Uniqueness of meromorphic functions sharing three values

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Abstract. We prove a result on the uniqueness of meromorphic functions sharing three values with weights and as a consequence of this result we improve a recent result of W. R. Lü and H. X. Yi.

1. Introduction, definitions and results. Let \( f \) and \( g \) be two non-constant meromorphic functions defined in the open complex plane \( \mathbb{C} \). For \( a \in \mathbb{C} \cup \{\infty\} \) we say that \( f \) and \( g \) share the value \( a \) CM (counting multiplicities) if \( f \) and \( g \) have the same set of \( a \)-points with the same multiplicities. If we do not take the multiplicities into account, we say that \( f \), \( g \) share the value \( a \) IM (ignoring multiplicities). For the standard notations and definitions of the value distribution theory we refer to [1].

We denote by \( N(r, a; f | \leq k) \) the counting function of \( a \)-points of \( f \) with multiplicities not exceeding \( k \), where \( a \in \mathbb{C} \cup \{\infty\} \) and \( k \) is a positive integer or infinity. Also we define

\[
\delta_k(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f | \leq k)}{T(r, f)}.
\]

In this paper \( I \) denotes a set of nonnegative real numbers of infinite linear measure, not necessarily the same in each of its occurrences.

In 1976 M. Ozawa [8] proved the following result.

**Theorem A.** Let \( f \) and \( g \) be two nonconstant entire functions of finite order sharing \( 0, 1 \) CM. If \( \delta(0; f) > 1/2 \) then either \( f \equiv g \) or \( fg \equiv 1 \).

Improving Theorem A, H. Ueda [9] proved the following result.

**Theorem B.** Let \( f \) and \( g \) be two nonconstant meromorphic functions sharing \( 0, 1, \infty \) CM. If

\[
\limsup_{r \to \infty} \frac{N(r, 0; f) + N(r, \infty; f)}{T(r, f)} < \frac{1}{2}
\]

then either \( f \equiv g \) or \( fg \equiv 1 \).

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In 1990 H. X. Yi [10] further improved Theorem B as follows:

**Theorem C.** Let \( f \) and \( g \) be two nonconstant meromorphic functions sharing 0, 1, \( \infty \) CM. If

\[
N(r, 0; f \mid \leq 1) + N(r, \infty; f \mid \leq 1) < \{\lambda + o(1)\} T(r)
\]

for \( r \in I \), where \( 0 < \lambda < 1/2 \) and \( T(r) = \max\{T(r, f), T(r, g)\} \), then either \( f \equiv g \) or \( fg \equiv 1 \).

Recently W. R. Lü and H. X. Yi [7] investigated the situation when the bound \( 1/2 \) in the above theorems is replaced by 1 and proved the following result.

**Theorem D.** Let \( f \) and \( g \) be two distinct nonconstant meromorphic functions sharing 0, 1, \( \infty \) CM. If

\[
\limsup_{r \to \infty, r \in I} \frac{N(r, 0; f \mid \leq 1) + N(r, \infty; f \mid \leq 1)}{T(r, f)} < 1
\]

then

\[
f = \frac{e^{s\gamma} - 1}{e^{-(k+1-s)\gamma} - 1} \quad \text{and} \quad g = \frac{e^{-s\gamma} - 1}{e^{(k+1-s)\gamma} - 1},
\]

where \( s \) and \( k \) are relatively prime positive integers with \( 1 \leq s \leq k \) and \( \gamma \) is a nonconstant entire function.

Considering \( f = (e^\gamma - 1)^2 \) and \( g = e^\gamma - 1 \), where \( \gamma \) is a nonconstant entire function, we see that in Theorem D it is not possible to relax the nature of sharing the value 0 from CM to IM. So one may naturally ask: *Is it possible in Theorem D to relax the nature of sharing the value 0?*

In this paper we answer this question with the help of the notion of weighted sharing of values which measures how close a shared value is to being shared CM or to being shared IM.

**Definition 1.1 ([2, 3]).** Let \( k \) be a nonnegative integer or infinity. For \( a \in \mathbb{C} \cup \{\infty\} \) we denote by \( E_k(a; f) \) the set of all \( a \)-points of \( f \) where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \). If \( E_k(a; f) = E_k(a; g) \), we say that \( f, g \) share the value \( a \) with weight \( k \).

The definition implies that if \( f, g \) share a value \( a \) with weight \( k \) then \( z_0 \) is a zero of \( f - a \) with multiplicity \( m \) (\( \leq k \)) if and only if it is a zero of \( g - a \) with multiplicity \( m \) (\( \leq k \)), and \( z_0 \) is a zero of \( f - a \) with multiplicity \( m \) (\( > k \)) if and only if it is a zero of \( g - a \) with multiplicity \( n \) (\( > k \)) where \( m \) is not necessarily equal to \( n \).

We write \( f, g \) share \((a, k)\) to mean that \( f, g \) share the value \( a \) with weight \( k \). Clearly if \( f, g \) share \((a, k)\) then \( f, g \) share \((a, p)\) for all integers \( p \) with \( 0 \leq p < k \). Also we note that \( f, g \) share a value \( a \) IM or CM if and only if \( f, g \) share \((a, 0)\) or \((a, \infty)\) respectively.
We prove the following result which enables us to improve Theorem D.

**Theorem 1.1.** Let $f$ and $g$ be two distinct nonconstant meromorphic functions sharing $(0,1)$, $(1,m)$, $(\infty,k)$, where

$$(m - 1)(mk - 1) > (1 + m)^2$$

and

$$\limsup_{r \to \infty} \frac{N(r,0; f \leq 1) + N(r,\infty; f \leq 1)}{T(r,f)} < 1.$$  

Then $f$ and $g$ satisfy the following relations:

$$(1.2) \quad \left(1 + \frac{\alpha}{f} - \frac{1}{f}\right)^s \equiv \alpha^{s+t},$$

$$(1.3) \quad \left(1 + \frac{1}{g\alpha} - \frac{1}{g}\right)^s \equiv \alpha^{-(s+t)},$$

where $\alpha$ is a nonconstant meromorphic function such that $\bar{N}(r,0; \alpha) + \bar{N}(r,\infty; \alpha) = S(r,f)$ and $s$, $t$ are relatively prime nonzero integers with $s > 0$ and $s + t \neq 0$.

The following corollary improves Theorem D.

**Corollary 1.1.** The assertion of Theorem D holds if $f$ and $g$ share $(0,1)$, $(1,\infty)$, $(\infty, \infty)$.

Considering the example mentioned earlier we can easily verify that in Corollary 1.1 sharing $(0,1)$ cannot be relaxed to sharing $(0,0)$.

2. **Lemmas.** In this section we present some lemmas which are required to prove the theorem and the corollary.

**Lemma 2.1 ([4]).** Let $f$ and $g$ be two nonconstant meromorphic functions sharing $(0,0)$, $(1,0)$, $(\infty,0)$. Then

$$T(r,f) \leq 3T(r,g) + S(r,f) \quad \text{and} \quad T(r,g) \leq 3T(r,f) + S(r,g).$$

Hence it follows that $S(r,f) = S(r,g)$ and we denote them by $S(r)$.

**Lemma 2.2 ([4]).** Let $f$ and $g$ be two distinct nonconstant meromorphic functions sharing $(0,1)$, $(1,m)$, $(\infty,k)$, where $(m - 1)(mk - 1) > (1 + m)^2$. Then for $a = 0, 1, \infty$,

1. $\bar{N}(r,a; f \geq 2) = S(r)$,
2. $\bar{N}(r,a; g \geq 2) = S(r),$

where $\bar{N}(r,a; f \geq 2)$ denotes the reduced counting function of multiple $a$-points of $f$. 
Lemma 2.3 ([6]). Let $f$ and $g$ be two distinct nonconstant meromorphic functions sharing $(0,0), (1,0), (\infty, 0)$. If $f$ is a bilinear transformation of $g$ then $f$ and $g$ satisfy one of the following:

(i) $fg \equiv 1$,
(ii) $(f - 1)(g - 1) \equiv 1$,
(iii) $f + g \equiv 1$,
(iv) $f \equiv cg$,
(v) $f - 1 \equiv c(g - 1)$,
(vi) $[(c - 1)f + 1][(c - 1)g - c] + c \equiv 0$, where $c \neq 0, 1$ is a constant.

Lemma 2.4 ([11]). Let $f_1$ and $f_2$ be nonconstant meromorphic functions satisfying $N(r, f_i) + N(r, \infty; f_i) = S_0(r)$ for $i = 1, 2$. Then either $N_0(r, 1; f_1, f_2) = S_0(r)$ or there exist two integers $s, t$ ($|s| + |t| > 0$) such that $f_1 f_2 \equiv 1$, where $N_0(r, 1; f_1, f_2)$ denotes the reduced counting function of $f_1$ and $f_2$ related to the common 1-points and $T(r) = T(r, f_1) + T(r, f_2)$, $S_0(r) = o(T(r))$ as $r \to \infty$ possibly outside a set of finite linear measure.

Lemma 2.5 ([5]). Let $f$ and $g$ be two distinct nonconstant meromorphic functions sharing $(0,1), (1,m), (\infty,k)$, where $(m-1)(mk-1) > (1+m)^2$. If $\alpha = (f - 1)/(g - 1)$ and $h = f/g$ then $N(r, a; \alpha) = S(r)$ and $N(r, a; h) = S(r)$ for $a = 0, \infty$.

Lemma 2.6 ([4]). Let $f$ and $g$ be two nonconstant meromorphic functions sharing $(0,1), (1,m), (\infty,k)$, where $(m-1)(mk-1) > (1+m)^2$. If

$$2\delta_1(0; f) + 2\delta_1(\infty; f) + \min \left\{ \sum_{a \neq 0,1,\infty} \delta_2(a; f), \sum_{a \neq 0,1,\infty} \delta_2(a; g) \right\} > 3$$

then either $f \equiv g$ or $fg \equiv 1$. If $f$ has at least one zero or pole then the case $fg \equiv 1$ does not arise.

Lemma 2.7 ([5]). Let $f$ and $g$ be two distinct nonconstant meromorphic functions sharing $(0,1), (1,m)$ and $(\infty,k)$, where $(m-1)(mk-1) > (1+m)^2$. If $f$ is not a bilinear transformation of $g$ then each of the following holds:

(i) $T(r, f) + T(r, g) = N(r, 0; g| \leq 1) + N(r, 1; g| \leq 1)$

$$+ N(r, \infty; g| \leq 1) + N_0(r) + S(r),$$

(ii) $T(r, f) + T(r, g) = N(r, 0; f| \leq 1) + N(r, 1; f| \leq 1)$

$$+ N(r, \infty; f| \leq 1) + N_0(r) + S(r),$$

where $N_0(r)$ denotes the counting function of those simple zeros of $f - g$ which are not the zeros of $g(g-1), 1/g$ and so are not the zeros of $f(f-1), 1/f$.

Lemma 2.8 ([11]). Let $s$ and $t$ be relatively prime integers with $s > 0$. Then $x^s - 1$ and $x^t - c$ have one and only one common factor, where $c$ is a constant satisfying $c^s = 1$. 
3. Proofs of the theorem and the corollary

Proof of Theorem 1.1. Let \( \alpha = (f - 1)/(g - 1) \) and \( h = f/g \). Then clearly \( \alpha \not\equiv 1 \) and \( h \not\equiv 1 \). Also we get

\[
(3.1) \quad f = h \frac{1 - \alpha}{h - \alpha} \quad \text{and} \quad g = \frac{1 - \alpha}{h - \alpha}.
\]

We now consider the following cases.

**Case I.** Let \( \delta_1(0; f) + \delta_1(\infty; f) > 3/2 \). Then by Lemma 2.6 we get \( fg \equiv 1 \) and so (1.2) and (1.3) hold for \( s = 1, t = -2 \) and \( \alpha = e^\beta \), where \( \beta \) is a nonconstant entire function.

**Case II.** Let \( \delta_1(0; f) + \delta_1(\infty; f) \leq 3/2 \). If possible, suppose that \( f \) is a bilinear transformation of \( g \). Then the possibilities (i)–(vi) of Lemma 2.3 will occur.

If \( fg \equiv 1 \) then \( 0, \infty \) are exceptional values of \( f \) in the sense of Picard (evP) and so \( \delta_1(0; f) + \delta_1(\infty; f) = 2 \), which is a contradiction.

If \( (f - 1)(g - 1) \equiv 1 \) then \( 1, \infty \) are evP of \( f \). So by the second fundamental theorem and Lemma 2.2 we get

\[
T(r, f) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + \overline{N}(r, 1; f) + S(r, f)
\]

\[
= N(r, 0; f \, | \, \leq 1) + S(r, f),
\]

which contradicts (1.1).

If \( f + g \equiv 1 \) then \( 0, 1 \) are evP of \( f \). So by the second fundamental theorem and Lemma 2.2 we get

\[
T(r, f) \leq N(r, \infty; f \, | \, \leq 1) + S(r, f),
\]

which contradicts (1.1).

If \( f \equiv cg \) then \( 1, c \) are evP of \( f \). So by the second fundamental theorem and Lemma 2.2 we get

\[
T(r, f) \leq \overline{N}(r, 1; f) + \overline{N}(r, c; f) + \overline{N}(r, \infty; f) + S(r, f)
\]

\[
= N(r, \infty; f \, | \, \leq 1) + S(r, f),
\]

which contradicts (1.1).

If \( f - 1 \equiv c(g - 1) \) then \( 0, 1 - c \) are evP of \( f \). So by the second fundamental theorem and Lemma 2.2 we get

\[
T(r, f) \leq N(r, \infty; f \, | \, \leq 1) + S(r, f),
\]

which contradicts (1.1).

If \([c - 1]f + 1\)[(c - 1)g - c] + c \equiv 0 then \( \infty, 1/(1 - c) \) are evP of \( f \). So by the second fundamental theorem and Lemma 2.2 we get

\[
T(r, f) \leq N(r, 0; f \, | \, \leq 1) + S(r, f),
\]

which contradicts (1.1).
Therefore $f$ is not a bilinear transformation of $g$. Noting that $f$, $g$ share $(1, m)$, it follows from Lemma 2.7(ii) that

$$T(r, f) \leq N(r, 0; f | \leq 1) + N(r, \infty; f | \leq 1) + N_0(r) + S(r)$$

and so by (1.1) we get $N_0(r) \neq S(r)$.

Again since $T(r, \alpha) + T(r, h) \leq 2T(r, f) + 2T(r, g) + O(1)$ and $N_0(r) \leq \overline{N}(r, 1; \alpha, h)$, it follows from Lemma 2.4 that there exist integers $s$ and $t$ $(|s| + |t| > 0)$ such that $\alpha^t h^s \equiv 1$. Without loss of generality we may assume that $s > 0$ and $s$, $t$ are relatively prime. Since $f$ is not a bilinear transformation of $g$, we see that $t \neq 0$ and $s + t \neq 0$. Now from (3.1) we get $h^s(f - 1 + \alpha)^s \equiv \alpha^s f^s$ and $h^s g^s \equiv (\alpha g + 1 - \alpha)^s$. Since $\alpha^t h^s \equiv 1$, we can deduce (1.2) and (1.3). Since $f$ and $g$ are nonconstant, clearly $\alpha$ is nonconstant. Also by Lemma 2.5 we get $\overline{N}(r, 0; \alpha) + \overline{N}(r, \infty; \alpha) = S(r, f)$. This proves the theorem. 

**Proof of Corollary 1.1.** Since $f$, $g$ share $(1, \infty)$, $(\infty, \infty)$, we can put $\alpha = (f - 1)/(g - 1) = e^\beta$, where $\beta$ is a nonconstant entire function. Then from (1.2) and (1.3) we get

$$f = \frac{e^{\gamma s} - 1}{e^{\gamma (s + t)} - 1} \quad \text{and} \quad g = \frac{e^{-\gamma s} - 1}{e^{-\gamma (s + t)} - 1}, \quad \text{where} \quad \beta = \gamma s.$$

We now consider the following cases.

**Case I.** Let $t > 0$ so that $s + t \geq 2$. Since $s$, $t$ are relatively prime and so are $s$, $s + t$, by Lemma 2.8 we get

$$T(r, f) = (s + t - 1)T(r, e^\gamma) + S(r),$$

$N(r, \infty; f | \leq 1) = (s + t - 1)T(r, e^\gamma) + S(r),$

$N(r, 0; f | \leq 1) = (s - 1)T(r, e^\gamma) + S(r).$

Then

$$\limsup_{r \to \infty, r \in I} \frac{N(r, 0; f | \leq 1) + N(r, \infty; f | \leq 1)}{T(r, f)} = 1 + \frac{s - 1}{s + t - 1} \geq 1,$$

which contradicts the given condition.

**Case II.** Let $t < 0$. If $s + t = 1$ then $s \geq 2$ and

$$f = \frac{e^{s \gamma} - 1}{e^\gamma - 1} = 1 + e^\gamma + e^{2 \gamma} + \cdots + e^{(s - 1) \gamma}.$$

Hence $T(r, f) = (s - 1)T(r, e^\gamma) + S(r), N(r, 0; f | \leq 1) = (s - 1)T(r, e^\gamma) + S(r)$ and $N(r, \infty; f | \leq 1) \equiv 0$. Therefore

$$\limsup_{r \to \infty, r \in I} \frac{N(r, 0; f | \leq 1) + N(r, \infty; f | \leq 1)}{T(r, f)} = 1,$$

which contradicts the given condition.
Let $s + t \geq 2$. Then as in Case I we arrive at a contradiction. Therefore $s + t \leq -1$. We now put $k = -1 - t$. Then $k > 0$ and $k - s = -t - 1 - s \geq 0$ so that $1 \leq s \leq k$. Also $s$ and $1 + k$ are relatively prime because $s$ and $t$ are so. Therefore we get

$$f = \frac{e^{s\gamma} - 1}{e^{(k+1-s)\gamma} - 1}$$

and

$$g = \frac{e^{-s\gamma} - 1}{e^{(k+1-s)\gamma} - 1}.$$

This proves the corollary.

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