup I(g) \cup \bigcup_{s\ne t} f^{-1}(H_s \cap H_t)$, it is easy to see that:

- If z is a zero of (f, H_i) then it is a zero of P_i with multiplicity at least $\min\{\nu_{(f,H_i)}, \nu_{(g,H_i)}\}$. Similarly, if z is a zero of $(f, H_{\sigma(i)})$ then it is a zero of P_i with multiplicity at least $\min\{\nu_{(f,H_{\sigma(i)})}, \nu_{(g,H_{\sigma(i)})}\}$.
- If z is a zero of (f, H_v) with $v \notin \{i, \sigma(i)\}$ then it is a zero of P_i (because $f(z) = g(z)$).

Thus, we have

$$\nu_{P_i}(z) \ge \min\{\nu_{(f,H_i)}, \nu_{(g,H_i)}\} + \min\{\nu_{(f,H_{\sigma(i)})}, \nu_{(g,H_{\sigma(i)})}\} + \sum_{\substack{v=1\\ v\ne i,\sigma(i)}}^{2n+3} \nu^{(1)}_{(f,H_v),\le k}(z)$$

for all z outside the analytic set $I(f) \cup I(g) \cup \bigcup_{s\ne t} f^{-1}(H_s \cap H_t)$ of dimension $\le m-2$.

Since $\min\{a, b\} \ge \min\{a, n\} + \min\{b, n\} - n$ for all positive integers a and b, the above inequality implies that

$$\begin{aligned}\nu_{P_i}(z) \ge \sum_{v=i,\sigma(i)} \big(&\min\{\nu_{(f,H_v)}(z), n\} + \min\{\nu_{(g,H_v)}(z), n\}\\ &- n\min\{\nu_{(f,H_v)}(z), 1\}\big) + \sum_{\substack{v=1\\ v\ne i,\sigma(i)}}^{2n+3} \nu^{(1)}_{(f,H_v),\le k}(z)\end{aligned}$$

for all z outside the analytic set $I(f) \cup I(g) \cup \bigcup_{s\ne t} f^{-1}(H_s \cap H_t)$.

Integrating both sides of the above inequality, we get

$$\begin{aligned}N_{P_i}(r) \ge &\sum_{v=i,\sigma(i)} (N^{(n)}_{(f,H_v),\le k}(r) + N^{(n)}_{(g,H_v),\le k}(r) - nN^{(1)}_{(f,H_v),\le k}(r))\\ &+ \sum_{\substack{v=1\\ v\ne i,\sigma(i)}}^{2n+3} N^{(1)}_{(f,H_v),\le k}(r).\end{aligned}$$

On the other hand, by Jensen's formula and the definition of the characteristic function, we have

$$\begin{aligned}N_{P_i}(r) &= \int_{S(r)} \log|P_i|\,\eta + O(1)\\ &\le \int_{S(r)} \log(|(f, H_i)|^2 + |(f, H_{\sigma(i)}|^2)^{1/2}\,\eta\\ &\quad + \int_{S(r)} \log(|(g, H_i)|^2 + |(g, H_{\sigma(i)}|^2)^{1/2}\,\eta + O(1)\end{aligned}$$

$$
\begin{aligned}
&\le \int\limits_{S(r)} \log(\|f\|(\|H_i\|^2+\|H_{\sigma(i)}\|^2)^{1/2})\eta \\
&\quad + \int\limits_{S(r)} \log(\|g\|(\|H_i\|^2+\|H_{\sigma(i)}\|^2)^{1/2})\eta + O(1) \\
&= \int\limits_{S(r)} \log\|f\|\,\eta + \int\limits_{S(r)} \log\|g\|\,\eta + O(1) \\
&= T_f(r)+T_g(r)+O(1).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
T_f(r)+T_g(r) \ge &\sum_{v=i,\sigma(i)} (N^{(n)}_{(f,H_v),\le k}(r)+N^{(n)}_{(g,H_v),\le k}(r) - nN^{(1)}_{(f,H_v),\le k}(r)) \\
&+ \sum_{\substack{v=1\\ v\ne i,\sigma(i)}}^{2n+3} N^{(1)}_{(f,H_v),\le k}(r) + o(T_f(r)).
\end{aligned}
$$

Summing both sides of the above inequality over $i=1,\dots,2n+3$, we have

$$
\begin{aligned}
(2n+3)(T_f(r)+T_g(r)) &\ge 2\sum_{v=1}^{2n+3}(N^{(n)}_{(f,H_v),\le k}(r)+N^{(n)}_{(g,H_v),\le k}(r)) \\
&\quad + \sum_{v=1}^{2n+3} N^{(1)}_{(f,H_v),\le k}(r) + o(T_f(r)) \\
&\ge \left(2+\frac{1}{2n}\right)\sum_{v=1}^{2n+3}(N^{(n)}_{(f,H_v),\le k}(r)+N^{(n)}_{(g,H_v),\le k}(r)) \\
&\quad + o(T_f(r)).
\end{aligned}
$$

By Lemma 3, it follows that

$$
\begin{aligned}
\Big\| \left(2+\frac{1}{2n}\right)&\sum_{v=i}^{2n+3}(N^{(n)}_{(f,H_v),\le k}(r)+N^{(n)}_{(g,H_v),\le k}(r)) \\
&\ge \left(2+\frac{1}{2n}\right)\frac{(k+1)(n+2)-2n^2-3n}{k+1-n}(T_f(r)+T_g(r)) + o(T_f(r)).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Big\| \ (2n+3)(T_f(r)+T_g(r)) \ge &\left(2+\frac{1}{2n}\right)\frac{(k+1)(n+2)-2n^2-3n}{k+1-n} \\
&\times (T_f(r)+T_g(r)) + o(T_f(r)).
\end{aligned}
$$

Letting $r\to\infty$, we get $k\le \frac{n(4n^2+11n+4)}{3n+2}-1$. This is a contradiction.

Hence $\sharp\mathcal{F}(f,\{H_i\}_{i=1}^{2n+3},k,1)=1$ for all $k>\frac{n(4n^2+11n+4)}{3n+2}-1$. ∎

3.4. *Proof of Theorem 2.* Suppose that $f \not\equiv g$. Then f and g belong to $\mathcal{F}(f, \{H_i\}_{i=1}^{2n+2}, \infty, 1)$. By repeating the same argument as in the proof of Theorem 1, we may assume that $P_i = (f, H_i)(g, H_{\sigma(i)}) - (g, H_i)(f, H_{\sigma(i)}) \not\equiv 0$ for all $1 \le i \le 2n+2$, where

$$\sigma(i) = \begin{cases} i+n & \text{if } i+n \le 2n+2, \\ i-n-2 & \text{if } i+n > 2n+2. \end{cases}$$

For each $1 \le i \le 2n+2$, we set $S_i = \{z \in \mathbb{C}^m : \nu_{(f,H_i)}(z) \ne \nu_{(g,H_i)}(z)\}$. Then $\overline{S}_i$ is an analytic subset of dimension $m-1$ and $\overline{S}_i \setminus S_i$ is an analytic subset of dimension $\le m-2$. Denote by ν_{S_i} the reduced divisor with support $\overline{S}_i$. For $z \in f^{-1}(H_i)$, it is easy to see that:

- If $z \in S_i$ then either
$$\max\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} < n \quad \text{or} \quad \min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} > n$$
by assumption (a) of the theorem. Because $\nu_{S_i}(z) = 1$, we have
$$\begin{aligned}\min\{\nu_{(f,H_i)}(z), n\} &+ \min\{\nu_{(g,H_i)}(z), n\} + \nu_{S_i}(z) \\ &\le \min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} + n\min\{\nu_{(f,H_i)}(z), 1\}.\end{aligned}$$
- If $z \notin S_i$ then $\nu_{(f,H_i)}(z) = \nu_{(g,H_i)}(z)$ and $\nu_{S_i}(z) = 0$. Hence
$$\begin{aligned}\min\{\nu_{(f,H_i)}(z), n\} + \min\{\nu_{(g,H_i)}(z), n\} + \nu_{S_i}(z) &\le \min\{\nu_{(f,H_i)}(z), n\} + n \\ &\le \min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} + n\min\{\nu_{(f,H_i)}(z), 1\}.\end{aligned}$$

This yields

$$\begin{aligned}\min\{\nu_{(f,H_i)}(z), n\} &+ \min\{\nu_{(g,H_i)}(z), n\} + \nu_{S_i}(z) \\ &\le \min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} + n\min\{\nu_{(f,H_i)}(z), 1\}\end{aligned}$$

for all $z \in f^{-1}(H_i)$ and hence for all $z \in \mathbb{C}^m$.

By using the same argument as in the proof of Theorem 1, we obtain

$$\begin{aligned}\nu_{P_i}(z) \ge\ &\min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} + \min\{\nu_{(f,H_{\sigma(i)})}(z), \nu_{(g,H_{\sigma(i)})}(z)\} \\ &+ \sum_{\substack{v=1 \\ v \ne i,\sigma(i)}}^{2n+2} \nu^{(1)}_{(f,H_v)}(z) \\ \ge &\sum_{v=i,\sigma(i)} \big(\min\{\nu_{(f,H_v)}(z), n\} + \min\{\nu_{(g,H_i)}(z), n\} + \nu_{S_v}(z) \\ &- n\min\{\nu_{(f,H_v)}(z), 1\}\big) + \sum_{\substack{v=1 \\ v \ne i,\sigma(i)}}^{2n+2} \nu^{(1)}_{(f,H_v)}(z)\end{aligned}$$

for all z outside an analytic set of dimension $\leq m-2$. This implies that

$$N_{P_i}(r) \geq \sum_{v=i,\sigma(i)} (N^{(n)}_{(f,H_v)}(r) + N^{(n)}_{(g,H_i)}(r) + N(r,\nu_{S_v}) - nN^{(1)}_{(f,H_v)}(r)) + \sum_{\substack{v=1\\ v\neq i,\sigma(i)}}^{2n+2} N^{(1)}_{(f,H_v)}(r).$$

By repeating the same argument as in the proof of Theorem 1, we have

$$\begin{aligned} &T_f(r) + T_g(r) \geq N_{P_i}(r) \\ &\geq \sum_{v=i,\sigma(i)} (N^{(n)}_{(f,H_v)}(r) + N^{(n)}_{(g,H_i)}(r) + N(r,\nu_{S_v}) - nN^{(1)}_{(f,H_v)}(r)) + \sum_{\substack{v=1\\ v\neq i,\sigma(i)}}^{2n+2} N^{(1)}_{(f,H_v)}(r). \end{aligned} \tag{3.5}$$

Summing over $i = 1, \ldots, 2n+2$ and using the Second Main Theorem, we obtain

$$\begin{aligned} \Big\| \ (2n+2)(T_f(r) + T_g(r)) & \\ &\geq 2\sum_{i=1}^{2n+2} (N^{(n)}_{(f,H_i)}(r) + N^{(n)}_{(g,H_i)}(r) + N(r,\nu_{S_i}) - nN^{(1)}_{(f,H_i)}(r)) \\ &\quad + 2n\sum_{i=1}^{2n+2} N^{(1)}_{(f,H_i)}(r) \\ &= 2\sum_{i=1}^{2n+2} (N^{(n)}_{(f,H_i)}(r) + N^{(n)}_{(g,H_i)}(r) + N(r,\nu_{S_i})) \\ &\geq (2n+2)(T_f(r) + T_g(r)) + 2\sum_{i=1}^{2n+2} N(r,\nu_{S_i}) + o(T_f(r)). \end{aligned} \tag{3.6}$$

Hence,

$$\| \ N(r,\nu_{S_i}) = o(T_f(r)), \tag{3.7}$$

and inequalities (3.5), (3.6) become equalities for all $1 \leq i \leq 2n+2$. Thus, for $1 \leq i \leq 2n+2$, we have

$$\begin{aligned} \Big\| \ N_{P_i}(r) = &\sum_{v=i,\sigma(i)} (N^{(n)}_{(f,H_v)}(r) + N^{(n)}_{(g,H_i)}(r) - nN^{(1)}_{(f,H_v)}(r)) \\ &+ \sum_{\substack{v=1\\ v\neq i,\sigma(i)}}^{2n+2} N^{(1)}_{(f,H_v)}(r) + o(T_f(r)) \end{aligned} \tag{3.8}$$

$$= \sum_{v=i,\sigma(i)} (2N^{(n)}_{(f,H_v)}(r) - nN^{(1)}_{(f,H_v)}(r)) + \sum_{\substack{v=1\\ v\neq i,\sigma(i)}}^{2n+2} N^{(1)}_{(f,H_v)}(r)$$
$$+ o(T_f(r)),$$

$$\| \; T_f(r) + T_g(r) = N_{P_i}(r) + o(T_f(r)), \tag{3.9}$$

$$\Big\| \; (n+1)T_f(r) = (n+1)T_g(r) + o(T_f(r)) \tag{3.10}$$
$$= \sum_{i=1}^{2n+2} N^{(n)}_{(f,H_i)}(r) + o(T_f(r)).$$

On the other hand, by (3.7) we also have

$$\Big\| \; N_{P_i}(r) \geq \sum_{v=i,\sigma(i)} N_{(f,H_v)}(r) + \sum_{\substack{v=1\\ v\neq i,\sigma(i)}}^{2n+2} N^{(1)}_{(f,H_v)}(r) + o(T_f(r)). \tag{3.11}$$

From (3.8) and (3.11), it follows that

$$\Big\| \sum_{v=i,\sigma(i)} N_{(f,H_v)}(r) \leq \sum_{v=i,\sigma(i)} (2\; N^{(n)}_{(f,H_v)}(r) - n\; N^{(1)}_{(f,H_v)}(r)) + o(T_f(r)). \tag{3.12}$$

Since $N^{(n)}_{(f,H_v)}(r) \leq nN^{(1)}_{(f,H_v)}(r)$ and $N^{(n)}_{(f,H_v)}(r) \leq N_{(f,H_v)}(r)$, the inequality (3.12) implies that

$$\| \; N_{(f,H_i)}(r) = N^{(n)}_{(f,H_i)}(r) + o(T_f(r)) = nN^{(1)}_{(f,H_i)}(r) + o(T_f(r)) \tag{3.13}$$

for all $1 \leq i \leq 2n+2$.

Combining (3.8), (3.9), (3.10) and (3.13), we have the following:

$$\Big\| \; N_{P_i}(r) = \sum_{v=i,\sigma(i)} N^{(n)}_{(f,H_v)}(r) + \sum_{\substack{v=1\\ v\neq i,\sigma(i)}}^{2n+2} N^{(1)}_{(f,H_v)}(r) + o(T_f(r)), \tag{3.14}$$

$$\| \; T_f(r) + T_g(r) = N_{P_i}(r) + o(T_f(r)), \tag{3.15}$$

$$\Big\| \; T_f(r) = T_g(r) + o(T_f(r)) = \sum_{v=i,\sigma(i)} N^{(n)}_{(f,H_v)}(r) + o(T_f(r)). \tag{3.16}$$

Assume that $H_i = \{a_{i0}\omega_0 + \cdots + a_{in}\omega_n = 0\}$. We set $h_i = (f,H_i)/(g,H_i)$ $(1 \leq i \leq 2n+2)$. Then

$$h_i/h_j = \frac{(f,H_i)\cdot(g,H_j)}{(f,H_j)\cdot(g,H_i)}$$

does not depend on the representations of f and g respectively. Since

$\sum_{k=0}^{n} a_{ik} f_k - h_i \sum_{k=0}^{n} a_{ik} g_k = 0$ $(1 \le i \le 2n+2)$, this implies that $\det(a_{i0}, \dots, a_{in}, a_{i0} h_i, \dots, a_{in} h_i; 1 \le i \le 2n+2) = 0$.

For each subset $I \subset \{1, \dots, 2n+2\}$, put $h_I = \prod_{i \in I} h_i$. Denote by $\mathcal{I}$ the set of all combinations $I = (i_1, \dots, i_{n+1})$ with $1 \le i_1 < \dots < i_{n+1} \le 2n+2$.

For each $I = (i_1, \dots, i_{n+1}) \in \mathcal{I}$, define

$$\begin{aligned} A_I = (-1)^{(n+1)(n+2)/2 + i_1 + \dots + i_{n+1}} & \det(a_{i_r l}; 1 \le r \le n+1, 0 \le l \le n) \\ & \times \det(a_{j_s l}; 1 \le s \le n+1, 0 \le l \le n), \end{aligned}$$

where $J = (j_1, \dots, j_{n+1}) \in \mathcal{I}$ such that $I \cup J = \{1, \dots, 2n+2\}$. We have

$$\sum_{I \in \mathcal{I}} A_I h_I = 0.$$

Take $I_0 \in \mathcal{I}$. Then $A_{I_0} h_{I_0} = - \sum_{I \in \mathcal{I}, I \ne I_0} A_I h_I$, that is,

$$h_{I_0} = - \sum_{I \in \mathcal{I}, I \ne I_0} \frac{A_I}{A_{I_0}} h_I.$$

Observe then $A_I / A_{I_0} \not\equiv 0$ for each $I \in \mathcal{I}$.

Denote by t the minimal number satisfying the following: There exist t elements $I_1, \dots, I_t \in \mathcal{I} \setminus \{I_0\}$ and t nonzero constants $b_i \in \mathbb{C}$ such that $h_{I_0} = \sum_{i=1}^{t} b_i h_{I_i}$.

Since $h_{I_0} \not\equiv 0$ and by the minimality of t, it follows that the family $\{h_{I_1}, \dots, h_{I_t}\}$ is linearly independent over $\mathbb{C}$.

Case 1: $t = 1$. Then $h_{I_0} / h_{I_1} = o(T_f(r))$.

Case 2: $t \ge 2$. Consider the meromorphic mapping $h : \mathbb{C}^m \to \mathbb{P}^{t-1}(\mathbb{C})$ with a reduced representation $h = (dh_{I_1} : \dots : dh_{I_t})$, where d is meromorphic on $\mathbb{C}^m$.

If z is a zero of dh_{I_i}, then z must be either a zero or a pole of some h_v. Hence z belongs to S_v for some v. This yields

$$\Big\| \; N^{(1)}_{dh_{I_i}}(r) \le \sum_{v=1}^{2n+2} N(r, \nu_{S_v}) = o(T_f(r)).$$

By the Second Main Theorem, we have

$$\Big\| \; T_h(r) \le \sum_{i=1}^{t} N^{(t)}_{dh_{I_i}}(r) + N^{(t)}_{dh_{I_0}}(r) + o(T_f(r)) = o(T_f(r)) + o(T_f(r)).$$

This yields $\| \; T_h(r) = o(T_f(r))$. Then $h_{I_0} / h_{I_1} = o(T_f(r))$.

Hence, from Cases 1 and 2 we see that for each $I \in \mathcal{I}$, there is $J \in \mathcal{I} \setminus \{I\}$ such that $h_I / h_J \in \mathcal{R}_f^*$.

We now consider the torsion free abelian subgroup generated by the subset $\{[h_1], \dots, [h_{2n+2}]\}$ of the abelian group $\mathcal{M}_m^* / \mathcal{R}_f^*$. Then the tuple

$([h_1], \dots, [h_{2n+2}])$ has the property $(P_{2n+2,n+1})$. This implies that there exist $2n+2-2n = 2$ elements, say $[h_1], [h_2]$, such that $[h_1] = [h_2]$. Then $h_1/h_2 = \chi \in \mathcal{R}_f^*$.

Suppose that $\chi \not\equiv 1$.

Since $h_1(z)/h_2(z) = 1$ for each $z \in \bigcup_{i=3}^{2n+2} f^{-1}(H_i) \setminus (f^{-1}(H_1) \cup f^{-1}(H_2))$, it follows that $\bigcup_{i=3}^{2n+2} f^{-1}(H_i) \setminus (f^{-1}(H_1) \cup f^{-1}(H_2)) \subset \chi^{-1}\{1\}$. By the Second Main Theorem, we have

$$\begin{aligned}\Big\| \ (2n-n-1)T_f(r) &\le \sum_{i=3}^{2n+2} N^{(n)}_{(f,H_i)}(r) + o(T_f(r)) \\ &\le (2n+2)nN^{(1)}_{(\chi-1)}(r) + o(T_f(r)) = o(T_f(r)).\end{aligned}$$

This is a contradiction. Thus, $\chi \equiv 1$, i.e., $h_1 \equiv h_2$. Hence $\nu_{(f,H_i)} = \nu_{(g,H_i)}$, $i = 1, 2$. By changing the reduced representations of f_1, f_2 if necessary, we may assume that $(f, H_1) = (g, H_1)$. This yields $(f, H_2) = (g, H_2)$.

Now we consider

$$\begin{aligned}P_1 &= (f, H_1)(g, H_{n+1}) - (f, H_{n+1})(g, H_1) \\ &= (f, H_1)((f_1, H_{n+1}) - (g, H_{n+1})) \not\equiv 0.\end{aligned}$$

Since $(f, H_i)(z) = (g, H_i)(z)$ on $\bigcup_{j=1}^{2N+2} f^{-1}(H_j) \setminus (f^{-1}(H_1) \cap f^{-1}(H_2))$ $(1 \le i \le 2n+2)$, we have

$$\begin{aligned}\| \ N_{P_1}(r) \ge (N_{(f,H_1)}(r) &+ N^{(1)}_{(f,H_1)}(r)) + N_{(f,H_{n+1})}(r) \\ &+ \sum_{\substack{v=1 \\ v\ne 1,n+1}}^{2n+2} N^{(1)}_{(f,H_v)}(r) + o(T_f(r)).\end{aligned} \tag{3.17}$$

From (3.14) and (3.17), we have $\| \ N^{(1)}_{(f,H_1)}(r) = o(T_f(r))$. Then $\| \ T_f(r) = N^{(n)}_{(f,H_{n+1})}(r) + o(T_f(r))$ by (3.16).

We set $Q_i = (f, H_i)(g, H_{n+1}) - (g, H_i)(f, H_{n+1})$. Put $\mathcal{Q} = \{1 \le i \le 2n+2 : Q_i \not\equiv 0\}$. Suppose that $\sharp\mathcal{Q} \ge n+2$. Without loss of generality, we may assume that $i_j \in \mathcal{Q}$ $(1 \le j \le n+2)$. Repeating the same argument as in the proof of Theorem 1 and using the Second Main Theorem, we obtain

$$\begin{aligned}\| \ T_f(r) + T_g(r) &\ge N_{Q_i}(r) + O(1) \\ &\ge \sum_{v=n+1,i_j} N^{(n)}_{(f,H_v)}(r) + \sum_{\substack{v=1 \\ v\ne n+1,i_j}}^{2n+2} N^{(1)}_{(f,H_v)}(r) + o(T_f(r)) \\ &= \frac{n-1}{n} \sum_{v=n+1,i_j} N^{(n)}_{(f,H_v)}(r) + \sum_{v=1}^{2n+2} N^{(1)}_{(f,H_v)}(r) + o(T_f(r))\end{aligned}$$

$$= \frac{n-1}{n}T_f(r) + \frac{n-1}{n}N^{(n)}_{(f,H_{i_j})}(r) + \frac{n+1}{n}T_f(r) + o(T_f(r))$$
$$= T_f(r) + T_g(r) + \frac{n-1}{n}N^{(n)}_{(f,H_{i_j})}(r) + o(T_f(r)).$$

Thus, $\parallel N^{(n)}_{(f,H_{i_j})}(r) = o(T_f(r))$. By the Second Main Theorem again,

$$\Big\| \; T_f(r) \leq \sum_{j=1}^{n+2} N^{(n)}_{(f,H_{i_j})}(r) + o(T_f(r)) = o(T_f(r)).$$

This is a contradiction. Hence $\sharp\mathcal{Q} \leq n+1$. This means that there exist at least $n+1$ indices i such that $Q_i \equiv 0$. This implies that $f \equiv g$. This is a contradiction.

Hence $f \equiv g$. The theorem is proved. ■

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