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Estimates of capacity of self-similar measures

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Abstract. We give lower and upper estimates of the capacity of self-similar measures generated by iterated function systems $\{(S_i, p_i) : i = 1, \ldots, N\}$ where S_i are bi-lipschitzean transformations.

1. Introduction. The idea of dimension of measures is fundamental in measure theory and it also occurs in diverse branches of mathematics. For example, it is a basic tool in the study of attractors of dynamical systems, in particular in the study of attractors (also called fractals) generated by iterated function systems, or more generally, fractals generated by Markov chains (see [1–6, 15, 16, 18, 23]). Various notions of dimension have been proposed: Hausdorff dimension, box dimension, entropy dimension, correlation dimension. These concepts were widely investigated and used. Closely related to Hausdorff dimension is capacity, introduced by Kolmogorov (see [14]). This capacity, however, does not distinguish between a set and its closure. Ledrappier [17] has made some modification to correct this insensitivity. While the other concepts mentioned here have been extensively studied, Ledrappier's version of capacity does not seem to be sufficiently explored. In this paper we give lower and upper estimates of Ledrappier's capacity of measures invariant with respect to iterated function systems of functions which are bi-lipschitzean.

The calculation of dimensions has been performed by several authors inspired by Hutchinson's elegant treatment [13]. For an account of the technique involved, generalizations and improvements see [2–4, 12, 18, 20]. Our approach is also based on this idea.

2. Notations and preliminaries. Throughout this paper (X, ϱ) denotes a Polish space and this assumption will not be repeated in the state-

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ment of theorems. By $B(x,r)$ we denote the closed ball in X with center at x and radius r. For $A \subset X$, $A \neq \emptyset$, we denote by $\operatorname{diam} A$ the diameter of A and by 1_A the characteristic function of A. Moreover, for $A, B \subset X$, $A, B \neq \emptyset$, we define

$$\operatorname{dist}(A,B) = \inf\{\varrho(x,y) : x \in A,\ y \in B\}.$$

As usual, $\mathbb{R}$ stands for the set of all reals and $\mathbb{N}$ for the set of all positive integers. Moreover set $\mathbb{R}_+ = [0,\infty)$.

We denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X and by $\mathcal{M}$ the family of all finite Borel measures on X. Moreover, $\mathcal{M}_1$ denotes the family of all $\mu \in \mathcal{M}$ such that $\mu(X) = 1$, and $\mathcal{M}_{\mathrm{s}} = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}\}$ is the space of all finite signed measures.

Finally, $B(X)$ stands for the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$ and $C(X)$ for the subspace of $B(X)$ of all bounded continuous functions. In both spaces the norm is given by the formula

$$\|f\|_0 = \sup_{x\in X} |f(x)|.$$

For $f \in B(X)$ and $\nu \in \mathcal{M}_{\mathrm{s}}$ we write

$$\langle f, \nu\rangle = \int_X f(x)\,\nu(dx).$$

We say that a sequence $(\mu_n)_{n\ge 1} \subset \mathcal{M}$ converges weakly to a measure $\mu \in \mathcal{M}$ if $\lim_{n\to\infty}\langle f, \mu_n\rangle = \langle f, \mu\rangle$ for every $f \in C(X)$.

We endow $\mathcal{M}_{\mathrm{s}}$ with the *Fortet–Mourier norm* (see [11]) given by

$$\|\nu\| = \sup\{|\langle f, \nu\rangle| : f \in F\},$$

where F is the set of all functions $f \in C(X)$ such that $\|f\|_0 \le 1$ and $|f(x) - f(y)| \le \varrho(x,y)$ for $x, y \in X$. It is known that the convergence

$$\lim_{n\to\infty} \|\mu_n - \mu\| = 0 \quad \text{ for } \mu_n, \mu \in \mathcal{M}_1$$

is equivalent to the weak convergence of the sequence $(\mu_n)_{n\ge1}$ to μ (see [7]).

An operator $P : \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* if

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2 \quad \text{ for } \lambda_1, \lambda_2 \in \mathbb{R}_+ \text{ and } \mu_1, \mu_2 \in \mathcal{M}$$

and

$$P\mu(X) = \mu(X) \quad \text{ for } \mu \in \mathcal{M}.$$

A measure μ is called *stationary* (or *invariant*) with respect to the operator P if $P\mu = \mu$. A Markov operator P is called *asymptotically stable* if there exists an invariant probability measure μ_* such that

$$\lim_{n\to\infty} \|P^n\mu - \mu_*\| = 0 \quad \text{ for } \mu \in \mathcal{M}_1.$$

Clearly, the stationary measure is unique if P is asymptotically stable.

Let $\mu \in \mathcal{M}_1$. For given $\varepsilon > 0$ and $C \subset X$ we denote by $N_C(\varepsilon)$ the minimal number of ε-balls needed to cover the set C. Further, for $\varepsilon, \eta > 0$ we define

$$N(\varepsilon, \eta) = \inf\{N_C(\varepsilon) : C \subset X \text{ and } \mu(C) > 1 - \eta\}.$$

Then the quantities

$$\underline{\mathrm{Cap}}_L(\mu) = \sup_{\eta>0} \liminf_{\varepsilon\to 0} \frac{\log N(\varepsilon,\eta)}{-\log \varepsilon}$$

and

$$\overline{\mathrm{Cap}}_L(\mu) = \sup_{\eta>0} \limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon,\eta)}{-\log \varepsilon}$$

are called the *lower* and *upper capacity* of μ, respectively.

The above definitions were introduced by Ledrappier (see [17, 22, 25]) and are closely related to Kolmogorov dimensions.

REMARK 2.1. In the definitions of the lower and upper capacity we can replace the continuous variable ε by a decreasing sequence $(\varepsilon_n)_{n\geq 1}$ with $\log \varepsilon_{n+1}/\log \varepsilon_n \to 1$.

Assume now we are given a sequence of continuous transformations

$$S_i : X \to X \quad \text{ for } i = 1, \ldots, N$$

and a probability vector

$$p_i : X \to [0,1] \quad \text{ for } i = 1, \ldots, N,$$

where the p_i are continuous functions satisfying

$$p_i(x) > 0 \quad \text{and} \quad \sum_{i=1}^{N} p_i(x) = 1 \quad \text{for } x \in X.$$

Such a system is denoted by $(S, p)_N$ and called an *iterated function system* (briefly *IFS*).

Having an IFS $(S, p)_N$ we define the corresponding Markov operator $P : \mathcal{M} \to \mathcal{M}$ by

$$P\mu(A) = \sum_{i=1}^{N} \int_{S_i^{-1}(A)} p_i(x)\,\mu(dx) \quad \text{ for } A \in \mathcal{B}(X) \tag{2.1}$$

and its dual operator $U : B(X) \to B(X)$ by

$$Uf(x) = \sum_{i=1}^{N} p_i(x) f(S_i(x)). \tag{2.2}$$

We say that an IFS $(S,p)_N$ is *asymptotically stable* if the corresponding Markov operator P is asymptotically stable. A measure $\mu_* \in \mathcal{M}$ is called *invariant* for the IFS $(S,p)_N$ if it is invariant with respect to the corresponding Markov operator P.

We assume that $S_i : X \to X$, $i = 1,\dots,N$, are *bi-lipschitzean transformations*, i.e. there exist constants $l_i, L_i > 0$ such that

$$l_i\varrho(x,y) \le \varrho(S_i(x),S_i(y)) \le L_i\varrho(x,y) \quad \text{for } x,y\in X. \tag{2.3}$$

Throughout this paper $l_1,\dots,l_N$ and $L_1,\dots,L_N$ always stand for the constants satisfying (2.3). Moreover, we assume that

$$\varGamma_0 = \sup_{x\in X}\prod_{i=1}^N L_i^{p_i(x)} < 1, \tag{2.4}$$

$$\alpha_0 = \min_{1\le i\le N}\inf_{x\in X} p_i(x) > 0, \tag{2.5}$$

$$\sum_{i=1}^N |p_i(x)-p_i(y)| \le \omega(\varrho(x,y)) \quad \text{for } x,y\in X, \tag{2.6}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing concave function satisfying the Dini condition

$$\int_0^\eta \frac{\omega(t)}{t}\,dt < \infty \quad \text{for some } \eta > 0. \tag{2.7}$$

The following constants will play a crucial role:

$$\varDelta_0 = \sup_{x\in X}\prod_{i=1}^N p_i(x)^{p_i(x)}, \tag{2.8}$$

$$\delta_0 = \inf_{x\in X}\prod_{i=1}^N p_i(x)^{p_i(x)}, \tag{2.9}$$

$$\gamma_0 = \inf_{x\in X}\prod_{i=1}^N l_i^{p_i(x)}. \tag{2.10}$$

Obviously $\varDelta_0, \delta_0, \gamma_0 \in (0,1)$.

We say that a family of transformations $S_1,\dots,S_N$ satisfies the *strong Moran condition* (see [19]) if there exists a bounded closed subset F of X and a constant $\sigma > 0$ such that

$$\bigcup_{i=1}^N S_i(F) \subset F, \tag{2.11}$$

$$\operatorname{dist}(S_i(F),S_j(F)) \ge \sigma \quad \text{for } i\ne j. \tag{2.12}$$

PROPOSITION 2.1. *If an IFS* $(S,p)_N$ *satisfies conditions* (2.3)–(2.7), *then it is asymptotically stable.*

Proof. See [23].

Let $\Omega = \{1,\dots,N\}^\infty = \{(i_1,i_2,\dots) : i_k \in \{1,\dots,N\}$ for every $k \in \mathbb{N}\}$ and $\Omega_* = \bigcup_{n=1}^\infty \Omega_n$, where $\Omega_n = \{1,\dots,N\}^n$. Observe that Ω_* (resp. Ω) is the space of all finite (resp. infinite) sequences of elements $i_k \in \{1,\dots,N\}$. For $k \in \mathbb{N}$ we set $\Omega_{\le k} = \bigcup_{n=1}^k \Omega_n$ and $\Omega_{\ge k} = \bigcup_{n=k}^\infty \Omega_n$. For $\mathbf{i} = (i_1,\dots,i_n) \in \Omega_*$ let $|\mathbf{i}| = n$ denote the length of $\mathbf{i}$. If $\mathbf{i} \in \Omega$ we assume that $|\mathbf{i}| = \infty$. For $\mathbf{i} \in \Omega \cup \Omega_*$ and $m \in \mathbb{N}$, $m \le |\mathbf{i}|$, we set $\mathbf{i}|m = (i_1,\dots,i_m)$. We say that $\mathbf{i} < \mathbf{j}$ with $\mathbf{i} \in \Omega_*$ and $\mathbf{j} \in \Omega \cup \Omega_*$ if $|\mathbf{j}| > n$ and $\mathbf{j}|n = \mathbf{i}$, where $n = |\mathbf{i}|$. Finally, for $\mathbf{i} = (i_1,\dots,i_n) \in \Omega_*$, we write $\mathbf{i}^{-1} = (i_n,\dots,i_1)$.

A subset $\Lambda \subset \Omega$ is called a *cylinder* if there exists $\mathbf{i} = (i_1,\dots,i_n) \in \Omega_*$ such that

$$\Lambda = \Lambda(\mathbf{i}) = \{\mathbf{j} \in \Omega : \mathbf{j}|n = \mathbf{i}\}.$$

We denote by $\mathcal{A}$ the σ-algebra of subsets of Ω which is generated by such cylinders.

Given an IFS $(S,p)_N$ and a point $x \in X$ we denote by $\mathbb{P}_x$ the probability measure on $\mathcal{A}$ defined on the cylinder $\Lambda(\mathbf{i})$, $\mathbf{i} = (i_1,\dots,i_n) \in \Omega_*$, by the formula

$$\mathbb{P}_x(\Lambda(\mathbf{i})) = p_{i_1}(x)p_{i_2}(S_{i_1}(x))\dots p_{i_n}(S_{i_{n-1}} \circ \dots \circ S_{i_1}(x)). \tag{2.13}$$

It is clear that the above formula defines the unique probability measure for realization of the Markov process starting from the point x for the given IFS $(S,p)_N$ (see [2]).

For convenience, in what follows we write $\mathbb{P}_x(\mathbf{i})$ instead of $\mathbb{P}_x(\Lambda(\mathbf{i}))$ and $\mathbb{P}_x(A)$ instead of $\mathbb{P}_x(\Lambda(A))$, where $A \subset \Omega_*$ and $\Lambda(A) = \bigcup_{\mathbf{i}\in A} \Lambda(\mathbf{i})$. Moreover, for $\mathbf{i} \in \Omega_n$ we write

$$S_{\mathbf{i}} = S_{i_n} \circ \dots \circ S_{i_1}.$$

PROPOSITION 2.2. *For every* $\mathbf{i} = (i_1,\dots,i_n) \in \Omega_*$, *we have*

$$\mathbb{P}_x(\mathbf{i}) = p_{i_1}(x)\mathbb{P}_{S_{i_1}(x)}((i_2,\dots,i_n)), \tag{2.14}$$

$$\mathbb{P}_x(\mathbf{i}) = p_{i_n}(S_{i_{n-1}} \circ \dots \circ S_1(x))\mathbb{P}_x((i_1,\dots,i_{n-1})), \tag{2.15}$$

$$\sum_{i=1}^N \mathbb{P}_x((\mathbf{i},i)) = \mathbb{P}_x(\mathbf{i}), \tag{2.16}$$

$$\mathbb{P}_x(\mathbf{i}|k) \ge \mathbb{P}_x(\mathbf{i}|m) \quad \text{if } k \le m \le n. \tag{2.17}$$

Proof. Follows immediately from the definition of $\mathbb{P}_x$.

3. Auxiliary results. Throughout this section we assume that an IFS $(S,p)_N$ is given and $\mathbb{P}_x$ is the corresponding probability measure on Ω given by (2.13). Using a standard martingale argument we prove the following

Lemma 3.1. *Assume that an IFS $(S,p)_N$ satisfies conditions* (2.3)–(2.7). *Let $f_i : X \to \mathbb{R}_+$, $i = 1, \ldots, N$, be bounded continuous functions such that*

$$\min_{1\le i\le N} \inf_{x\in X} f_i(x) > 0.$$

Then for every $x \in X$ there exists a measurable set $\Omega_x \subset \Omega$ with $\mathbb{P}_x(\Omega_x) = 1$ such that, for all $(i_1, i_2, \ldots) \in \Omega_x$,

$$\limsup_{n\to\infty} \frac{1}{n} \log(f_{i_1}(x) f_{i_2}(S_{i_1}(x)) \ldots f_{i_n}(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x))) \le \log \Delta, \tag{3.1}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log(f_{i_1}(x) f_{i_2}(S_{i_1}(x)) \ldots f_{i_n}(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x))) \ge \log \delta \tag{3.2}$$

where

$$\Delta = \sup_{x\in X} \prod_{i=1}^{N} f_i(x)^{p_i(x)}, \tag{3.3}$$

$$\delta = \inf_{x\in X} \prod_{i=1}^{N} f_i(x)^{p_i(x)}. \tag{3.4}$$

Proof. To prove (3.1), fix $x \in X$ and for each $n \in \mathbb{N}$ define $X_n : \Omega \to \mathbb{R}$ by

$$X_n(\mathbf{i}) = \log(f_{i_n}(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x))).$$

For $\mathbf{i} = (i_1, \ldots, i_n) \in \Omega_*$, we denote by $\mathcal{A}(\mathbf{i})$ the σ-algebra generated by the cylinders $\{\Lambda(\mathbf{j}) : \mathbf{j} \in \Omega_*,\ \mathbf{j} \ge \mathbf{i}\}$. Moreover, let $\mathbb{E}_x$ denote the expectation with respect to the probability measure $\mathbb{P}_x$ on Ω.

Fix $\mathbf{i} = (i_1, i_2, \ldots) \in \Omega$. We have

$$\mathbb{E}_x(X_n \,|\, \mathcal{A}(i_1, \ldots, i_{n-1})) = \sum_{i=1}^{N} p_i(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x)) X_n((i_1, \ldots, i_{n-1}, i)).$$

By (3.3) we have

$$\sum_{i=1}^{N} p_i(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x)) \log(f_i(S_{i_{n-1}} \circ \ldots \circ S_{i_1}(x))) \le \log \Delta.$$

Now let $Y_n = X_n - \mathbb{E}_x(X_n \,|\, \mathcal{A}(i_1, \ldots, i_{n-1}))$. Then

$$|Y_n(\mathbf{i})| \le 2 \sup_{\mathbf{i}\in\Omega} |X_n(\mathbf{i})| \quad \mathbb{P}_x\text{-a.s.}$$

Write

$$M = 2 \sup_{\mathbf{i}\in\Omega} |X_n(\mathbf{i})| < \infty.$$

Define

$$Z_n = \sum_{k=1}^{n} \frac{Y_k}{k} \quad \text{for } n \in \mathbb{N}.$$

It is easy to see that $(Z_n)_{n\ge 1}$ is a martingale. Since Y_k and Y_l for $k \neq l$ are mutually orthogonal, we have

$$\mathbb{E}_x(Z_n^2) \le M^2 \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Hence $(Z_n)_{n\ge 1}$ is an $\mathbb{L}^2$-bounded martingale, and so $(Z_n)_{n\ge 1}$ is convergent a.s. Then by Kronecker's lemma (see [7])

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k = 0 \qquad \mathbb{P}_x\text{-a.s.}$$

Thus

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} X_k - \log \varDelta \le 0 \qquad \mathbb{P}_x\text{-a.s.},$$

whence (3.1) follows immediately.

Replacing f_i with $1/f_i$ and using the same argument gives (3.2). ■

A finite set $\mathcal{L} \subset \varOmega_*$ is called *fundamental* for the IFS $(S,p)_N$ if

$$\sum_{\mathbf{i}\in\mathcal{L}} \mathbb{P}_x(\mathbf{i}) = 1 \quad \text{ for every } x \in X \tag{3.5}$$

and there are no $\mathbf{i}, \mathbf{j} \in \mathcal{L}$ such that $\mathbf{i} < \mathbf{j}$. Set

$$|\mathcal{L}| = \max\{|\mathbf{i}| : \mathbf{i} \in \mathcal{L}\}.$$

LEMMA 3.2. *Let $\mathcal{L} \subset \varOmega_*$ be a fundamental set for the IFS $(S,p)_N$. If $\mathbf{i} = (i_1,\dots,i_n) \in \mathcal{L}$ and $n = |\mathcal{L}|$, then $(i_1,\dots,i_{n-1},i) \in \mathcal{L}$ for every $i \in \{1,\dots,N\}$.*

Proof. First observe that $\varLambda(\mathbf{i}) \cap \varLambda(\mathbf{j}) = \emptyset$ for every $\mathbf{i},\mathbf{j} \in \mathcal{L}$, $\mathbf{i} \neq \mathbf{j}$. Now, suppose for a contradiction that there is $\mathbf{i} = (i_1,\dots,i_n) \in \mathcal{L}$ such that $(i_1,\dots,i_{n-1},i) \notin \mathcal{L}$ for some $i \in \{1,\dots,N\}$. It is easy to verify that $\varLambda(i_1,\dots,i_{n-1},i) \cap \varLambda(\mathbf{j}) = \emptyset$ for every $\mathbf{j} \in \mathcal{L}$. Since $\mathbb{P}_x$ is a probability measure and $\mathbb{P}_x(\mathbf{i}) > 0$ for every $\mathbf{i} \in \varOmega_*$, we have

$$\sum_{\mathbf{i}\in\mathcal{L}} \mathbb{P}_x(\mathbf{i}) \le 1 - \mathbb{P}_x((i_1,\dots,i_{n-1},i)) < 1,$$

which contradicts (3.5). ■

REMARK 3.1. Note that for every $n \in \mathbb{N}$ there exists a fundamental set $\mathcal{L}$ for $(S,p)_N$ such that $\mathcal{L} \subset \varOmega_{\le n}$.

LEMMA 3.3. *Assume that an IFS $(S,p)_N$ satisfies conditions* (2.3)–(2.7). *Let $\mu_* \in \mathcal{M}_1$ be its unique invariant measure. Then for every fundamental* *set* $\mathcal{L} \subset \Omega_*$ *we have*

$$\mu_*(A) = \sum_{\mathbf{i}\in\mathcal{L}} \int_X \mathbb{P}_x(\mathbf{i}^{-1}) 1_A(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx) \quad \textit{for } A \in \mathcal{B}(X). \tag{3.6}$$

Proof. We use induction on n, where $\mathcal{L} \subset \Omega_{\le n}$.

Suppose first that $\mathcal{L} \subset \Omega_1$. Since $p_i(x) > 0$ for $x \in X$ and $i = 1, \dots, N$, it follows immediately that $\mathcal{L} = \{1, \dots, N\}$ and (3.6) is obviously satisfied.

Now suppose that (3.6) holds for every $\mathcal{L} \subset \Omega_{\le n}$, and take $\mathcal{L} \subset \Omega_{\le n+1}$. Using the invariance of μ_* and (2.2), for $f \in B(X)$ we have

$$\begin{aligned} \int_X f(x)\mu_*(dx) &= \int_X f(x)\, P\mu_*(dx) = \int_X Uf(x)\, \mu_*(dx) \\ &= \sum_{i=1}^N \int_X p_i(x) f(S_i(x))\, \mu_*(dx). \end{aligned} \tag{3.7}$$

Set

$$\mathcal{L}_{n+1} = \{\mathbf{i} \in \mathcal{L} : |\mathbf{i}| = n+1\}, \qquad \mathcal{L}_{n+1}^n = \{\mathbf{i}|n : \mathbf{i} \in \mathcal{L}_{n+1}\}. \tag{3.8}$$

We assume that $\mathcal{L}_{n+1} \neq \emptyset$ (otherwise there is nothing to prove). Let $A \in \mathcal{B}(X)$. Using in succession Lemma 3.2, formula (2.14) and (3.7) we have

$$\begin{aligned} &\sum_{\mathbf{i}\in\mathcal{L}_{n+1}} \int_X \mathbb{P}_x(\mathbf{i}^{-1}) 1_A(S_{\mathbf{i}^{-1}}(x))\mu_*(dx) \\ &\quad = \sum_{\mathbf{j}\in\mathcal{L}_{n+1}^n} \sum_{i=1}^N \int_X \mathbb{P}_x((\mathbf{j}, i)^{-1}) 1_A(S_{(\mathbf{j},i)^{-1}}(x))\, \mu_*(dx) \\ &\quad = \sum_{\mathbf{j}\in\mathcal{L}_{n+1}^n} \sum_{i=1}^N \int_X p_i(x) \mathbb{P}_{S_i(x)}(\mathbf{j}^{-1}) 1_A(S_{\mathbf{j}^{-1}}(S_i(x)))\, \mu_*(dx) \\ &\quad = \sum_{\mathbf{j}\in\mathcal{L}_{n+1}^n} \int_X \mathbb{P}_x(\mathbf{j}^{-1}) \mathbf{1}_A(S_{\mathbf{j}^{-1}}(x))\, \mu_*(dx). \end{aligned} \tag{3.9}$$

Now setting $\mathcal{L}^* = (\mathcal{L} \setminus \mathcal{L}_{n+1}) \cup \mathcal{L}_{n+1}^n$ and using (3.9) we obtain

$$\begin{aligned} &\sum_{\mathbf{i}\in\mathcal{L}} \int_X \mathbb{P}_x(\mathbf{i}^{-1}) 1_A(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx) \\ &\quad = \sum_{\mathbf{i}\in\mathcal{L}_{n+1}} \int_X \mathbb{P}_x(\mathbf{i}^{-1}) 1_A(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx) \\ &\qquad + \sum_{\mathbf{i}\in\mathcal{L}\setminus\mathcal{L}_{n+1}} \int_X \mathbb{P}_x(\mathbf{i}^{-1}) 1_A(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx) \\ &\quad = \sum_{\mathbf{i}\in\mathcal{L}^*} \int_X \mathbb{P}_x(\mathbf{i}^{-1}) 1_A(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx). \end{aligned} \tag{3.10}$$

Clearly $\mathcal{L}^*$ is fundamental and $\mathcal{L}^* \subset \Omega_{\le n}$. Now apply the induction hypothesis. ■

Let L_i, $i = 1, \ldots, N$, be Lipschitz constants of S_i and let Γ_0 be given by (2.4). For $\mathbf{i} = (i_1, \ldots, i_k) \in \Omega_*$, we write

$$L_{\mathbf{i}} = L_{i_1} \ldots L_{i_k}. \tag{3.11}$$

For $\Gamma > \Gamma_0$ and $n_0, n \in \mathbb{N}$, $n \ge n_0$ we define

$$Q^n_{n_0}(\Gamma) = \{\mathbf{i} \in \Omega_{\ge n} : L_{\mathbf{i}|k} \le \Gamma^k \text{ for } n_0 \le k \le n\}. \tag{3.12}$$

LEMMA 3.4. *Let F be a bounded subset of X. Then for every $\Gamma \in (\Gamma_0, 1)$ and $n_0 \in \mathbb{N}$ there exists $\alpha > 0$ such that*

$$\mathbb{P}_x(\mathbf{i}) \ge \alpha \mathbb{P}_y(\mathbf{i}) \quad \textit{for all } \mathbf{i} \in Q^n_{n_0}(\Gamma) \cap \Omega_n,\ n \ge n_0 \textit{ and } x, y \in F.$$

Proof. Fix $\Gamma \in (\Gamma_0, 1)$ and let $d = \operatorname{diam} F$. Let ω be as in (2.6), (2.7). Set

$$\omega_0 = \sum_{k=1}^{\infty} \omega(\Gamma^k d).$$

Clearly $\omega_0 < \infty$. Fix $n_0 \in \mathbb{N}$. Let $n \ge n_0$ and $x, y \in F$. For $\mathbf{i} \in Q^n_{n_0}(\Gamma) \cap \Omega_n$ we have

$$\begin{aligned}
\mathbb{P}_y(\mathbf{i}) &= p_{i_1}(y) p_{i_2}(S_{i_1}(y)) \ldots p_{i_n}(S_{i_{n-1}} \ldots S_{i_1}(y)) \\
&= \frac{p_{i_1}(y) \ldots p_{i_{n_0}}(S_{i_{n_0-1}} \circ \ldots \circ S_{i_1}(y))}{p_{i_1}(x) \ldots p_{i_{n_0}}(S_{i_{n_0-1}} \circ \ldots \circ S_{i_1}(x))} \\
&\quad \times p_{i_1}(x) \ldots p_{i_{n_0}}(S_{i_{n_0-1}} \circ \ldots \circ S_{i_1}(x)) \\
&\quad \times \prod_{k=n_0+1}^{n} \Bigg[\left(1 + \frac{p_{i_k}(S_{i_{k-1}} \circ \ldots \circ S_{i_1}(y)) - p_{i_k}(S_{i_{k-1}} \circ \ldots \circ S_{i_1}(x))}{p_{i_k}(S_{i_{k-1}} \circ \ldots \circ S_{i_1}(x))} \right) \\
&\qquad\qquad \times p_{i_k}(S_{i_{k-1}} \circ \ldots \circ S_{i_1}(x)) \Bigg].
\end{aligned}$$

Using the inequality $p_i(x) \ge \alpha_0$, conditions (2.6), (2.3) and definition (3.12) we obtain

$$\begin{aligned}
\mathbb{P}_y(\mathbf{i}) &\le \frac{(1-\alpha_0)^{n_0}}{\alpha_0^{n_0}} \\
&\quad \times \prod_{k=n_0+1}^{n} \left(1 + \frac{\omega(\varrho(S_{i_{k-1}} \circ \ldots \circ S_{i_1}(x), S_{i_{k-1}} \circ \ldots \circ S_{i_1}(y)))}{\alpha_0} \right) \mathbb{P}_x(\mathbf{i}) \\
&\le \left(\frac{1-\alpha_0}{\alpha_0} \right)^{n_0} \prod_{k=n_0+1}^{n} \left(1 + \frac{\omega(\Gamma^{k-1} d)}{\alpha_0} \right) \mathbb{P}_x(\mathbf{i}).
\end{aligned}$$

Consequently,

$$\mathbb{P}_y(\mathbf{i}) \le \left(\frac{1-\alpha_0}{\alpha_0}\right)^{n_0} \prod_{k=n_0+1}^{\infty} e^{\omega(\Gamma^{k-1}d)/\alpha_0} \mathbb{P}_x(\mathbf{i}) = \left(\frac{1-\alpha_0}{\alpha_0}\right)^{n_0} e^{\omega_0/\alpha_0} \mathbb{P}_x(\mathbf{i}).$$

Setting $\alpha = \alpha_0^{n_0}(1-\alpha_0)^{-n_0}e^{-\omega_0/\alpha_0}$ we complete the proof. ∎

From now on assume that the constants $l_i \in (0,1)$, $i = 1,\dots,N$, satisfy (2.3). Let γ_0 be given by (2.10). For $\gamma \in (0,\gamma_0)$ and $n \in \mathbb{N}$ define

$$J_n(\gamma) = \{i \in \{1,\dots,N\} : l_i \le \gamma^n\} \cup \{\mathbf{i} \in \Omega_* : |\mathbf{i}| > 1 \text{ and } l_{\mathbf{i}} \le \gamma^n < l_{\mathbf{i}||\mathbf{i}|-1}\},$$

where $l_{\mathbf{i}}$ is given by (3.11).

Lemma 3.5. *For every $\gamma \in (0,\gamma_0)$ and $n \in \mathbb{N}$ the set $J_n(\gamma)$ is fundamental for $(S,p)_N$.*

Proof. Fix $\gamma \in (0,\gamma_0)$ and $n \in \mathbb{N}$. It is easy to verify that $J_n(\gamma) \subset \Omega_{\le m}$, where m is the least integer such that $(\max_{1\le i\le N} l_i)^m \le \gamma^n$. Consequently, $J_n(\gamma)$ is finite. Moreover, from the definition of $J_n(\gamma)$ it follows that if $\mathbf{i} \in J_n(\gamma)$, $\mathbf{j} \in \Omega_*$ and $\mathbf{j} > \mathbf{i}$, then $\mathbf{j} \notin J_n(\gamma)$. This implies that

$$\Lambda(\mathbf{i}) \cap \Lambda(\mathbf{j}) = \emptyset \quad \text{for } \mathbf{i},\mathbf{j} \in J_n(\gamma),\ \mathbf{i} \ne \mathbf{j}. \tag{3.13}$$

Finally, observe that for every $\mathbf{i} \in \Omega$ there is $k \in \mathbb{N}$ such that $\mathbf{i}|k \in J_n(\gamma)$. Consequently,

$$\Omega = \bigcup_{\mathbf{i}\in J_n(\gamma)} \Lambda(\mathbf{i}). \tag{3.14}$$

By (3.13) and (3.14) for all $x \in X$ we have

$$\sum_{\mathbf{i}\in J_n(\gamma)} \mathbb{P}_x(\mathbf{i}) = \mathbb{P}_x\Big(\bigcup_{\mathbf{i}\in J_n(\gamma)} \Lambda(\mathbf{i})\Big) = \mathbb{P}_x(\Omega) = 1. \ \blacksquare$$

Lemma 3.6. *Assume that a family $S_1,\dots,S_N$ satisfies the strong Moran condition and condition* (2.3) *with $l_i \in (0,1)$. Then for every $\gamma \in (0,\gamma_0)$, $n \in \mathbb{N}$ and $\mathbf{i},\mathbf{j} \in J_n(\gamma)$, $\mathbf{i} \ne \mathbf{j}$, we have*

$$\operatorname{dist}(S_{\mathbf{i}^{-1}}(F), S_{\mathbf{j}^{-1}}(F)) \ge \gamma^n\sigma, \tag{3.15}$$

where the set F and the constant σ satisfy (2.11), (2.12).

Proof. Fix $\gamma \in (0,\gamma_0)$, $n \in \mathbb{N}$ and $\mathbf{i},\mathbf{j} \in J_n(\gamma)$, $\mathbf{i} \ne \mathbf{j}$. Suppose $\mathbf{i} = (i_1,\dots,i_p)$ and $\mathbf{j} = (j_1,\dots,j_q)$. Since $J_n(\gamma)$ is fundamental, there exists an integer $m \le \min\{p,q\}$ such that $i_m \ne j_m$, but $i_k = j_k$ for $k < m$. From the strong Moran condition it follows immediately that $S_{\mathbf{i}^{-1}}(F) \subset S_{i_1}\circ\ldots\circ S_{i_m}(F)$, $S_{\mathbf{j}^{-1}}(F) \subset S_{j_1}\circ\ldots\circ S_{j_m}(F)$ and $\operatorname{dist}(S_{i_m}(F), S_{j_m}(F)) \ge \sigma$. Consequently,

$$\begin{aligned}\operatorname{dist}(S_{\mathbf{i}^{-1}}(F), S_{\mathbf{j}^{-1}}(F)) &\ge \operatorname{dist}(S_{i_1}\circ\ldots\circ S_{i_m}(F), S_{j_1}\circ\ldots\circ S_{j_m}(F))\\ &\ge l_{i_1}\ldots l_{i_m-1}\sigma \ge \gamma^n\sigma. \ \blacksquare\end{aligned}$$

Lemma 3.7. *Assume that an IFS (S, p_N) satisfies* (2.3) *with $l_i \in (0,1)$, $i = 1, \ldots, N$. Let γ_0 be given by* (2.10). *Then for every $\gamma \in (0, \gamma_0)$ there is $\beta > 0$ such that*

$$\sum_{\mathbf{i} \in J_n^*(\gamma)} \mathbb{P}_x(\mathbf{i}) \geq \beta \quad \textit{for every } x \in X,$$

where

$$J_n^*(\gamma) = \{\mathbf{i} \in J_n(\gamma) : \mathbf{i}^{-1} \in J_n(\gamma)\}. \tag{3.16}$$

Proof. Without any loss of generality we can assume that

$$l_1 \leq l_i \leq l_N \quad \text{for } i = 1, \ldots, N.$$

Observe that if $\mathbf{i} = (i_1, \ldots, i_k) \in J_n(\gamma)$ and $i_k = N$ then $\mathbf{i}^{-1} \in J_n(\gamma)$. Moreover, for every $\mathbf{i} = (i_1, \ldots, i_k) \in J_n(\gamma)$ there exists a unique element

$$\tau(\mathbf{i}) = (i_1, \ldots, i_{k-1}, N, \ldots, N) \tag{3.17}$$

which belongs to $J_n(\gamma)$. In this way formula (3.17) defines a one-to-one map from $J_n(\gamma)$ into $J_n(\gamma)$. Note also that

$$(\tau(\mathbf{i}))^{-1} \in J_n(\gamma) \quad \text{for every } \mathbf{i} \in J_n(\gamma). \tag{3.18}$$

Fix $\mathbf{i} = (i_1, \ldots, i_k) \in J_n(\gamma)$ and let m_0 be such that $l_N^{m_0} \leq l_1$. Since $l_{\mathbf{i}} \leq \gamma^n$ and $l_1 \leq l_i$ for $i = 1, \ldots, N$, we have

$$|\tau(\mathbf{i})| \leq |\mathbf{i}| - 1 + m_0,$$

which means that in (3.17) the number N appears at most m_0 times. Now it is easy to see that for every $\mathbf{i} \in J_n(\gamma)$,

$$\operatorname{card}\{\mathbf{j} \in J_n(\gamma) : \tau(\mathbf{j}) = \tau(\mathbf{i})\} \leq N^{m_0} \tag{3.19}$$

(card stands for cardinality). By (3.17), (2.15) and (2.17) for every $x \in X$ we have

$$\begin{aligned} \mathbb{P}_x(\tau(\mathbf{i})) &= \mathbb{P}_x(i_1, \ldots, i_{k-1}, N, \ldots, N) \\ &\geq \mathbb{P}_x(i_1, \ldots, i_{k-1})\alpha_0^{m_0} \geq \mathbb{P}_x(\mathbf{i})\alpha_0^{m_0}, \end{aligned} \tag{3.20}$$

where α_0 is given by (2.5). By Lemma 3.5 and (3.5), (3.20) and (3.19) we have

$$1 = \sum_{\mathbf{i} \in J_n(\gamma)} \mathbb{P}_x(\mathbf{i}) \leq \alpha_0^{-m_0} \sum_{\mathbf{i} \in J_n(\gamma)} \mathbb{P}_x(\tau(\mathbf{i})) \leq N^{m_0}\alpha_0^{-m_0} \sum_{\mathbf{i} \in J_n^*(\gamma)} \mathbb{P}_x(\mathbf{i}).$$

Setting $\beta = (\alpha_0/N)^{m_0}$ completes the proof. ∎

4. Upper estimate of capacity

THEOREM 4.1. *Assume that an IFS* $(S,p)_N$ *satisfies* (2.3)–(2.7) *and let* $\mu_* \in \mathcal{M}_1$ *be the corresponding invariant measure. Then*

$$\overline{\mathrm{Cap}}_L(\mu_*) \le \frac{\log \delta_0}{\log \Gamma_0}, \tag{4.1}$$

where Γ_0 *and* δ_0 *are given by* (2.4) *and* (2.9), *respectively.*

Proof. Fix $\eta > 0$. Let μ_* be the unique invariant probability measure for the IFS $(S,p)_N$ and let K be a compact subset of X such that $\mu_*(K) \ge 1-\eta/4$. Set $d = \operatorname{diam} K$. Fix $x_0 \in K$. By Lemma 3.1 (with L_i and $p_i(x)$ in place of $f_i(x)$ and Γ_0 and δ_0 in place of Δ and δ, respectively) there exists a measurable set $\Omega_0 \subset \Omega$ with $\mathbb{P}_{x_0}(\Omega_0) = 1$ such that, for all $\mathbf{i} \in \Omega_0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log(L_{\mathbf{i}|n}) \le \log \Gamma_0, \tag{4.2}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log(\mathbb{P}_{x_0}(\mathbf{i}|n)) \ge \log \delta_0, \tag{4.3}$$

where $L_{\mathbf{i}|n} = L_{i_1}\ldots L_{i_n}$. By Lemma 3 of [8] the measure $\mathbb{P}_x$, $x \in X$, is absolutely continuous with respect to $\mathbb{P}_{x_0}$. Thus $\mathbb{P}_x(\Omega_0) = 1$ for $x \in X$.

Choose $\Gamma \in (\Gamma_0, 1)$, $\delta \in (0,\delta_0)$ and define the sequence $(\Omega_0(n))_{n\ge1}$ of measurable subsets of Ω_0 by

$$\Omega_0(n) = \{\mathbf{i} \in \Omega_0 : \mathbb{P}_{x_0}(\mathbf{i}|k) \ge \delta^k \text{ and } L_{\mathbf{i}|k} \le \Gamma^k \text{ for } k \ge n\}.$$

Obviously $\Omega_0(n) \subset \Omega_0(n+1)$ for $n \in \mathbb{N}$. Moreover, from (4.2), (4.3) and the choice of Γ and δ it follows that $\Omega_0 = \bigcup_{n=1}^{\infty} \Omega_0(n)$. Consequently,

$$\lim_{n\to\infty} \mathbb{P}_x(\Omega_0(n)) = 1 \quad \text{for } x \in X. \tag{4.4}$$

By Lemma 4.1 of [24] the function $x \mapsto \mathbb{P}_x(\Omega_0(n))$ is Borel measurable for each $n \in \mathbb{N}$. By (4.4) the sequence $(\mathbb{P}_x(\Omega_0(n)))_{n\ge1}$ is a.s. convergent and so convergent with respect to the measure μ_*. Hence there exists $n_0 \in \mathbb{N}$ such that

$$\mu_*(\{x \in K : \mathbb{P}_x(\Omega_0(n)) > 1-\eta/2\}) \ge 1-\eta/2 \quad \text{for } n \ge n_0. \tag{4.5}$$

By the invariance of μ_*, for all $n \in \mathbb{N}$ and $A \in \mathcal{B}(X)$ we have

$$\mu_*(A) = P^n\mu_*(A) = \sum_{\mathbf{i}\in\Omega_n} \int_X \mathbb{P}_x(\mathbf{i}) 1_A(S_{\mathbf{i}}(x))\, \mu_*(dx). \tag{4.6}$$

Now, for $n \in \mathbb{N}$ define

$$D_n = \bigcup_{\mathbf{i}\in\Omega_0^n(n)} B(S_{\mathbf{i}}(x_0), \varepsilon_n),$$

where

$$\Omega_0^n(n) = \{\mathbf{i}|n : \mathbf{i} \in \Omega_0(n)\} \quad \text{and} \quad \varepsilon_n = \Gamma^n \operatorname{diam} K.$$

Observe that

$$(4.7)\qquad \varrho(S_{\mathbf{i}}(x), S_{\mathbf{i}}(x_0)) \le \varepsilon_n \quad \text{for } \mathbf{i} \in \Omega_0^n(n) \text{ and } x \in K.$$

Using (4.5)–(4.7) and the inclusion $\Omega_0(n) \subset \Lambda(\Omega_0^n(n))$, for every $n \ge n_0$ we have

$$\begin{aligned}
\mu_*(D_n) &= \sum_{\mathbf{i}\in\Omega_n} \int_X \mathbb{P}_x(\mathbf{i}) 1_{D_n}(S_{\mathbf{i}}(x))\, \mu_*(dx) \\
&\ge \sum_{\mathbf{i}\in\Omega_0^n(n)} \int_X \mathbb{P}_x(\mathbf{i}) 1_{D_n}(S_{\mathbf{i}}(x))\, \mu_*(dx) \\
&\ge \int_K \sum_{\mathbf{i}\in\Omega_0^n(n)} \mathbb{P}_x(\mathbf{i})\, \mu_*(dx) = \int_K \mathbb{P}_x(\Omega_0^n(n))\, \mu_*(dx) \\
&\ge \int_K \mathbb{P}_x(\Omega_0(n))\, \mu_*(dx) \ge (1-\eta/2)(1-\eta/2) > 1-\eta.
\end{aligned}$$

From this inequality and the definition of D_n it follows that $N(\varepsilon_n, \eta) \le N_0$, where $N_0 = \operatorname{card} \Omega_0^n(n)$ and $N(\varepsilon_n, \eta)$ comes from the definition of capacity. Since $\mathbb{P}_{x_0}(\mathbf{i}) \ge \delta^n$ for $\mathbf{i} \in \Omega_0^n(n)$ and $\sum_{\mathbf{i}\in\Omega_0^n(n)} \mathbb{P}_{x_0}(\mathbf{i}) \le 1$, we have $N_0\delta^n \le 1$. Consequently, $N(\varepsilon_n, \eta) \le \delta^{-n}$. By Remark 2.1 we now have

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon,\eta)}{-\log \varepsilon} &= \limsup_{n\to\infty} \frac{\log N(\varepsilon_n,\eta)}{-\log \varepsilon_n} \\
&\le \limsup_{n\to\infty} \frac{\log \delta^{-n}}{-\log(\Gamma^n \operatorname{diam} K)} = \frac{\log \delta}{\log \Gamma}.
\end{aligned}$$

Letting $\delta \to \delta_0$ and $\Gamma \to \Gamma_0$ we conclude that

$$\limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon,\eta)}{-\log \varepsilon} \le \frac{\log \delta_0}{\log \Gamma_0}.$$

Since $\eta > 0$ was arbitrary, the proof is complete. ■

5. Lower estimate of capacity

THEOREM 5.1. *Assume that an IFS* $(S,p)_N$ *satisfies* (2.3)–(2.7) *and* $\mu_* \in \mathcal{M}_1$ *is the corresponding unique invariant measure. Moreover, assume that the functions* $S_1, \dots, S_N$ *satisfy the strong Moran condition. Then*

$$(5.1)\qquad \underline{\operatorname{Cap}}_L(\mu_*) \ge \frac{\log \Delta_0}{\log \gamma_0},$$

where Δ_0 *and* γ_0 *are given by* (2.8) *and* (2.9), *respectively.*

Proof. Consider first the case $l_i < 1$, $i = 1, \dots, N$, where the l_i satisfy (2.3). Let F be a closed set satisfying (2.11) and (2.12). Since F is invariant for $S_1, \dots, S_N$, we have $\operatorname{supp} \mu_* \subset F$. Choose $x_0 \in F$. By Lemma 3.1

(with p_i and Δ_0 or L_i and Γ_0 in place of f_i and Δ, and l_i and γ in place of f_i and γ_0, respectively) we have

$$\limsup_{n\to\infty} \frac{1}{n} \log(\mathbb{P}_{x_0}(\mathbf{i}|n)) \le \log \Delta_0 \quad \mathbb{P}_{x_0}\text{-a.s.,} \tag{5.2}$$

$$\limsup_{n\to\infty} \frac{1}{n} \log(L_{\mathbf{i}|n}) \le \log \Gamma_0 \quad \mathbb{P}_{x_0}\text{-a.s.,} \tag{5.3}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log(l_{\mathbf{i}|n}) \ge \log \gamma_0 \quad \mathbb{P}_{x_0}\text{-a.s.} \tag{5.4}$$

Fix $\gamma \in (0, \gamma_0)$, $\Gamma \in (\Gamma_0, 1)$ and $\Delta \in (\Delta_0, 1)$. Let $n_0 \in \mathbb{N}$ and let α be as in Lemma 3.4. By (5.2)–(5.4) there exists $n_1 \ge n_0$ such that

$$\mathbb{P}_{x_0}(\{\mathbf{i} \in \Omega : \mathbb{P}_{x_0}(\mathbf{i}|n) \le \Delta^n \text{ for } n \ge n_1\}) \ge 1 - \beta/6, \tag{5.5}$$

$$\mathbb{P}_{x_0}(\{\mathbf{i} \in \Omega : L_{\mathbf{i}|n} \le \Gamma^n \text{ for } n \ge n_1\}) \ge 1 - \beta/6, \tag{5.6}$$

$$\mathbb{P}_{x_0}(\{\mathbf{i} \in \Omega : l_{\mathbf{i}|n} \ge \gamma^n \text{ for } n \ge n_1\}) \ge 1 - \beta/6, \tag{5.7}$$

where β is as in Lemma 3.7. Now choose $n_* \in \mathbb{N}$ such that

$$\min\{|\mathbf{i}| : \mathbf{i} \in J_n(\gamma)\} \ge n_1 \quad \text{for } n \ge n_*. \tag{5.8}$$

For $n \ge n_*$ define

$$J_n^0(\gamma) = \{\mathbf{i} \in J_n(\gamma) : \mathbb{P}_{x_0}(\mathbf{i}^{-1}|k) \le \Delta^k,\ L_{\mathbf{i}^{-1}|k} \le \Gamma^k \\ \text{and } l_{\mathbf{i}^{-1}|k} \ge \gamma^k \text{ for } k \in \mathbb{N},\ n_1 \le k \le |\mathbf{i}|\}.$$

By (5.5)–(5.8) and Lemma 3.7 we have

$$\sum_{\mathbf{i}\in J_n^0(\gamma)} \mathbb{P}_{x_0}(\mathbf{i}^{-1}) \ge \beta/2 \quad \text{for } n \ge n_1. \tag{5.9}$$

Since $\mathbf{i}^{-1} \in Q_{n_1}^n(\gamma)$ for $\mathbf{i} \in J_n^0(\gamma)$, Lemma 3.4 yields

$$\alpha^{-1}\mathbb{P}_{x_0}(\mathbf{i}^{-1}) \ge \mathbb{P}_x(\mathbf{i}^{-1}) \ge \alpha\mathbb{P}_{x_0}(\mathbf{i}^{-1}) \quad \text{for } \mathbf{i} \in J_n^0(\gamma) \text{ and } x \in F. \tag{5.10}$$

From (5.9) and (5.10) it follows that

$$\sum_{\mathbf{i}\in J_n^0(\gamma)} \mathbb{P}_x(\mathbf{i}^{-1}) \ge \alpha\beta/2, \tag{5.11}$$

$$\mathbb{P}_x(\mathbf{i}^{-1}) \le \alpha^{-1}\Delta^{|\mathbf{i}|}, \tag{5.12}$$

for all $n \ge n_*$, $x \in F$ and $\mathbf{i} \in J_n^0(\gamma)$. Since for $\mathbf{i} \in J_n^0(\gamma)$ we have $l_{\mathbf{i}} \le \gamma^n$ and $l_{\mathbf{i}^{-1}|k} \ge \gamma^k$ for $n_1 \le k \le |\mathbf{i}|$, it follows that $|\mathbf{i}| \ge n$. Hence

$$\mathbb{P}_x(\mathbf{i}^{-1}) \le \alpha^{-1}\Delta^n \quad \text{for } \mathbf{i} \in J_n^0(\gamma). \tag{5.13}$$

For $n \ge n_*$ define

$$D_n = \bigcup_{\mathbf{i}\in J_n^0(\gamma)} S_{\mathbf{i}^{-1}}(F). \tag{5.14}$$

Since by Lemma 3.5 the set $J_n(\gamma)$ is fundamental, Lemma 3.3 and (5.11) show that

$$\mu_*(D_n) = \sum_{\mathbf{i}\in J_n(\gamma)} \int_X \mathbb{P}_x(\mathbf{i}^{-1}) 1_{D_n}(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx) \tag{5.15}$$
$$\geq \int_F \sum_{\mathbf{i}\in J_n^0(\gamma)} \mathbb{P}_x(\mathbf{i}^{-1}) 1_{D_n}(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx)$$
$$= \int_F \sum_{\mathbf{i}\in J_n^0(\gamma)} \mathbb{P}_x(\mathbf{i}^{-1})\, \mu_*(dx) \geq \alpha\beta/2$$

for $n \geq n_*$.

By Lemmas 3.3, 3.5 and 3.6, the inclusion $\operatorname{supp} \mu_* \subset F$ and inequality (5.13), for every $\mathbf{j} \in J_n^0(\gamma)$ and $n \geq n_*$, we have

$$\mu_*(S_{\mathbf{j}^{-1}}(F)) = \sum_{\mathbf{i}\in J_n(\gamma)} \int_X \mathbb{P}_x(\mathbf{i}^{-1}) 1_{S_{\mathbf{j}^{-1}}(F)}(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx)$$
$$= \sum_{\mathbf{i}\in J_n^0(\gamma)} \int_F \mathbb{P}_x(\mathbf{i}^{-1}) 1_{S_{\mathbf{j}^{-1}}(F)}(S_{\mathbf{i}^{-1}}(x))\, \mu_*(dx)$$
$$= \int_F \mathbb{P}_x(\mathbf{j}^{-1})\, \mu_*(dx) \leq \Delta^n/\alpha,$$

since $S_{\mathbf{i}^{-1}}(F) \cap S_{\mathbf{j}^{-1}}(F) = \emptyset$ for $\mathbf{i} \neq \mathbf{j}$, $\mathbf{i}, \mathbf{j} \in J_n^0(\gamma)$. Define $\varepsilon_n = \gamma^n\sigma/2$ for $n \geq n_*$, where $\sigma > 0$ is given by (2.12). By (3.15) every ball B with radius ε_n meets at most one set $S_{\mathbf{i}^{-1}}(F)$ for $\mathbf{i} \in J_n^0(\gamma)$. The inclusion $\operatorname{supp} \mu_* \subset \bigcup_{\mathbf{i}\in J_n(r)} S_{\mathbf{i}^{-1}}(F)$ then implies that to cover a set of μ_*-measure greater than or equal to $1-\eta$ (with $\eta \leq \alpha\beta/2$) we need at least $\alpha(1-\eta)\Delta^{-n}$ balls with radius ε_n. Thus $N(\varepsilon_n, \eta) \geq \alpha(1-\eta)\Delta^{-n}$ for $n \geq n_*$. Consequently,

$$\liminf_{\varepsilon\to 0} \frac{\log N(\varepsilon,\eta)}{-\log \varepsilon} \geq \liminf_{n\to\infty} \frac{\log(\alpha(1-\eta)\Delta^{-n})}{-\log \varepsilon_n} = \frac{\log \Delta}{\log \gamma}.$$

Thus

$$\underline{\mathrm{Cap}}_L(\mu_*) \geq \frac{\log \Delta}{\log \gamma}$$

and letting $\Delta \to \Delta_0$ and $\gamma \to \gamma_0$ we conclude that

$$\underline{\mathrm{Cap}}_L(\mu_*) \geq \frac{\log \Delta_0}{\log \gamma_0}.$$

Suppose now that some of the l_i's are equal to 1. Choose $\bar{l}_i < l_i$, $i = 1, \ldots, N$. Since

$$\sup_{x\in X} \prod_{i=1}^N \bar{l}_i^{\,p_i(x)} \to \sup_{x\in X} \prod_{i=1}^N l_i^{p_i(x)}$$

as $\bar{l}_i \to l_i$ for $i \in \{1, \ldots, N\}$, the statement of Theorem 5.1 follows. ∎

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