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Conformal nullity of isotropic submanifolds

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Abstract. We introduce and study submanifolds with extrinsic curvature and second fundamental form related by an inequality that holds for isotropic submanifolds and becomes equality for totally umbilical submanifolds. The dimension of umbilical subspaces and the index of conformal nullity of these submanifolds with low codimension are estimated from below. The corollaries are characterizations of extrinsic spheres in Riemannian spaces of positive curvature.

1. Introduction. Isotropic submanifolds with low codimension have not been extensively studied apart from the case of isometric immersions between space forms (see [8], [6], [2], [3]). We introduce the submanifolds whose extrinsic curvature and second fundamental form are related by an inequality that holds for *isotropic submanifolds* and becomes equality for *totally umbilical submanifolds*. These submanifolds include the class of submanifolds with nonpositive extrinsic curvature. We study the problem of *characterizing totally umbilical and isotropic submanifolds among all submanifolds satisfying an inequality of the above type.*

In 1994 Florit obtained the best estimate for the index of relative nullity of a submanifold with nonpositive *extrinsic sectional curvature* and low codimension. In 1987 Borisenko studied submanifolds with nonpositive extrinsic sectional curvature and obtained characterizations of totally geodesic submanifolds in Riemannian spaces of positive curvature. These results were generalized in [11] to the case of nonpositive *extrinsic qth Ricci curvature*.

The present paper continues these studies. Its main goals are estimates for conformal nullity of isotropic submanifolds with low codimension and submanifolds with the above mentioned inequality (local Theorems 1, 2) and characterizations of extrinsic spheres in Riemannian spaces of positive curvature (Theorems 3, 4). These results presented in Section 2 are based on the local lemmas of Section 3, where the symmetric bilinear forms with

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qth Ricci curvature bounded from above are studied, and lower estimates of the index of conformal nullity of a submanifold with extrinsic qth Ricci curvature bounded from above and small codimension are given. Section 3.2 contains estimates from below of the dimension of umbilical subspaces of these bilinear forms. In Section 3.3 we find a lower estimate of the index of conformal nullity of these forms.

2. Conformal nullity of submanifolds. Let M^n be a Riemannian manifold, $q \in [1, n)$ some integer, $V = \{x_i\}_{1\le i\le q} \subset T_m M$ an orthonormal system of q vectors at $m \in M$ and $x_0 \perp V$ a unit vector. The *qth Ricci curvature* of M is defined as $\mathrm{Ric}^q(x_0; V) = \sum_{1\le i\le q} K(x_0, x_i)$ where K stands for sectional curvature. Let $M^n \to \overline{M}^{n+p}$ be a submanifold with the second fundamental form h. The *extrinsic qth Ricci curvature* Ric^q_h is (see [11])

$$\mathrm{Ric}^q_h(x_0; V) = \sum_{1\le i\le q} K_h(x_0, x_i), \tag{1}$$

where

$$K_h(x_0, x_i) = \langle h(x_0, x_0), h(x_i, x_i)\rangle - h^2(x_0, x_i)$$

is the *extrinsic sectional curvature* of h on the plane $\sigma = \{x_0, x_i\}$. Define

$$\lambda_h(m) = \min\{|h(x, x)| : \ x \in T_m M,\ |x| = 1\} \quad \text{and} \quad \overline{\lambda}_h = \min_{m\in M} \lambda_h(m).$$

The totally umbilical submanifolds obey the equality $K_h = \lambda_h^2$.

The function $\lambda_h : M \to \mathbb{R}$ also arises in the study of isotropic submanifolds. A submanifold M^n is *isotropic at* $m \in M$ if $|h(x, x)|$ is positive and does not depend on the choice of the unit vector $x \in T_m M$ (see [8]). In this case the function $x \mapsto |h(x, x)|/x^2$ at m is constant and equal to $\lambda_h(m)$. If the function λ_h of an isotropic submanifold M^n is constant, then M^n is a *constant isotropic submanifold*. Every totally umbilical submanifold is isotropic, but not vice versa. For $p = 1$ the isotropic submanifolds are precisely the totally umbilical ones.

Proposition 1 (see [8]). *Every isotropic submanifold $M \subset \overline{M}$ obeys the inequality $K_h \le \lambda_h^2$. If equality holds then M is totally umbilical.*

Proof. Let $x, y \in TM$ be unit orthogonal vectors. Then

$$K_h(x, y) = \langle h(x, x), h(y, y)\rangle - h^2(x, y) \le |h(x, x)| \cdot |h(y, y)| = \lambda_h^2.$$

To prove the second statement note that equality holds if and only if $h(x, x) = h(y, y)$ and $h(x, y) = 0$. From the arbitrariness of x, y it follows that M is totally umbilical. ∎

Consider now the submanifolds with weaker inequality $\mathrm{Ric}^q_h \le q\lambda_h^2$ for some $q \ge 1$.

Theorem 1. *Let $M^n \subset \overline{M}^{n+p}$ be a submanifold with $\lambda_h > 0$ and*

$$\mathrm{Ric}_h^q(m) \le q\lambda_h^2(m), \qquad m \in M^n,$$

for some q with $1 + \delta_{1q} < p \le n - 2q + \delta_{1q}$. Then the conformal nullity index satisfies

$$\mu_\xi \ge n - 2p - q + 2 + \delta_{1q}$$

for some normal ξ with $|\xi| = \lambda_h$.

Corollary 1. *Let $M^n \subset \overline{M}^{n+p}$ be a submanifold with $\mathrm{Ric}_h^q \le 0$ for some q with $1 + \delta_{1q} \le p < n - 2q + \delta_{1q}$. Then the relative nullity index satisfies*

$$\mu(M) \ge n - 2p - q + \delta_{1q}.$$

Here δ_{ij} is Kronecker's symbol. Theorem 1 and Corollary 1 are local (we do not assume completeness and a "curvature-invariant" condition), i.e. the estimates depend on the second fundamental form at any point $m \in M$. They follow from Lemmas 4 and 5, respectively. Using Proposition 1 and Theorem 1 with $q = 1$ we estimate the conformal nullity of isotropic submanifolds.

Theorem 2. *Let $M^n \subset \overline{M}^{n+p}$ $(2 < p < n)$ be an isotropic submanifold and $m \in M$. Then $\mu_\xi \ge n - 2p + 2$ for some nonzero normal ξ at m.*

Recall some important properties of conformal nullity distributions.

Definition 1. A vector $\xi \in TM^\perp$ is the *principal curvature normal* if the *conformal nullity subspace* $T_\xi \subseteq TM$ associated to ξ and given by

$$T_\xi = \{x \in TM : h(x, y) = \xi\langle x, y\rangle,\ \forall y \in TM\}$$

is at least one-dimensional. The integer $\mu_\xi = \dim T_\xi$ is the *conformal nullity index* associated to ξ, and $|\xi|$ is the *normal curvature* of T_ξ (see also definition in Section 3.1). For $\xi = 0$ we obtain the *relative nullity subspace* $T_\mu(m)$ and the *relative nullity index* $\mu(m) = \dim\, T_\mu(m)$ *at a point* $m \in M$. Denote by $\mu(M) = \min_{m\in M} \mu(m)$ the *relative nullity index of* M.

Proposition 2 ([10]). *Let $M^n \to \mathbb{R}^{n+p}$ be a submanifold with a non-vanishing proper principal curvature normal ξ of multiplicity s. Then the following holds*:

(1) *E_ξ is a umbilical distribution on M^n; its leaves are s-dimensional round spheres in $\mathbb{R}^{n+p}$ if and only if ξ is parallel in the normal connection of E_ξ.*
(2) *If $s > 1$ then ξ is parallel in the normal connection of E_ξ.*
(3) *The leaves are totally geodesic in M^n if and only if $\nabla_x(\xi^2) = 0$ $(x \perp E_\xi)$.*

For a *curvature-invariant* submanifold $M \subset \overline{M}$, i.e. with the condition

$$\overline{R}(x,y)z^{\perp} = 0 \quad (x,y,z \in TM), \tag{2}$$

the distribution E_ξ (E_μ) has the same properties (1)–(3) as in Proposition 2; moreover, if M is *complete* then the leaves are *complete* extrinsic spheres (totally geodesic submanifolds) in $\overline{M}$ (see [5]).

Using Theorem 1 and Proposition 2 we obtain a characterization of extrinsic spheres, and generalize results in [1], [11].

THEOREM 3. *Let $M^n \subset \overline{M}^{n+p}$ be a complete curvature-invariant submanifold with* $\mathrm{Ric}^q_h \le q\overline{\lambda}^2_h$ ($\overline{\lambda}_h > 0$) *for some q with $2+\delta_{1q} \le p \le n-2q+\delta_{1q}$. Then M is an extrinsic sphere with normal curvature $\overline{\lambda}_h$ if one of the following conditions holds*:

(a) $\mathrm{Ric}^s(M) > 0$ *for some* $s \le n-4p-2q+5+2\delta_{1q}$,
(b) $\mathrm{Ric}^s(\overline{M})_{|M} > 0$ *for some* $s \le n-5p-2q+5+2\delta_{1q}$.

Proof. By Theorem 1, the conformal nullity index satisfies the inequality $\mu_\xi \ge n-2p-q+2+\delta_{1q}$ for some continuous normal vector field ξ of constant length $\overline{\lambda}_h$. By Proposition 2, the leaves $\{L\}$ of the conformal nullity distribution E_ξ are μ_ξ-dimensional extrinsic spheres in $\overline{M}$ of normal curvature $\overline{\lambda}_h$, and they are totally geodesic in M. Assume that M is not totally umbilical (and hence is not an extrinsic sphere) of normal curvature $\overline{\lambda}_h$, i.e., $\mu_\xi < n$. Let L_1, L_2 be two sufficiently close leaves. The shortest geodesic $\gamma(t)$ $(0 \le t \le 1)$ of length $\mathrm{dist}(L_1, L_2) > 0$ between the points $m_i \in L_i$ is orthogonal to L_1 and L_2. Since the normals $\xi_1 = \dot\gamma(0)$, $\xi_2 = \dot\gamma(1)$ to L_1, L_2 are tangent to M and hence are orthogonal to the mean curvature vectors of the leaves, we have $h_i(x_i,x_i) \perp \xi_i$ for all $x_i \in T_{m_i}L_i$. By the asumption and the estimate of μ_ξ we have

$$\text{(a) } \mu_\xi \ge (n-1)+s, \qquad \text{(b) } \mu_\xi \ge (n+p-1)+s.$$

Hence, in both cases (a) and (b) there is an s-dimensional subspace $V_1 \subset T_{m_1}L$ whose parallel translation $V_1(t)$ along γ lies in $T_{m_2}L$. Let $e_j(t)$ be any orthonormal basis of $V_1(t)$. Since the second variation of the energy of γ along $e_j(t)$ is nonnegative, we have

$$\text{(a) } \sum_{j=1}^{s}\int_0^1 K(\dot\gamma, e_j(t))\,dt \le 0, \qquad \text{(b) } \sum_{j=1}^{s}\int_0^1 \overline{K}(\dot\gamma, e_j(t))\,dt \le 0, \tag{3}$$

which contradicts the positivity of the sth Ricci curvature. ■

For $q=2$ and $\gamma^q_M \le 1$ Theorem 3 has been proved in [1] under a stronger assumption on the curvature tensor. From the proof of Theorem 3 with $q=1$ and Theorem 2 follows

THEOREM 4. *A complete curvature-invariant constant isotropic submanifold* $M^n \subset \overline{M}^{n+p}$ $(2 < p < n)$ *is an extrinsic sphere if one of the following conditions holds*:

(a) $\mathrm{Ric}^s(M) > 0$ *for some* $s \le n - 4p + 5$,
(b) $\mathrm{Ric}^s(\overline{M})_{|M} > 0$ *for some* $s \le n - 5p + 5$.

By Theorem 4(a) with $s = q = 1$, a complete curvature-invariant constant isotropic embedding of a sphere $S^n(c) \subset \overline{M}^{n+p}$ $(c > 0)$ with codimension $2 < p \le n/4 + 1$ is totally umbilical. Note that isotropic isometric immersions between space forms $N^n(c) \subset M^{n+p}(\tilde{c})$ $(c \ge \tilde{c})$ with codimension $p < \frac{1}{2}n(n+1)$ are totally umbilical [2].

3. Symmetric bilinear forms with qth Ricci curvature bounded from above. In 1953 Otsuki proved that the symmetric bilinear form h with nonpositive sectional curvature and small codimension has a nonzero asymptotic vector. In 1994 Florit obtained the best estimate for the dimension of an asymptotic subspace of h and for the index of relative nullity, $\mu_h \ge n - p$. In 1998 the author proved results similar to Florit's but for nonnegative qth Ricci curvature. In this section we study the symmetric bilinear forms with $\mathrm{Ric}_h^q \le q\lambda_h^2$ and small codimension. We generalize results by Florit and the author [11, Lemma 6]. For convenience, we collect the resulting estimates in Table 1.

Table 1. Umbilical and asymptotic subspaces (dimension)

$\mathrm{Ric}_h^q \le q\,\lambda_h^2$	$\mathrm{Ric}_h^q \le 0$	Statement
$\dim T \ge n{-}p{-}q{+}1{+}\delta_{1q}$	$\dim T \ge n{-}p{-}q{+}\delta_{1q}$	Lemma 2, Corollary 2
$\mu \ge \dim T - p + 1$	$\mu \ge \dim T - p$	Propositions 4, 5
$\mu \ge n - 2p - q + 2 + \delta_{1q}$	$\mu \ge n - 2p - q + \delta_{1q}$	Lemmas 4, 5

Since in the proofs we use only the second fundamental forms at a chosen point, the results of this section hold for submanifolds in a Riemannian space with extrinsic qth Ricci curvature bounded from above.

3.1. *Preliminaries*

DEFINITION 2. Let $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map (e.g. the second fundamental form h of a submanifold $M^n \subset \overline{M}^{n+p}$), $V = \{x_i\}_{1 \le i \le q} \subset \mathbb{R}^n$ an orthonormal system of q vectors and $x_0 \perp V$ a unit vector. Define the *extrinsic qth Ricci curvature* Ric_h^q (see [11]) by

$$\mathrm{Ric}_h^q(x_0; V) = \sum_{1 \le i \le q} K_h(x_0, x_i), \tag{4}$$

where $K_h(x_0, x_i) = \langle h(x_0, x_0), h(x_i, x_i)\rangle - h^2(x_0, x_i)$ is the *extrinsic sectional curvature* on the plane $\sigma = \{x_0, x_i\}$, and $\langle\cdot,\cdot\rangle$ a scalar product in $\mathbb{R}^p$.

DEFINITION 3. A subspace $T \subseteq \mathbb{R}^n$ is called an *umbilical subspace* of h relative to $\xi \in \mathbb{R}^p$ if $h(x, y) = \xi\langle x, y\rangle$ for $x, y \in T$. A vector $\xi \in \mathbb{R}^p$ is called a *principal curvature normal* of h if the *conformal nullity subspace* $E_\xi \subset \mathbb{R}^n$ associated to ξ, given by

$$E_\xi = \{x \in \mathbb{R}^n : h(x, y) = \xi\langle x, y\rangle,\ \forall y \in \mathbb{R}^n\},$$

is at least one-dimensional. The integer $\mu_\xi = \dim\, E_\xi$ is the *conformal nullity index* of h associated to ξ, and $|\xi|$ is the *normal curvature* of E_ξ. (For $\xi = 0$ we obtain the notions of *asymptotic subspace*, *relative nullity subspace* and *nullity index* of h, respectively).

Define $\lambda_h = \min\{|h(x, x)| : |x\,| = 1\}$ and $A_h = \{|x\,| = 1 : |h(x, x)| = \lambda_h\}$, the "minimum set" of h. Note that h has no asymptotic vectors when $\lambda_h > 0$, and if $\lambda_h = 0$ then A_h coincides with the set of asymptotic vectors of h.

PROPOSITION 3. *Let $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with $\lambda_h > 0$, and suppose two unit vectors x_1, $x_2 \in A_h$ satisfy*

$$|h(x_1, x_1)| = |h(x_2, x_2)| = \lambda_h, \qquad h(x_1, x_2) = 0.$$

Then $h(x_1, x_1) = h(x_2, x_2)$ and $K_h(x_1, x_2) = \lambda_h^2$.

Proof. Assume the opposite, $h(x_1, x_1) \neq h(x_2, x_2)$, and consider the unit vector $x_0 = (x_1 + x_2)/\sqrt{2}$. In view of the triangle inequality, we obtain a contradiction

$$|h(x_0, x_0)| = |h(x_1, x_1) + h(x_2, x_2)|/2 < \lambda_h. \ \blacksquare$$

The following result has been proved by T. Otsuki [9] for $q = 1$ and $c = 0$.

LEMMA 1. *Let $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with $\mathrm{Ric}_h^q \leq qc^2$. If $p \leq n - q$, then $\lambda_h \leq c$.*

Proof. Assume the opposite, i.e., $\lambda_h > c \geq 0$. The minimum of a smooth positive function $f(x) = h^2(x, x)$ on the unit sphere S^{n-1} is reached at some vector x_0. Let $F(x) = f(x) - \lambda\langle x, x\rangle$. Then

$$\frac{1}{2}\, dF(x_0)x = 2\langle h(x_0, x_0), h(x_0, x)\rangle - \lambda(x_0, x) = 0, \tag{5}$$

$$\frac{1}{2}\, d^2F(x_0)(x, x) = 2\langle h(x_0, x_0), h(x, x)\rangle + 4h^2(x_0, x) - \lambda\langle x, x\rangle \geq 0, \tag{6}$$

where $x \in \mathbb{R}^n$. From (5) we have $\lambda/2 = |h(x_0, x_0)|^2 \geq \lambda_h^2$ and the subspace $V = \ker h(x_0, \cdot)$ is orthogonal to x_0. For unit vectors $x \in V$, in view of (6), we have $\langle h(x_0, x_0), h(x, x)\rangle \geq \lambda/2$. Since $\dim V \geq n - p \geq q$, we can find an orthonormal system $\{x_i\}_{1\le i\le q}\subset V$. Hence

$$qc^2\ge \mathrm{Ric}_h^q(x_0;x_1,\dots,x_q)=\sum_{i=1}^q\langle h(x_0,x_0),h(x_i,x_i)\rangle\ge q\lambda/2\ge q\lambda_h^2,$$

and we get a contradiction $c\ge\lambda_h$. ■

3.2. *Umbilical and asymptotic subspaces.* In this section we estimate from below the dimensions of umbilical or asymptotic subspaces. The following result completes Lemma 8 in [11] (corresponding to $\lambda_h=0$).

LEMMA 2. *Suppose that a symmetric bilinear map* $h:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^p$ *with* $\lambda_h>0$ *obeys the inequality* $\mathrm{Ric}_h^q\le q\lambda_h^2$. *If* $p\le n-q$ *then there exists an umbilical subspace* $T_\xi\subseteq\mathbb{R}^n$ *with* $|\xi|=\lambda_h$ *and* $\dim T_\xi\ge n-p-q+1+\delta_{1q}$.

Proof. From the proof of Lemma 1 it follows that there is a unit vector $x_0\in A_h$ such that for unit vectors $x\in\ker h(x_0)$,

$$K_h(x_0,x)=\langle h(x_0,x_0),h(x,x)\rangle=\lambda_h^2. \tag{7}$$

Here a linear transformation $h(x):\mathbb{R}^n\to\mathbb{R}^p$ is defined for each $x\in A_h$ by $h(x)y=h(x,y)$. Hence for $n-p-q+1+\delta_{1q}\le 1$ the proof is complete.

Now assume $n-p-q+1+\delta_{1q}\ge 2$ and hence $\dim\ker h(x_0)\ge q+1-\delta_{1q}$. We set $V_1=\ker h(x_0)\oplus x_0$, $W_1=\{\mathrm{Im}\,h(x_0)\}^\perp\oplus\xi_0$, where $\xi_0=h(x_0,x_0)$.

For an isotropic form h and $q=1$, in view of Proposition 3, $h(x,x)=h(x_0,x_0)$ for unit vectors of V_1. Hence V_1 is an $(n-p+1)$-dimensional umbilical subspace.

For general h define $h_1=h_{|V_1\times V_1}$. Note that $|\xi_0|=\lambda_h$. With the above notations we claim that $\mathrm{Im}\,h_1\subseteq W_1$. To prove this, take orthonormal vectors $\{z_i\}_{\delta_{1q}\le i\le q}\subset\ker h(x_0)$. From (7) it follows that $\langle h(x_0,x_0),h(z_i,z_i)\rangle=K_h(x_0,z_i)=\lambda_h^2$. Then for any unit $y\in\mathbb{R}^n$ with $y\perp x_0$ and for all t we have

$$\begin{aligned}&\langle h(x_0+ty,x_0+ty),h(z_i,z_i)\rangle-\langle h(x_0+ty,z_i),h(x_0+ty,z_i)\rangle\\&\quad=\lambda_h^2+2t\langle h(x_0,y),h(z_i,z_i)\rangle+t^2\left[\langle h(y,y),h(z_i,z_i)\rangle-\langle h(y,z_i),h(y,z_i)\rangle\right].\end{aligned}\tag{8}$$

We can assume $y\perp V_1$ (see the coefficient of t in (8)), and then the unit vector $x_t=\frac{1}{\sqrt{1+t^2}}(x_0+ty)$ is orthogonal to $U=\{z_1,\dots,z_q\}$. Hence for all t,

$$(1+t^2)\mathrm{Ric}_h^q(x_t;U)=q\lambda_h^2+2t\Big\langle h(x_0,y),\sum_{i=1}^q h(z_i,z_i)\Big\rangle+t^2\mathrm{Ric}_h^q(y;U).$$

In view of $\mathrm{Ric}_h^q\le q\lambda_h^2$ we have $\langle h(x_0,y),\sum_{i=1}^q h(z_i,z_i)\rangle=0$ ($y\in\mathbb{R}^n$, $y\perp x_0$), i.e., $\sum_{i=1}^q h(z_i,z_i)\perp\mathrm{Im}\,h(x_0)$. Note that for $q=1$ we have $h(z,z)\in W_1$ for all $z\in V_1$ and in view of symmetry of h the claim is proved for this case. So assume $q>1$, i.e., $\delta_{1q}=0$. Since the analogous property $\sum_{i=0}^{q-1} h(z_i, z_i) \in W_1$ is true, we have $h(z_0, z_0) - h(z_q, z_q) \in W_1$. In the same way we obtain $h(z_0, z_0) - h(z_i, z_i) \in W_1$ for each i and hence

$$h(z_0, z_0) = \frac{1}{q}\sum_{i=1}^{q}[h(z_0, z_0) - h(z_i, z_i)] + \frac{1}{q}\sum_{i=1}^{q} h(z_i, z_i) \in W_1.$$

Since z_0 is an arbitrary unit vector in V_1, in view of symmetry of h we have $\operatorname{Im} h_1 \subseteq W_1$.

The above claim allows us to proceed inductively as follows. Set $V_0 = \mathbb{R}^n$ and $W_0 = \mathbb{R}^p$. Given $k \geq 0$, for the symmetric bilinear map $h_k = h_{|V_k \times V_k} : V_k \times V_k \to W_k$ with $\operatorname{Ric}^q_{h_k} \leq q\lambda_h^2$, define a nonnegative integer

$$r_k = \max\{\dim \operatorname{Im} h_k(x) : x \in A_{h_k}\} - 1,$$

and suppose that if $k \geq 1$, then

$$n_k = \dim V_k \geq n - \sum_{i=0}^{k-1} r_i, \qquad p_k = \dim W_k \leq p - \sum_{i=0}^{k-1} r_i.$$

Picking a unit vector $x_k \in V_k$ such that $|h(x_k, x_k)| = \lambda_{h_k}$ and $\dim \operatorname{Im} h_k(x_k) = r_k + 1$, set $V_{k+1} = \ker h_k(x_k) \oplus x_k \subseteq V_k$, and then $n_{k+1} = \dim V_{k+1} \geq (n_k+1)-(r_k+1) \geq n - \sum_{i=0}^{k} r_i$. Note that $n_k - p_k \geq n - p \geq q + 1 - \delta_{1q}$. The above claim implies that $\operatorname{Im} h_{k+1} \subseteq W_{k+1}$, where $W_{k+1} = \{\operatorname{Im} h_k(x_k)\}^\perp \oplus \xi_0 \subseteq W_k$ and $h_{k+1} = h_{|V_{k+1} \times V_{k+1}}$. In view of Proposition 3, $h(x_k, x_k) = h(x_0, x_0) = \xi_0$. Since

$$0 \leq p_{k+1} = \dim W_{k+1} = p_k - r_k = p - \sum_{i=0}^{k} r_i,$$

there exists an integer $m > 0$ such that $r_m = 0$. This tells us that $A_{h_m} = \ker h_m(x_m) \oplus x_m$. By Lemma 1 for each subspace $S \subseteq V_m$ with $\dim S > p_m + q - 1 - \delta_{1q}$, we have $S \cap A_{h_m} \neq \{0\}$. Hence, $\dim A_{h_m} \geq n_m - (p_m + q - 1 - \delta_{1q}) \geq n - p - q + 1 + \delta_{1q}$. Moreover, since $h_m = h_{|V_m \times V_m}$, we see that $T_{\xi_0} = A_{h_m}$ is an umbilical subspace of h (with $|\xi_0| = \lambda_h$) and this concludes the proof. ∎

Corollary 2 ([11]). *Suppose a symmetric bilinear map* $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ *obeys the inequality* $\operatorname{Ric}^q_h \leq 0$. *Then there is an asymptotic subspace* $T \subset \mathbb{R}^n$ *with* $\dim T \geq n - p - q + \delta_{1q}$.

Proof. Extend Euclidean subspace $\mathbb{R}^{p+1} = \mathbb{R}^p \times \mathbb{R}(e_{p+1})$ so that $|e_{p+1}| = 1$, and consider the bilinear form

$$\widetilde{h}\langle x, y\rangle = h\langle x, y\rangle + \langle x, y\rangle\, e_{p+1} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{p+1}.$$

Then $\lambda^2_{\widetilde{h}} \geq \lambda^2_h + 1 \geq 1$ and $\operatorname{Ric}^q_{\widetilde{h}} = \operatorname{Ric}^q_h + q \leq q$. Applying Lemma 2, we obtain an umbilical subspace $T \subset \mathbb{R}^n$ of $\widetilde{h}$ relative to e_{p+1}, and $\dim \widetilde{T} \geq$

$n-(p+1)-q+1+\delta_{1q} = n-p-q+\delta_{1q}$. From $\widetilde{h}(x,x) = h(x,x) + x^2\, e_{p+1}$ since $h(x,x) \perp e_{p+1}$ we get $h(x,x) = 0$ $(x \in T)$. Hence T is the required asymptotic subspace of h. ■

3.3. *Indices of conformal and relative nullity.* In this section we estimate the index of relative or conformal nullity of the symmetric bilinear form with extrinsic qth Ricci curvature bounded above.

We say that $y \in \mathbb{R}^n$ is a *regular element* of a bilinear map $\beta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ if $\dim \operatorname{Im} \beta(y) = \max\{\dim \operatorname{Im} \beta(z) : z \in \mathbb{R}^n\}$ (see [7]). Note that the set $\mathrm{RE}(\beta)$ of regular elements of β is open and dense in $\mathbb{R}^n$.

LEMMA 3 ([7]). *Let $\beta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ be a bilinear map and let $y_0 \in \mathrm{RE}(\beta)$. Then $\beta(y, \ker(\beta(y_0))) \subseteq \operatorname{Im} \beta(y_0)$ for all $y \in \mathbb{R}^n$.*

In 1994 Florit [4] obtained the best estimate for the index of relative nullity $\mu(h)$ of a submanifold with nonpositive extrinsic sectional curvature and small codimension. This result was generalized to the case of nonpositive extrinsic qth Ricci curvature in [11, Lemma 6]. Lemma 2 and Proposition 4 yield Lemma 4 below that estimates the conformal nullity index of submanifolds with $\lambda_h > 0$ and completes the above mentioned results.

LEMMA 4. *Let $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with $\lambda_h > 0$ and $\mathrm{Ric}^q_h \le q\lambda_h^2$ with $2+\delta_{1q} \le p \le n-2q+\delta_{1q}$. Then the conformal nullity index satisfies $\mu_\xi \ge n-2p-q+2+\delta_{1q}$ for some $\xi \in \mathbb{R}^p$ with $|\xi| = \lambda_h$.*

PROPOSITION 4. *Let $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with $\lambda_h > 0$ and $\mathrm{Ric}^q_h \le q\lambda_h^2$, and suppose there is an umbilical subspace $T \subset \mathbb{R}^n$ $(n-q > \dim T > q)$ for some $\xi \in \mathbb{R}^p$ with $|\xi| = \lambda_h$. Then $\mu_\xi \ge \dim T - p + 1$.*

Proof. Define a bilinear map $\beta : T' \times T \to \mathbb{R}^p$ by $\beta = h_{|T' \times T}$. Take unit vectors $y_0 \in \mathrm{RE}(\beta) \subset T'$, $x_0 \in T$, and orthonormal systems $\{x_i\}_{1 \le i \le q} \subset \ker \beta(y_0) \subseteq T$, $\{y_i\}_{1 \le i \le q} \subset T'$ that are also orthogonal to x_0 and y_0, resp. Using only the assumption $K_h(x_i, x_j) = \langle h(x_j,x_j), h(x_i,x_i)\rangle = \lambda_h^2$ and $h(x_j, x_i) = \delta_{ij}\xi$ on T and $h(x_i, y_0) = 0$ $(i > 0)$, we see for small $s, t \in \mathbb{R}$ that

$$\begin{aligned}\langle h(ty_0+x_0, ty_0+x_0), h(sy_i+x_i, sy_i+x_i)\rangle - h^2(ty_0+x_0, sy_i+x_i)\\ = \lambda_h^2 + 2s\{2t\langle h(x_0,y_0), h(x_i,y_i)\rangle + \langle h(x_i,y_i),\xi\rangle\} + 2t\langle h(x_0,y_0),\xi\rangle + A_i,\end{aligned}$$

where A_i contains the terms with s^2 or t^2. Then for the orthonormal system $\widetilde{y}_0 = \frac{1}{\sqrt{1+t^2}}(ty_0 + x_0)$, $\widetilde{y}_i = \frac{1}{\sqrt{1+s^2}}(sy_i + x_i)$ and $\widetilde{V} = \{\widetilde{y}_i\}_{1 \le i \le q}$ we obtain

$$\mathrm{Ric}^q_h(\widetilde{y}_0; \widetilde{V}) = q\lambda_h^2 + 2s[2t\langle \xi, \eta\rangle + \langle \xi, \eta\rangle] + 2t\langle h(x_0,y_0), (q-1)\xi\rangle + A, \tag{9}$$

where $\eta = \sum_{i=1}^q h(x_i, y_i)$ and A contains the terms with s^2 or t^2. Since $\mathrm{Ric}^q_h \le q\lambda_h^2$, equate to zero the linear terms in s and t. This implies

$\langle h(x_0, y_0), \xi\rangle = 0$, i.e.,

$$h(x_0, y_0) \perp \xi \ \Leftrightarrow\ \dim \operatorname{Im} \beta(y_0) \le p - 1 \tag{10}$$

and

$$2t\langle h(x_0, y_0), \eta\rangle + \langle \xi, \eta\rangle = 0. \tag{11}$$

Equate to zero the linear and constant terms in t of (11) to obtain

$$\langle h(x_0, y_0), \eta\rangle = 0, \qquad \langle \xi, \eta\rangle = 0. \tag{12}$$

Note that terms in η of (12) change sign under the transformation $x_i \Rightarrow -x_i$ or $y_i \Rightarrow -y_i$ for each i, while the equations themselves still hold. Thus from (12) we obtain

$$\langle h(x, y), h(x_0, y_0)\rangle = 0 \qquad (y \in T',\ x \in \ker \beta(y_0)) \tag{13}$$

and, as $h(x, y_0) = 0$, one can drop the assumption $x_0 \perp x$. Now from the arbitrariness of $x_0 \in T$ and $x \in \ker \beta(y_0)$ it follows that $\beta(y, \ker \beta(y_0)) \perp \operatorname{Im} \beta(y_0)$. This together with Lemma 3 tells us that

$$h(x, y) = 0 \qquad (y \in T',\ x \in \ker \beta(y_0)). \tag{14}$$

But since $\ker \beta(y_0) \subseteq T$, we obtain $\ker \beta(y_0) \subseteq E_\xi$. (Note that a principal curvature normal ξ of E_ξ is the same as for T.) Then, in view of the equivalence (10),

$$\mu_\xi \ge \dim \ker \beta(y_0) = \dim T - \dim \operatorname{Im} \beta(y_0) \ge \dim T - p + 1. \ \blacksquare$$

From Corollary 2 and Proposition 4 we obtain

Proposition 5. *Let $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with $\mathrm{Ric}^q_h \le 0$, and suppose there is an asymptotic subspace $T \subset \mathbb{R}^n$ $(n-q > \dim T > q)$. Then the relative nullity index satisfies $\mu_h \ge \dim T - p$.*

Corollary 2 and Proposition 5 yield Lemma 5 below that estimates the relative nullity index of submanifolds and improves results of [11].

Lemma 5. *Let $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map such that $\mathrm{Ric}^q_h \le 0$ for some q with $1 + \delta_{1q} \le p < n - 2q + \delta_{1q}$. Then $\mu_h \ge n - 2p - q + \delta_{1q}$.*

Note that Lemma 6 of [11] was formulated without the condition $1+\delta_{1q} \le p < n-2q+\delta_{1q}$. The proof of Lemma 5 is similar to the proof of Corollary 2.

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