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A note on Bierstone–Milman–Pawłucki's paper "Composite differentiable functions"

by Krzysztof Jan Nowak (Kraków)

Abstract. We demonstrate that the composite function theorems of Bierstone–Milman–Pawłucki and of Glaeser carry over to any polynomially bounded, o-minimal structure which admits smooth cell decomposition. Moreover, the assumptions of the o-minimal versions can be considerably relaxed compared with the classical analytic ones.

Due to the classical theorem of Glaeser [7], every semiproper real analytic mapping φ which is generically a submersion enjoys the composite function property. Bierstone–Milman–Pawłucki [3] introduced a new point of view on Glaeser's theorem by considering a $\mathcal{C}^k$ composite function property, k being a positive integer, which is satisfied by every semiproper real analytic mapping. In this paper, we demonstrate that the theorem of Bierstone–Milman–Pawłucki and hence that of Glaeser carry over to the case of smooth mappings definable in any polynomially bounded, o-minimal structure $\mathcal{R}$ which admits smooth cell decomposition. Moreover, we considerably relax the assumptions of the o-minimal versions of those theorems in comparison with the classical analytic ones.

Before formulating an o-minimal version of the main theorem from [3], we recall the indispensable terminology. Let $M \subset \mathbb{R}^p$ be a smooth submanifold, $A \subset M$ a locally closed subset, B a closed subset of A and $k \in \mathbb{N} \cup \{\infty\}$. We denote by $\mathcal{C}^k(A; B)$ the Fréchet algebra of restrictions to A of real functions of class $\mathcal{C}^k$ in the vicinity of A that are k-flat on B. Let $\varphi : M \to N$ be a smooth mapping between two real smooth manifolds with closed image $T := \varphi(M) \subset N$. Denote by $(\varphi^* \mathcal{C}^k(T))^\wedge$ the subalgebra of all functions $f \in \mathcal{C}^k(M)$ that are formally composites with φ (i.e., for each $b \in T$, there is $g \in \mathcal{C}^k(T)$ such that the function $f - \varphi^*(g)$ is k-flat on the fibre $\varphi^{-1}(b)$).

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For a closed subset Z of T, put

$$(\varphi^* \mathcal{C}^k(T;Z))^\wedge := (\varphi^* \mathcal{C}^k(T))^\wedge \cap \mathcal{C}^k(M;\varphi^{-1}(Z)).$$

Main Theorem. *Consider a polynomially bounded, o-minimal structure $\mathcal{R}$ which admits smooth cell decomposition. Let $M \subset \mathbb{R}^p$ and $N \subset \mathbb{R}^q$ be smooth definable submanifolds, $\varphi : M \to N$ a smooth definable mapping with closed image $T := \varphi(M) \subset N$, and Z a closed definable subset of T. Then, for each $k \in \mathbb{N}$, there is an integer $l = l(k) \geq k$ such that*

$$(\varphi^* \mathcal{C}^l(T;Z))^\wedge \subset \varphi^* \mathcal{C}^k(T;Z).$$

Remark. Let us emphasize that the assumptions in the o-minimal version of the $\mathcal{C}^k$ composite function theorem have been considerably reduced in comparison with the classical version. The mapping φ under study does not need to be semiproper, but should have closed image $T := \varphi(M) \subset N$ instead. Further, we do not require that T be compact.

The above result, along with Whitney's extension theorem, immediately yields an o-minimal version of Glaeser's composite function theorem:

Composite Function Theorem. *Under the assumptions of the Main Theorem, suppose $\varphi : M \to N$ is a smooth definable mapping with closed image which is generically a submersion. Then*

$$(\varphi^* \mathcal{C}^\infty(T))^\wedge = \varphi^* \mathcal{C}^\infty(T).$$

Yet another direct consequence of the Main Theorem is the following

Corollary. *We have the equality*

$$(\varphi^* \mathcal{C}^\infty(T;Z))^\wedge = \varphi^* \mathcal{C}^{(\infty)}(T;Z) \quad \textit{where} \quad \mathcal{C}^{(\infty)}(T;Z) := \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(T;Z).$$

In particular, φ has the composite function property iff $\mathcal{C}^{(\infty)}(T;Z) = \mathcal{C}^\infty(T;Z)$; the latter condition depends only on T.

We assume familiarity with the paper by Bierstone–Milman–Pawłucki. The specific arguments from subanalytic geometry applied in [3] include the following issues: the Łojasiewicz inequality, Whitney regularity and stratifications of subanalytic mappings. By inspection of the proofs, one can conclude that the approach of [3] adapts to an o-minimal structure $\mathcal{R}$ whenever the above three issues are valid for $\mathcal{R}$.

Let $\mathcal{R}$ be an o-minimal expansion of the real field. It is well known that the foregoing first two issues hold in the structure $\mathcal{R}$ whenever $\mathcal{R}$ is polynomially bounded (see e.g. [1, Section 6] for the subanalytic version and [6, Section 4] for the o-minimal one). As for the third, a basic tool applied in [3] is the theorem on stratification of a proper continuous subanalytic mapping. Below we state the theorems on trivialization and stratification of a continuous mapping definable in the structure $\mathcal{R}$, which are valid provided that

$\mathcal{R}$ admits smooth cell decomposition. In the o-minimal case, however, the assumption of properness is superfluous.

TRIVIALIZATION THEOREM. *Suppose the structure $\mathcal{R}$ admits smooth cell decomposition. Let $S \subset \mathbb{R}^p$ be a definable subset, $\varphi : S \to \mathbb{R}^q$ a continuous definable mapping and $T := \varphi(S)$. Further, let $\mathcal{V} = (V_i)$ and $\mathcal{W} = (W_j)$ be finite families of definable subsets of S and T, respectively. Then there exist finite smooth definable cell decompositions $\mathcal{S}$ and $\mathcal{T}$ of S and T, respectively, which enjoy the following properties:*

(i) *$\varphi(C) \in \mathcal{T}$ for every $C \in \mathcal{T}$;*
(ii) *$\mathcal{S}$ and $\mathcal{T}$ are compatible with $\mathcal{V}$ and $\mathcal{W}$, respectively;*
(iii) *for each cell $D \in \mathcal{T}$, one can find a definable set F and a definable homeomorphism*
$$\theta : D \times F \to \varphi^{-1}(D)$$
such that $\varphi \circ \theta$ is the canonical projection (obviously, F is homeomorphic to the fibre $\varphi^{-1}(b)$ for every $b \in D$); moreover, there is a finite smooth definable cell decomposition $\mathcal{F} = (F_k)$ of F such that for each k,
$$\theta(D \times F_k) =: C \in \mathcal{S} \quad \textit{and} \quad \operatorname{res}\theta : D \times F_k \to C$$
is a diffeomorphism of $D \times F_k$ onto C.

We immediately obtain the following

COROLLARY (Stratification of a definable mapping). *Under the above assumptions, there exist finite smooth definable stratifications $\mathcal{S}$ and $\mathcal{T}$ of S and T, respectively, which enjoy the following properties:*

(i) *$\varphi(C) \in \mathcal{T}$ for every $C \in \mathcal{T}$;*
(ii) *$\mathcal{S}$ and $\mathcal{T}$ are compatible with $\mathcal{V}$ and $\mathcal{W}$, respectively;*
(iii) *for each cell $D \in \mathcal{T}$, one can find a definable set F and a definable homeomorphism*
$$\theta : D \times F \to \varphi^{-1}(D)$$
such that $\varphi \circ \theta$ is the canonical projection (obviously, F is homeomorphic to the fibre $\varphi^{-1}(b)$ for every $b \in D$); moreover, there is a finite smooth definable stratification $\mathcal{F} = (F_k)$ of F such that for each k,
$$\theta(D \times F_k) =: C \in \mathcal{S} \quad \textit{and} \quad \operatorname{res}\theta : D \times F_k \to C$$
is a diffeomorphism of $D \times F_k$ onto C.

A pair $(\mathcal{S}, \mathcal{T})$ of definable stratifications as in the above Corollary will be called a *stratification* of the mapping φ. If we require that $\mathcal{S}$ be only a finite partition into definable leaves (i.e. connected definable subsets of $\mathbb{R}^p$ which are smooth submanifolds), we call the pair $(\mathcal{S}, \mathcal{T})$ a *semistratification* of φ.

The foregoing Trivialization Theorem is a strengthening of the classical trivialization, and can be established by combining the latter (cf. [8], [9], [4, Chap. 9], [5, Chap. 9]) with the technique of cell decompositions. More precisely, the proof runs via the classical trivialization along with the following two elementary lemmas. We leave the detailed verification to the reader.

Lemma 1. *If $\theta : A \to E$ is a definable homeomorphism, then there are finite smooth cell decompositions $\mathcal{A} = (A_i)$ and $\mathcal{E} = (E_i)$ of A and E, respectively, such that each restriction $\theta|A_i$ is a smooth diffeomorphism of A_i onto E_i.*

Lemma 2. *Consider a definable set F, a smooth definable cell B and a finite smooth cell decomposition $\mathcal{C} = \{C_1, \ldots, C_k\}$ of $B \times F$ with all cells C_j lying over B. Then there exist a smooth cell decomposition $\mathcal{F} = \{F_1, \ldots, F_k\}$ of F and a definable homeomorphism $h : B \times F \to B \times F$ which commutes with the projection onto B, and such that each restriction $h|C_j$ is a smooth diffeomorphism of C_j onto $B \times F_j$.*

Finally, we return to the main result of this paper, which is an o-minimal version of the classical $\mathcal{C}^k$ composite function theorem by Bierstone–Milman–Pawłucki (cf. [3, Theorem 1.2]). The reason why we are able to drop the assumption that the mapping φ under study is semiproper and only to require that the image $T := \varphi(M)$ be closed is that the integer $l(k)$ constructed in [3] depends merely on some exponents in the Łojasiewicz inequalities and Whitney regularity conditions which occur in the proof. We shall outline how to achieve this result as follows.

Since every definable bounded open subset of $\mathbb{R}^n$ is a finite union of open cells (cf. [15]), it is not difficult to check that a definable submanifold $N \subset \mathbb{R}^q$ of dimension n is a finite union of definable open subsets each of which is diffeomorphic to $\mathbb{R}^n$. Therefore, without loss of generality, we may assume that $N = \mathbb{R}^n$. Now, in view of the following two crucial observations, the proofs of Theorem 1.2 and Proposition 6.1 from [3] can be repeated almost verbatim.

Observation 1 (Łojasiewicz inequality). *Let $f, g : A \to \mathbb{R}$ be two continuous definable functions on a locally closed subset A of $\mathbb{R}^n$ such that*

$$\{x \in A : f(x) = 0\} \subset \{x \in A : g(x) = 0\}.$$

Then there exists a common Łojasiewicz exponent $s > 0$ for all compact definable subsets K of A. More precisely, there exists an exponent $s > 0$ such that, for each compact definable subset K of A, there is a constant $C = C(K) > 0$ such that

$$|g(x)| \leq C\,|f(x)|^s \quad \text{for all } x \in K.$$

Observation 2 (Whitney regularity). *Let A be a connected closed definable subset of $\mathbb{R}^n$. Then one can find an exhaustion of A by connected compact definable subsets A_ν of $\mathbb{R}^n$, $\nu \in \mathbb{N}$, i.e.*

$$A = \bigcup_{\nu=1}^{\infty} A_\nu \quad \text{and} \quad A_1 \subset A_2 \subset \cdots,$$

and a positive integer r with the following property. For each A_ν, $\nu \in \mathbb{N}$, there is a constant $C = C(A_\nu) > 0$ such that any two points $x, y \in A_\nu$ can be joined by a rectifiable (*definable*) *curve γ in A_ν of length*

$$|\gamma| \le C\,\|x-y\|^{1/r}.$$

Observation 1 is a special case of the Łojasiewicz inequality with parameter, recalled below:

Łojasiewicz Inequality with Parameter (cf. [10] and [11, Section 1]). *Consider two definable functions $f, g : A \to \mathbb{R}$ on a set $A \subset \mathbb{R}^m_u \times \mathbb{R}^n_x$. Assume that all sections $A_u := \{x \in \mathbb{R}^n : (u,x) \in A\}$, $u \in \mathbb{R}^m$, are compact and that all functions*

$$f_u, g_u : A_u \to \mathbb{R}, \qquad f_u(x) := f(u,x), \quad g_u(x) := g(u,x)$$

are continuous. If $\{f = 0\} \subset \{g = 0\}$, then there exist an exponent $s > 0$ and a definable function $c : \mathbb{R}^m \to (0,\infty)$ such that

$$|g(u,x)| \le c(u)|f(u,x)|^s \quad \text{for all } (u,x) \in A.$$

As an immediate consequence, we obtain the following

Hölder Continuity with Parameter. *Under the above assumptions, there exist a positive integer r and a definable function $c : \mathbb{R}^m \to (0,\infty)$ such that*

$$|f(u,x) - f(u,y)| \le c(u)\|x-y\|^{1/r} \quad \text{for all } (u,x) \in A.$$

The above Hölder continuity, along with the cell triangulation in the sense of Shiota (cf. [14, Chap. II, Theorem II.4.2]), can be applied to the proof of Observation 2, outlined below. By a *cell complex* we mean—after Shiota (op. cit., Chap. I, §3)—a finite family of semilinear, possibly unbounded cells which satisfies conditions corresponding to the standard ones for a simplicial complex.

Let (K,τ) be a cell triangulation of the closed definable subset A of $\mathbb{R}^n$; here K is a cell complex and $\tau : |K| \to A$ is a definable homeomorphism. It is not difficult to find a constant $C > 0$ and a semilinear mapping

$$\omega : |K| \times |K| \times [0,1] \to |K|$$

such that for all points $c, d \in |K|$ the mapping

$$\omega_{c,d} : [0,1] \to |K|, \qquad \omega_{c,d}(t) := \omega(c,d,t),$$

is an arc from c to d of length $\leq C\|c-d\|$. Define a mapping

$$\gamma : A \times A \times [0,1] \to A, \quad \gamma(a,b,t) := \tau(\omega(\tau^{-1}(a), \tau^{-1}(b), t)).$$

Then for all points $a, b \in A$ the mapping

$$\gamma_{a,b} : [0,1] \to A, \quad \gamma_{a,b}(t) := \gamma(a,b,t),$$

is an arc from a to b. We may regard γ as a definable family of curves parametrized by the set $A \times A$. Now, the assertion follows from the Hölder continuity with parameter.

REMARKS. 1) Bierstone and Milman [2, Theorem 1.13]) established the equivalence of the composite function property with many other natural, metric, differential and algebro-geometric properties of a closed subanalytic set, including semicoherence, the uniform Chevalley estimate as well as the semicontinuity of the Hironaka diagram of initial exponents and of the Hilbert–Samuel function.

2) The quasianalytic version of the composite function theorem plays a crucial role in our subsequent papers [12], [13], which are concerned with quasianalytic multi-parameter perturbation theory and the singular locus of a quasi-subanalytic set, respectively.

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Krzysztof Jan Nowak
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: nowak@im.uj.edu.pl

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