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On approximation by Chebyshevian box splines

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Abstract. Chebyshevian box splines were introduced in [5]. The purpose of this paper is to show some new properties of them in the case when the weight functions w_j are of the form

$$w_j(x) = W_j(v_{n+j} \cdot x),$$

where the functions W_j are periodic functions of one variable. Then we consider the problem of approximation of continuous functions by Chebyshevian box splines.

1. Introduction. Let an integer $n \times s$ matrix $V_s = \{v_1, \ldots, v_s\}$, $v_i \in \mathbb{Z}^n \setminus \{0\}$, $i = 1, \ldots, s$, be admissible, i.e. $\operatorname{rank} V_n = n$ (the first n columns of V_s are linearly independent), $s > n$ and let a sequence $W = \{w_{n+1}, \ldots, w_s\}$ of continuous functions on $\mathbb{R}^n$ be given such that

(i) each w_j is periodic, i.e. $w_j(x + \alpha) = w_j(x)$ for $\alpha \in \mathbb{Z}^n$,

(ii) $0 < a_j \le w_j(x) \le b_j < \infty$ for $x \in \mathbb{R}^n$, where a_j and b_j are some constants.

We define Chebyshevian box splines similarly to box splines, using the functions w_j as weights (see [5]):

$$B(x \mid V_n, W) = \frac{\chi_{\langle V_n \rangle}(x)}{|\det V_n|}, \tag{1}$$

where $\chi_{\langle V_n \rangle}$ is the characteristic function of the set $\langle V_n \rangle$ and

$$B(x \mid V_{n+k}, W) = \int_0^1 w_k(x - tv_{n+k}) B(x - tv_{n+k} \mid V_{n+k-1}, W)\, dt, \tag{2}$$

for $k = 1, \ldots, s - n$, where

$$\langle V_s \rangle = \Big\{ \sum_{j=1}^{s} t_j v_j : 0 \le t_j < 1,\ j = 1, \ldots, s \Big\}.$$

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For $w_j = 1$, $j = n+1, \ldots, s$, we obtain algebraic box splines, denoted by $B(x \mid V_s)$ (see [1–3]).

The purpose of this paper is to show some new properties of Chebyshevian box splines in the case when the functions w_j are of the form

$$w_j(x) = W_j(v_j \cdot x), \qquad j = n+1, \ldots, s, \tag{3}$$

and the W_j are periodic functions of one variable. Then we consider the problem of approximation of continuous functions by Chebyshevian box splines.

2. "Polynomials" and box splines. Now we shall define some generalization of homogeneous polynomials. We define them similarly to the construction of the Chebyshev system of one variable (see [4]).

Let π_i, $i = n+1, \ldots, s$, be hyperplanes in $\mathbb{R}^n$ defined by means of the vectors v_i:

$$\pi_i : \quad v_i \cdot x = 0, \qquad i = n+1, \ldots, s, \tag{4}$$

$X_i(x)$ the orthogonal projection of x on the hyperplane π_i and $t_i(x)$ the relative distance from x to π divided by the length of v_i,

$$t_i(x) = \frac{x \cdot v_i}{|v_i|^2}, \qquad i = n+1, \ldots, s. \tag{5}$$

Using the following scheme:

$$\begin{array}{ccccccc}
\text{step } 1 & & \text{step } 2 & & \text{step } 3 & & \ldots \\
u_1 & & & & & & \\
u_{1,1} & \to & u_2 & & & & \\
u_{1,2} & \to & u_{2,1} & \to & u_3 & & \\
\ldots & & & & & &
\end{array}$$

we define the following system of functions:

$$\begin{aligned}
&u_0(x) = 1,\\
&u_1(x) = \int_0^{t_s(x)} w_s[X_s(x) + \tau_1 v_s]\, d\tau_1,\\
&u_2(x) = \int_0^{t_s(x)} w_s[X_s(x) + \tau_1 v_s]\\
&\qquad \times \int_0^{t_{s-1}[X_s(x)+\tau_1 v_s]} w_{s-1}\{X_{s-1}[X_s(x) + \tau_1 v_s] + \tau_2 v_{s-1}\}\, d\tau_2\, d\tau_1,\\
&\ldots\\
&u_j(x) = \int_0^{t_s(x)} w_s[X_s(x) + \tau_1 v_s] u_{j-1,1}[X_s(x) + \tau_1 v_s]\, d\tau_1,
\end{aligned} \tag{6}$$

for $j = n+1, \ldots, s$, where

$$u_{0,1}(x) = 1,$$

$$u_{1,1}(x) = \int_0^{t_{s-1}(x)} w_{s-1}[X_{s-1}(x) + \tau_1 v_{s-1}]\, d\tau_1,$$

$$\begin{aligned} u_{2,1}(x) = {} & \int_0^{t_{s-1}(x)} w_{s-1}[X_{s-1}(x) + \tau_1 v_s] \\ & \times \int_0^{t_{s-2}[X_{s-1}(x)+\tau_1 v_s]} w_{s-2}\{X_{s-2}[X_{s-1}(x) + \tau_1 v_{s-1}] + \tau_2 v_{s-2}\}\, d\tau_2\, d\tau_1, \end{aligned} \tag{7}$$

$$\ldots$$

$$\begin{aligned} u_{j,1}(x) = {} & \int_0^{t_{s-1}(x)} w_{s-1}[X_{s-1}(x) + \tau_1 v_{s-1}] \\ & \times u_{j-1,2}[X_{s-1}(x) + \tau_1 v_{s-1}]\, d\tau_1, \end{aligned}$$

where $u_{j,2}$ is defined similarly to u_j, starting from $j = s - 2$.

Because of (3) we may write the systems (6) and (7) as follows:

$$u_0(x) = 1,$$

$$u_1(x) = \int_0^{t_s(x)} w_s[X_s(x) + \tau_1 v_s]\, d\tau_1,$$

$$\begin{aligned} u_2(x) = {} & \int_0^{t_s(x)} w_s[X_s(x) + \tau_1 v_s] \\ & \times \int_0^{t_{s-1}[X_s(x)+\tau_1 v_s]} w_{s-1}[X_{s-1}(x)\tau_2 v_{s-1}]\, d\tau_2\, d\tau_1, \end{aligned} \tag{8}$$

$$\ldots$$

$$u_j(x) = \int_0^{t_s(x)} w_s[X_s(x) + \tau_1 v_s] u_{j-1,1}[X_s(x) + \tau_1 v_s]\, d\tau_1,$$

and

$$u_{0,1}(x) = 1,$$

$$u_{1,1}(x) = \int_0^{t_{s-1}(x)} w_{s-1}[X_{s-1}(x) + \tau_1 v_{s-1}]\, d\tau_1,$$

$$\begin{aligned} u_{2,1}(x) = {} & \int_0^{t_{s-1}(x)} w_{s-1}[X_{s-1}(x) + \tau_1 v_s] \\ & \times \int_0^{t_{s-2}[X_{s-1}(x)+\tau_1 v_s]} w_{s-2}[X_{s-2}(x) + \tau_2 v_{s-2}]\, d\tau_2\, d\tau_1, \end{aligned} \tag{9}$$

$$\ldots$$

$$u_{j,1}(x) = \int\limits_0^{t_{s-1}(x)} w_{s-1}[X_{s-1}(x) + \tau_1 v_{s-1}] u_{j-1,2}[X_{s-1}(x) + \tau_1 v_{s-1}]\, d\tau_1. \tag{9 [cont.]}$$

We have the following

LEMMA 1. *Let*

$$D_v f(x) = \lim_{t \to 0+} \frac{1}{t}[f(x+tv) - f(x)], \qquad D_{v_i, w_j} f(x) = \frac{1}{w_j(x)} D_{v_i} f(x). \tag{10}$$

Then $D_{V_S} u_j(x) = u_{j-1,1}(x)$ *for* $j = 1, \ldots, s-n$.

The proof follows directly from (8) and (10).

EXAMPLE 1. $n = 2$, $v_5 = (1,1)$, $v_4 = (-1,3)$, $v_3 = (-2,1)$,

$$w_3 = w_2 = w_1 = 1, \quad \pi_5 : x_1 + x_2 = 0, \quad \pi_4 : -x_1 + 3x_2 = 0,$$

$$\pi_3 : -2x_1 + x_2 = 0, \quad u_0 = u_{0,1} = u_{0,2} = 1,$$

$$u_1(x) = \frac{x_1 + x_2}{2}, \quad u_{1,1}(x) = \frac{-x_1 + 3x_2}{10}, \quad u_{1,2}(x) = \frac{-2x_1 + x_2}{5},$$

$$u_2(x) = \frac{-3x_1^2 + 2x_1x_2 + 5x_2^2}{40}, \quad u_{2,1}(x) = \frac{7x_1^2 - 22x_1x_2 + 3x_2^2}{200},$$

$$u_3(x) = \frac{4x_1^3 - 5x_1^2x_2 - 6x_1x_2^2 + 3x_2^3}{200}, \quad \frac{\partial u_3}{\partial v_5} = u_{2,1}, \quad \frac{\partial u_2}{\partial v_5} = u_{1,1},$$

$$\frac{\partial u_{2,1}}{\partial v_4} = u_{1,2}, \quad \frac{\partial u_1}{\partial v_5} = u_0, \quad \frac{\partial u_{1,1}}{\partial v_4} = u_0, \quad \frac{\partial u_{1,2}}{\partial v_3} = u_0.$$

Assume that

$$\int\limits_0^1 W_k(t)\, dt = 1 \quad \text{for } k = n+1, \ldots, s. \tag{11}$$

As in the algebraic case the shifts of the Chebyshevian box spline $B(x \mid V_s, W)$ form a partition of unity.

THEOREM 1.

$$\sum_{\alpha \in \mathbb{Z}^n} B(x - \alpha \mid V_k, W) = 1, \quad x \in \mathbb{R}^n, \ k = n, \ldots, s. \tag{12}$$

Proof (by induction on k). For $k = 1$ this follows from the definition of $B(x \mid V_n)$ and $\langle V_n \rangle$ (see [2]). Assume that (12) holds for some $k \geq n$. Applying (3) and the fact that w_k is periodic we obtain

$$\sum_{\alpha\in\mathbb{Z}^n} B(x-\alpha\,|\,V_{k+1},W)$$

$$= \sum_{\alpha\in\mathbb{Z}^n}\int_0^1 w_{k+1}(x-tv_{k+1}-\alpha)B(x-tv_{k+1}-\alpha\,|\,V_k,W)\,dt$$

$$= \int_0^1 w_{k+1}(x-tv_{k+1})\sum_{\alpha\in\mathbb{Z}^n} B(x-tv_{k+1}-\alpha\,|\,V_k,W)\,dt$$

$$= \int_0^1 w_{k+1}(x-tv_{k+1})\,dt = \int_0^1 W_{k+1}(x\cdot v_{k+1}-t|v_{k+1}|^2)\,dt$$

$$= \frac{1}{|v_{k+1}|^2}\int_0^{|v_{k+1}|^2} W_{k+1}(x\cdot v_{k+1}+u)\,du$$

$$= \frac{1}{|v_{k+1}|^2}\int_0^{|v_{k+1}|^2} W_{k+1}(u)\,du = 1.$$

Further we need the following (see [5])

THEOREM 2.

$$D_{v_s,w_s}B(x\,|\,V_s,W) = B(x\,|\,V_{s-1},W) - B(x-v_s\,|\,V_{s-1},W) \tag{13}$$

at every point of continuity of $B(x\,|\,V_{s-1},W)$.

LEMMA 2. *Let*

$$f(x) = \sum_{\alpha\in\mathbb{Z}^n} u_1(\alpha)B(x-\alpha\,|\,V_s,W). \tag{14}$$

Then for $s\ge n+1$, $f=u_1+C$ *on each line* $l: x = x_0+v_st$, *where* C *is a constant.*

Proof. Let x_0 be a point of continuity of $B(x\,|\,V_{s-1},W)$. Then by (13),

$$D_{v_s,w_s}f(x_0) = \sum_{\alpha\in\mathbb{Z}^n}[u_1(\alpha)-u_1(\alpha-v_s)]B(x_0-\alpha\,|\,V_{s-1},W),$$

$$u_1(\alpha)-u_1(\alpha-v_s) = \int_0^{t_s(\alpha)} w_s[X_s(\alpha)+\tau v_s]\,d\tau - \int_0^{t_s(\alpha)-1} w_s[X_s(\alpha)+\tau v_s]\,d\tau$$

$$= \int_{t_s(\alpha)-1}^{t_s(\alpha)} w_s[X_s(\alpha)+\tau v_s]\,d\tau = \int_0^1 W_s(\tau|v_s|^2)\,d\tau = 1.$$

Hence by Theorem 1 we obtain

$$D_{v_s,w_s}f(x_0)=1 \quad \text{and} \quad f(x)=u_1(x)+C \text{ on each line } l: x=x_0+v_st,$$

where C is some constant.

If x_0 is not a point of continuity of $B(x\,|\,V_{s-1},W)$ then the vectors $v_{n+1},\ldots,v_{s-1}$ are parallel to some hyperplane π of dimension $n-1$ such that $x_0\in\pi$. If v_s is not parallel to π then $B(x\,|\,v_s,W)$ is continuous and the function (14) is continuous. If v_s is parallel to π then the function (14) is continuous on the line $x=x_0+v_st$, and from the definition of $\langle V_s\rangle$ we obtain the lemma.

Lemma 3. *Let*

$$f(x)=\sum_{\alpha\in\mathbb{Z}^n}u_2(\alpha)B(x-\alpha\,|\,V_s,W). \tag{15}$$

Then for $s\ge n+2$, $f=u_2+C_1u_1+C_0u_0$ on each line $l: x=x_0+v_st$, where C_0 and C_1 are some constants.

Proof. Let x be a point of continuity of $B(x\,|\,V_{s-1},W)$. Then by (13),

$$D_{v_s,w_s}f(x)=\sum_{\alpha\in\mathbb{Z}^n}[u_2(\alpha)-u_2(\alpha-v_s)]B(x-\alpha\,|\,V_{s-1},W).$$

By (6), (8) and (4) we obtain

$$\begin{aligned}
&u_2(\alpha)-u_2(\alpha-v_s)\\
&\quad=\int\limits_{t_s(\alpha)-1}^{t_s(\alpha)} w_s[X_s(\alpha)+\tau_1v_s]\\
&\qquad\times\int\limits_0^{t_{s-1}[X_s(\alpha)+\tau_1v_s]} w_{s-1}[X_{s-1}(\alpha)+\tau_2v_{s-1}]\,d\tau_2d\tau_1\\
&\quad=\int\limits_0^1 w_s(\alpha-v_s+uv_s)\\
&\qquad\times\int\limits_0^{t_{s-1}(\alpha-v_s+uv_s)} w_{s-1}[X_{s-1}(\alpha-v_s+uv_s)+\tau_2v_{s-1}]\,d\tau_2\,du\\
&\quad=\int\limits_0^1 w_s(\alpha-v_s+uv_s)\int\limits_0^{t_{s-1}(\alpha-v_s+uv_s)} w_{s-1}[X_{s-1}(\alpha)+\tau_2v_{s-1}]\,d\tau_2\,du\\
&\quad=\int\limits_0^1 w_s(\alpha-v_s+uv_s)\int\limits_0^{t_{s-1}(\alpha)} w_{s-1}[X_{s-1}(\alpha)+\tau_2v_{s-1}]\,d\tau_2\,du\\
&\qquad+\int\limits_0^1 w_s(\alpha-v_s+uv_s)\int\limits_{t_{s-1}(\alpha)}^{t_{s-1}(\alpha-v_s+uv_s)} w_{s-1}[X_{s-1}(\alpha)+\tau_2v_{s-1}]\,d\tau_2\,du.
\end{aligned}$$

Applying the periodicity of w_s and w_{s-1}, (3) and (11) we prove that

$$u_2(\alpha) - u_2(\alpha - v_s) = u_{1,1}(\alpha) + C,$$

where C is some constant. Now using Lemma 2 we obtain the assertion.

If x is not a point of continuity of $B(x \mid V_{s-1}, W)$ we proceed as in the proof of Lemma 2.

REMARK 1. Lemmas 2 and 3 cannot be generalized to $k \ge 3$ as the following example shows.

EXAMPLE 2. Let u_j and $u_{j,1}$ be as in Example 1. Then

$$u_3(x) - u_3(x - v_5) = u_{2,1}(x) + \frac{x_1 + 2x_2}{50} - \frac{1}{50}$$

and it cannot be written in the form $u_{2,1} + \alpha u_{1,1} + \beta$.

3. Approximation by Chebyshevian box splines. Let $f_h(x) = f(x/h)$, $h > 0$. We have the following

THEOREM 3. *There exists a constant C depending only on the matrix V_s such that for any function f defined on $\mathbb{R}^n$,*

$$\Big| f(x) - \sum_{\alpha \in h\mathbb{Z}^n} f(\alpha) B_h(x - \alpha \mid V_s, W) \Big| \le \omega(f, h),$$

where $\omega(f,h) = \sup\{|f(x+\delta) - f(x)| : x, \delta \in \mathbb{R}^n,\ |\delta| \le h\}$ is the modulus of continuity of f.

Proof. Applying Theorem 1 and properties of the modulus of continuity we obtain

$$\begin{aligned}\Big| f(x) - \sum_{\alpha \in h\mathbb{Z}^n} f(\alpha) B_h(x - \alpha \mid V_s, W) \Big| &\le \sum_{\alpha \in h\mathbb{Z}^n} |f(x) - f(\alpha)| B_h(x - \alpha \mid V_s, W) \\ &= \sum_{\alpha \in A} |f(x) - f(\alpha)| B_h(x - \alpha \mid V_s, W) \\ &\le (\operatorname{diam} A + 1)\omega(f,h) \sum_{\alpha \in A} B_h(x - \alpha \mid V_s, W) = C\omega(f,h),\end{aligned}$$

where $A = \{\alpha \in h\mathbb{Z}^n : B_h(x - \alpha \mid V_s, W) \ne 0\}$ and $C = \operatorname{diam} A + 1$.

Now we need some definitions, lemmas and theorems.

DEFINITION 1 (see [1–3 ,5]). The family of columns of the matrix V_s is called *unimodular* if the first n columns are linearly independent and $\forall_{Y \subset V_s,\, \sharp Y = n}\, |\det Y| \le 1$.

Let $\widehat{f}$ denote the Fourier transform of f, i.e.

$$\widehat{f}(x) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i t \cdot x}\, dt.$$

We have the following

THEOREM 4 (see [6]). *Let the family $V = V_s$ be admissible and unimodular and the functions W_j, $j = n+1, \ldots, s$, be trigonometric polynomials, i.e.*

$$W_j(x) = \sum_{k=-n_j}^{n_j} a_{j,k} e^{2\pi i k\cdot x}$$

where $a_{j,-k} = \overline{a}_{j,k}$, $k \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$. Then

$$\{x \in \mathbb{R}^n : \forall_{\alpha\in\mathbb{Z}^n}\ \widehat{B}(x-\alpha \mid V_s, W) = 0\} = \emptyset. \tag{16}$$

Let $X = V \cup -V = \{v_1, \ldots, v_s, -v_1, \ldots, -v_s\}$ and

$$\widetilde{B}(x \mid X, W) = \int_{\mathbb{R}^n} B(x-t \mid V, W) B(t \mid -V, W^-)\, dt,$$

where $-V = \{-v_1, \ldots, -v_s\}$ and $W^- = \{f : f(-x) \in W\}$.

THEOREM 5 ([5, Theorem 4], cf. [3]). *Let the family V be admissible and satisfy* (16). *Then for every $x \in \mathbb{R}^n$,*

$$P_{X,W}(x) = \sum_{\alpha\in\mathbb{Z}^n} \widetilde{B}(\alpha \mid X, W) e^{2\pi i\alpha\cdot x} \neq 0. \tag{17}$$

Now we may define the fundamental function $\Phi_{X,W}$ as follows (see [3, 5]):

$$\Phi_{X,W}(x) = \sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha) \widetilde{B}(x-\alpha \mid X, W), \tag{18}$$

where $b_{X,W}(\alpha)$ are the coefficients of the Fourier series of the function $1/P_{X,W}$, i.e.

$$1/P_{X,W}(x) = \sum_{\alpha\in\mathbb{R}^n} b_{X,W}(\alpha) e^{2\pi i\alpha\cdot x}, \quad b_{X,W}(\alpha) = \int_{[0,1]^n} \frac{1}{P_{X,W}(x)} e^{-2\pi i\alpha\cdot x}\, dx.$$

LEMMA 4 ([5], cf. [3]). *For every $\alpha \in \mathbb{Z}^n$,*

$$\Phi_{X,W}(\alpha) = \delta_{0,\alpha}. \tag{19}$$

LEMMA 5 ([5], cf. [6]). *There exist constants $C > 0$ and $0 < q < 1$ such that*

$$|\Phi_{X,W}(x)| \le C q^{\|x\|}, \quad x \in \mathbb{R}^n. \tag{20}$$

Now we may define interpolating operators I and I_h (see [3, 5]) as follows: for every function g defined on $\mathbb{Z}^n$ we put

$$Ig(x) = \sum_{\alpha\in\mathbb{Z}^n} g(\alpha)\Phi_{X,W}(x-\alpha), \quad I_h(x) = \sum_{\alpha\in h\mathbb{Z}^n} g(\alpha)\Phi_{X,W}\left(\frac{x-\alpha}{h}\right).$$

LEMMA 6 (cf. [3]). *We have*

$$B(x \mid V, W) = \sum_{\alpha\in\mathbb{Z}^n} B(\alpha \mid V, W)\Phi_{X,W}(x-\alpha), \tag{21}$$

$$\sum_{\beta\in\mathbb{Z}^n} \Phi_{X,W}(x-\beta) = 1. \tag{22}$$

Proof. Using the Fourier transforms of $\Phi_{X,W}$ and $\widetilde{B}(\cdot \mid X, W)$ in (18) we obtain (21) exactly as in [3]. Further

$$\begin{aligned}\sum_{\beta\in\mathbb{Z}^n} \Phi_{X,W}(x-\beta) &= \sum_{\beta\in\mathbb{Z}^n}\sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha)\widetilde{B}(x-\beta-\alpha \mid X, W)\\ &= \sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha) \sum_{\beta\in\mathbb{Z}^n} \widetilde{B}(x-\beta-\alpha \mid X, W)\\ &= \sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha) = \frac{1}{P_{X,W}(0)}\\ &= \frac{1}{\sum_{\alpha\in\mathbb{Z}^n} \widetilde{B}(\alpha \mid X, W)} = 1\end{aligned}$$

and we have proved (22).

THEOREM 6. *There exists a constant $C > 0$ such that for any function f defined on $\mathbb{R}^n$,*

$$|f(x) - I_h f(x)| \le C\omega(f, h).$$

Proof. Let $A_k = \{\alpha \in \mathbb{Z}^n : k-1 \le |x_i - h\alpha_i| < k,\ i = 1, \ldots, n\}$. Then $\sharp A_k = (2k+1)^n - (2k-1)^n$, $k = 1, 2, \ldots$

Using (20), (22) and properties of the modulus of continuity we obtain

$$\begin{aligned}|f(x) - I_h f(x)| &\le \sum_{\alpha\in\mathbb{Z}^n} |f(x) - f(h\alpha)||\Phi_{X,W}(x/h-\alpha)|\\ &\le \sum_{k=1}^{\infty}\sum_{\alpha\in A_k} |f(x) - f(h\alpha)|\Phi_{X,W}(x/h-\alpha)|\\ &\le C_1 \sum_{k=1}^{\infty} k[(2k+1)^n - (2k-1)^n]q^k\omega(f,h) \le C\omega(f,h),\end{aligned}$$

where $C = C_1\sum_{k=1}^{\infty} k[(2k+1)^n - (2k-1)^n]q^k$. Since $0 < q < 1$ the series is convergent and we have proved the theorem.

Let

$$B^*_{V,W}(x) = \sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha)B(x-\alpha \mid V, W), \quad x \in \mathbb{R}^n.$$

We have the following

LEMMA 7 (see [5]). *For any $\beta \in \mathbb{Z}^n$,*

$$(B^*_{V,W}, B(\cdot - \beta \,|\, V, W)) = \delta_{0,\beta}. \tag{23}$$

Moreover, there exist constants $C > 0$ and $0 < q < 1$ such that

$$|B^*_{V,W}(x)| \le C q^{\|x\|}, \quad x \in \mathbb{R}^n, \tag{24}$$

where $(f, g) = \int_{\mathbb{R}^n} f\overline{g}\, dx$.

Let

$$Pf(x) = \sum_{\alpha \in \mathbb{Z}^n} (f, B^*_{V,W}(\cdot - \alpha)) B(x - \alpha \,|\, V, W),$$

$$P_h f(x) = \sum_{\alpha \in h\mathbb{Z}^n} \left(f, B^*_{V,W}\left(\frac{\cdot - \alpha}{h}\right)\right) B\left(\frac{x - \alpha}{h} \,\middle|\, V, W\right).$$

By (24) we obtain

$$\|P_h f\|_\infty \le C \|F\|_\infty, \tag{25}$$

where $\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|$ and C is some constant depending only on the matrix V.

THEOREM 7. *There exists a constant $C > 0$ such that for any function $f \in L^2(\mathbb{R}^n)$,*

$$\|f - P_h f\|_\infty \le C\omega(f, h).$$

Proof. Let

$$S_f(x) = \sum_{\alpha \in \mathbb{Z}^n} f(\alpha) B(x - \alpha \,|\, V, W).$$

Using (23) we obtain $P_h S_f(x) = S_f(x)$. Hence by (24), (25) and Theorem 3

$$\begin{aligned} \|f - P_h f\|_\infty &\le \|f - S_f\|_\infty + \|S_f - P_h f\|_\infty \\ &= \|f - S_f\|_\infty + \|P_h(f - S_f)\|_\infty \\ &\le C_1 \|f - S_f\|_\infty \le C\omega(f, h). \end{aligned}$$

PROBLEM. Find the order of approximation of a function f by Chebyshevian box splines according to the regularity of f.

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