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Recurrent point set of the shift on Σ and strong chaos

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Abstract. Let (Σ, ϱ) be the one-sided symbolic space (with two symbols), and let σ be the shift on Σ. We use $A(\cdot)$, $R(\cdot)$ to denote the set of almost periodic points and the set of recurrent points respectively. In this paper, we prove that the one-sided shift is strongly chaotic (in the sense of Schweizer–Smítal) and there is a strongly chaotic set $\mathcal{J}$ satisfying $\mathcal{J} \subset R(\sigma) - A(\sigma)$.

1. Introduction. Throughout this paper, X will denote a compact metric space with metric d; I is the closed interval $[0,1]$.

For a continuous map $f : X \to X$, denote the sets of periodic points, nonwandering points and ω-limit points of f by $P(f)$, $\Omega(f)$ and $\omega(f)$ respectively; f^n will denote the n-fold iterate of f.

$D \subset X$ is said to be a *chaotic set* of f if for any different points $x, y \in D$,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0, \qquad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

f is said to be *chaotic* if it has a chaotic set which is uncountable.

The notion of strong chaos first occurred in [18] where it is characterised by the distribution function of distances between trajectories of two points. The concrete version is as follows.

Let $x, y \in X$. For any real $t > 0$, let

$$F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,t)}(d(f^i(x), f^i(y))),$$

$$F^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,t)}(d(f^i(x), f^i(y))),$$

where χ_A is the characteristic function of the set A. Obviously, F_{xy} and

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F^*_{xy} are both nondecreasing functions; for $t \leq 0$, define F_{xy} and F^*_{xy} as probability distribution functions (see [18] for details, where F_{xy} and F^*_{xy} are called, respectively, the lower and upper distribution function of x and y).

Call $x, y \in X$ a *pair of points displaying strong chaos* if

1) $F^*_{xy} = \chi_{[0,\infty)}$, i.e., $F^*_{xy}(t) = 1$ for all $t > 0$,
2) $F_{xy}(\varepsilon) = 0$ for some $\varepsilon > 0$.

f is said to *display strong chaos* if there exists an uncountable set $D \subset X$ such that any two different points in D display strong chaos.

Clearly any map displaying strong chaos must be Li–Yorke chaotic.

For a continuous map $f : I \to I$, Li and Yorke [10] has proved that if f has a periodic point of period 3, then it is chaotic.

Later, many sharpened results came into being in succession (see [7]–[9], [11], [13], [16], [17], [19]). One can find in [11] and [1] equivalent conditions for f to be chaotic, and in [16] or [19] a chaotic map with topological entropy zero, which has shown that positive topological entropy and chaos are not equivalent. On the other hand, it is known that by restricting the uncountable chaotic set to $R(f)$ or to $\overline{P(f)}$ or to $\Omega(f)$, the equivalence holds (see [4], [20], [21]).

This leaves open two questions:

1) Is the existence of an uncountable strongly chaotic set of f in $R(f)$ or in $A(f)$ equivalent to $\mathrm{ent}(f) > 0$?

2) Is there a map f such that $R(f) - A(f)$ contains an uncountable strongly chaotic set?

For a continuous map $f : I \to I$, Schweizer and Smítal [18] have proved:

(i) If f has zero topological entropy, then no pair of points can form a strongly chaotic set. (This implies that strong chaos and Li–Yorke chaos are not the same notion.)

(ii) If f has positive entropy, then there exists an uncountable strongly chaotic set in which each member is an ω-limit point of f.

The aim of this paper is to learn whether there exists a map f such that $R(f) - A(f)$ contains an uncountable strongly chaotic set of f.

In fact, we will prove

THEOREM. *The one-sided shift is strongly chaotic and there is a strongly chaotic set $\mathcal{J}$ satisfying $\mathcal{J} \subset R(\sigma) - A(\sigma)$.*

2. Basic definitions and preparations. Let $S = \{0, 1\}$, $\Sigma = \{x = x_1x_2\ldots \mid x_i \in S,\ i = 1, 2, \ldots\}$ and define $\varrho : \Sigma \times \Sigma \to \mathbb{R}$ as follows: for any

$x, y \in \Sigma$, if $x = x_1x_2\ldots$ and $y = y_1y_2\ldots$, then

$$\varrho(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1/2^k & \text{if } x \neq y, \end{cases} \quad \text{and} \quad k = \min\{n \mid x_n \neq y_n\} - 1.$$

It is not difficult to check that ϱ is a metric on Σ. The space (Σ, ϱ) is compact and called the *one-sided symbolic space.*

Define $\sigma : \Sigma \to \Sigma$ by $\sigma(x) = x_2x_3\ldots$ for any $x = x_1x_2\ldots \in \Sigma$. Then σ is continuous and called the *shift* on Σ. Call A a *tuple* (over S) if it is a finite arrangement of elements in S. If $A = a_1a_2\ldots a_n$ where $a_i \in S, 1 \leq i \leq n$, then the length of A is said to be n, denoted by $|A| = n$.

Let $B = b_1b_2\ldots b_m$ be another tuple. Set $AB = a_1a_2\ldots a_nb_1b_2\ldots b_m$; then AB is also a tuple.

Let $x \in \Sigma$ with $x = x_1x_2\ldots$. It is called a *repeating sequence* with *recurring period* of length m if $x_{i+m} = x_i$ for any $i \in \{1, 2, \ldots\}$. We then write $x = (\dot{x}_1\dot{x}_2\ldots\dot{x}_m)$.

Let $x \in X$. Then $y \in X$ is said to be an *ω-limit point of x* if the sequence $f(x), f^2(x), \ldots$ has a subsequence converging to y. The set of ω-limit points of x is denoted by $\omega(x, f)$. Each point in $\bigcup_{x\in X} \omega(x, f)$ is called an *ω-limit point of f*. The set of ω-limit points of f is denoted by $\omega(f)$.

$x \in X$ is called *almost periodic* for f if for any $\varepsilon > 0$, one can find $k > 0$ such that for any integer $q \geq 0$, there is an integer r with $q \leq r < k + q$ satisfying $d(f^r(x), x) < \varepsilon$. Denote by $A(f)$ the set of almost periodic points of f.

$x \in X$ is called a *recurrent point* for f if the sequence $f(x), f^2(x), \ldots$ has a subsequence converging to x. The set of recurrent points for f is denoted by $R(f)$.

$Y \subset X$ is called a *minimal set* of f if for any $x \in Y$, $\omega(x, f) = Y$.

LEMMA 2.1. *For any $x \in X$ and any $N > 0$, the following are equivalent.*

1) $x \in A(f)$.
2) $x \in A(f^N)$.
3) $x \in \omega(x, f)$ *and* $\omega(x, f)$ *is a minimal set of* f.

For a proof see [5] and [6].

LEMMA 2.2. *Σ has an uncountable subset E such that for any different points $x = x_1x_2\ldots$, $y = y_1y_2\ldots$ in E, $x_n = y_n$ for infinitely many n and $x_m \neq y_m$ for infinitely many m.*

Proof. For any $x = x_1x_2\ldots, y = y_1y_2\ldots \in \Sigma$, write $x \sim y$ if either $x_n = y_n$ holds only for finitely many n, or $x_m \neq y_m$ holds only for finitely many m. We easily check that $\sim$ is an equivalence relation on Σ. Let $x \in \Sigma$. It is easy to see that the set $\{y \in \Sigma \mid y \sim x\}$ is countable and so the quotient set $\Sigma/\sim$ is uncountable. Taking a representative in each equivalence class of $\Sigma/\sim$, we get an uncountable set E which satisfies the requirement.

LEMMA 2.3. *Let (Σ, ϱ) be the one-sided symbolic space, and let σ be the shift on Σ. Then*:

(i) *For all $s \in \Sigma$ and $m > 0, \sigma^m(s) = s$ if and only if $s = (\dot{s}_0\dot{s}_1 \ldots \dot{s}_{m-1})$ (i.e. s is a repeating sequence with recurring period of length m).*

(ii) *There are exactly 2^n elements s in Σ such that $\sigma^n(s) = s$.*

For a proof see [3].

3. Proof of the Theorem. First we construct the set $\mathcal{J}$ as in the conclusion of the Theorem.

Let $A = a_1a_2 \ldots a_n$ be a tuple (over $S = \{0, 1\}$). Define the inverse of A to be $\overline{A} = \overline{a}_1 \ldots \overline{a}_n$, where

$$\overline{a}_i = \begin{cases} 0 & \text{if } a_i = 1, \\ 1 & \text{if } a_i = 0, \end{cases} \qquad \text{for } i = 1, \ldots, n.$$

Clearly, $|\overline{A}| = |A|$ and $\overline{\overline{A}} = A$.

Take an arbitrary tuple A_1. Let A_2 be an arrangement of A_1 and $\overline{A}_1$, say $A_2 = A_1\overline{A}_1$ (or $\overline{A}_1A_1$). Define inductively the tuples $A_2, A_3, \ldots$ such that A_n is an arrangement of all the tuples of the finite set

$$\mathcal{J}_n = \{J_1 \ldots J_{n-1} \mid J_i \in \{A_i, \overline{A}_i\},\ 1 \le i \le n-1\}.$$

For each $n = 1, 2, \ldots$, put $m_n = |A_1A_2 \ldots A_n|$. Then $m_n - m_{n-1} = 2^{n-1}m_{n-1}$ for all $n > 1$, as can be easily derived from the definition.

Choose an uncountable subset E in Σ such that for any different points $x = x_1x_2 \ldots$, $y = y_1y_2 \ldots$, both $x_n = y_n$ holds for infinitely many n and $x_m \ne y_m$ holds for infinitely many m. By Lemma 2.2, such a subset exists. Define $\varphi : E \to E$ by $\varphi(x) = J_1J_2 \ldots = \langle J_i \rangle$, $i = 1, 2, \ldots$ for all $x = x_1x_2 \ldots \in E$, where

$$J_i = \begin{cases} A_i & \text{if } x_i = 1, \\ \overline{A}_i & \text{if } x_i = 0, \end{cases} \qquad \text{for } i = 1, 2, \ldots$$

Let $D = \varphi(E)$; then $D \subset E$. Since E is uncountable and φ is injective, D is uncountable.

By Lemma 2.3, for any $n \in \mathbb{N}$, the elements of Σ satisfying $\sigma^n(s) = s$ are the repeating sequences with recurring period of length n; there are exactly 2^n such elements.

Let p_1 be an arrangement of the recurring periods of the two repeating sequences of length 2 such that $\sigma^2(s) = s$, e.g. $p_1 = 01$.

Let p_2 be an arrangement of the recurring periods of the 2^2 repeating sequences of length 2 such that $\sigma^2(s) = s$, e.g. $p_2 = 00\,01\,10\,11$.

p_n is an arrangement of the recurring periods of the 2^n repeating sequences of length n such that $\sigma^n(s) = s$, e.g. $p_n = 000\ldots 0\ldots 11\ldots 1$.

Let $a = p_1p_2\ldots p_n\ldots = 0100011011000\ldots 001\ldots = a_1a_2\ldots$ It is easy to see that $\omega(a,\sigma) = \Sigma$. In fact, for any $x = x_1x_2\ldots \in \Sigma$, let T_n be a periodic point of σ with period n and with recurring period $(x_1,\ldots,x_n)$. Then $T_n \to x$ $(n\to\infty)$. By the construction of a, for any $\varepsilon > 0$, there exists $N_i(\varepsilon)$ such that

$$|\sigma^{N_i}(a) - x| < \sum_{n=N_i}^{\infty} \frac{1}{2^n} < \varepsilon.$$

So, $\sigma^{N_i}(a) \to x$ $(i\to\infty)$. This shows $x\in\omega(a,\sigma)$, i.e. $\Sigma\subset\omega(a,\sigma)$. On the other hand, $\omega(a,\sigma)\subset\Sigma$, hence $\Sigma = \omega(a,\sigma)$. Let

$$\mathcal{J} = \{J_1a_1J_2a_1a_2\ldots J_{n-1}a_1a_2\ldots a_{n-1}J_n\ldots \mid \langle J_i\rangle \in D\}.$$

Since D is uncountable, $\mathcal{J}$ is uncountable, and for any $x,y\in\mathcal{J}$, by Lemma 2.2 we have $x\neq y$.

Secondly, we prove $\mathcal{J}\subseteq R(\sigma) - A(\sigma)$.

Since $\omega(a,\sigma) = \Sigma$, for any $x\in\Sigma$, there exists an infinite sequence $\{p_i\}$ of positive integers such that $\sigma^{p_i}(a)\to x$ $(i\to\infty)$. For any $C\in\mathcal{J}$, $C = C_1a_1C_2a_1a_2\ldots$, put

$$q_i = p_i + m_{p_i} + \frac{(p_i-1)(p_i-2)}{2}.$$

Then $\sigma^{q_i}(C)\to x$ $(i\to\infty)$.

Thus we have proved that $\omega(C,\sigma) = \Sigma$ and $C\in R(\sigma)$. Since $\omega(C,\sigma) = \Sigma$ and Σ is not a minimal set of f, we know that $\omega(C,\sigma)$ is not a minimal set of f. By Lemma 2.1, $C\notin A(\sigma)$. Summing up, we obtain

$$\mathcal{J}\subset R(\sigma) - A(\sigma).$$

Finally, we will prove that $\sigma|_{\mathcal{J}}$ displays strong chaos.

Let $b = B_1a_1B_2a_1a_2\ldots$ and $c = C_1a_1C_2a_1a_2\ldots$ be different points in $\mathcal{J}$, where $B_i, C_i\in\{A_i,\overline{A}_i\}$, $i=1,2,\ldots$ By Lemma 2.2 and the construction of $\mathcal{J}$, there are sequences of positive integers $p_i\to\infty$ and $q_i\to\infty$ such that

$$B_{p_i} = C_{p_i},\quad B_{q_i} = \overline{C}_{q_i}\quad \text{for all } i.$$

Put, for simplicity,

$$\delta_{bc}(j) = \varrho(\sigma^j(b),\sigma^j(c)),\qquad j=1,2,\ldots$$

First, it is easily seen that for given $p_i\geq 3$, the first m_{p_i-1} symbols of $\sigma^j(b)$

and $\sigma^j(c)$ coincide for

$$m_{p_i-1} + \frac{(p_i-2)(p_i-3)}{2} \le j \le \left(m_{p_i} + \frac{p_i(p_i-1)}{2}\right) - \left(m_{p_i-1} + \frac{(p_i-2)(p_i-3)}{2}\right),$$

so

$$\delta_{bc}(j) \le \frac{1}{2^{m_{p_i-1}}}.$$

Thus $\delta_{bc}(j) < t$ for given $t > 0$, provided p_i is large enough. Furthermore $\chi_{[0,t)}(\delta_{bc}(j)) = 1$. Let

$$N_{p_i} = m_{p_i} + \frac{(p_i-1)p_i}{2} - \left(m_{p_i-1} + \frac{(p_i-2)(p_i-3)}{2}\right),$$
$$K_{p_i} = m_{p_i-1} + \frac{(p_i-2)(p_i-3)}{2}.$$

Thus, we have

$$\frac{1}{N_{p_i}} \sum_{j=1}^{N_{p_i}} \chi_{[0,t)}(\delta_{cb}(j)) \ge \frac{1}{N_{p_i}} \sum_{j=K_{p_i}}^{N_{p_i}} \chi_{[0,t)}(\delta_{cb}(j))$$

and

$$\begin{aligned}\frac{N_{p_i}-K_{p_i}}{N_{p_i}} &= 1 - \frac{m_{p_i-1} + \frac{(p_i-2)(p_i-3)}{2}}{\left(m_{p_i} + \frac{(p_i-1)p_i}{2}\right) - \left(m_{p_i-1} + \frac{(p_i-2)(p_i-3)}{2}\right)} \\ &= 1 - \frac{m_{p_i-1} + \frac{(p_i-2)(p_i-3)}{2}}{2^{p_i-1}m_{p_i-1} + \frac{(p_i-1)p_i - (p_i-2)(p_i-3)}{2}} \\ &\to 1 \quad (p_i \to \infty).\end{aligned}$$

Hence

$$F^*_{bc}(t) = 1. \tag{1}$$

Secondly, it is easy to see that for given q_i large enough, the first m_{q_i-1} symbols of $\sigma^j(b)$ and $\sigma^j(c)$ are all distinct for

$$m_{q_i-1} + \frac{(q_i-1)(q_i-2)}{2} \le j \le m_{q_i} - m_{q_i-1} + \frac{(q_i-1(q_i-2)}{2},$$

so

$$\delta_{bc}(j) = 1.$$

Take any $t \in (0,1]$; then $\chi_{[0,t)}(\delta_{bc}(j)) = 0$. Furthermore let

$$T_{q_i} = \left(m_{q_i} + \frac{(q_i-1)(q_i-2)}{2}\right) - \left(m_{q_i-1} + \frac{(q_i-1)(q_i-2)}{2}\right).$$

Then

$$\sum_{j=1}^{T_{q_i}} \chi_{[0,t)}(\delta_{bc}(j)) \le \frac{1}{m_{q_i} - m_{q_i-1}} \sum_{j=1}^{m_{q_i-1}} \chi_{[0,t)}(\delta_{bc}(j))$$
$$\le \frac{m_{q_i-1}}{2^{q_i-1} m_{q_i-1}} \to 0 \qquad (q_i \to \infty).$$

This shows

$$F_{bc}(t) = 0. \tag{2}$$

(1) and (2) prove that b and c are a pair of points displaying strong chaos. By the arbitrariness of b and c, $\sigma|_{\mathcal{J}}$ displays strong chaos.

The proof of the Theorem is complete.

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