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On certain subclasses of analytic functions involving a linear operator

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Abstract. A certain general class $\mathcal{S}(a,c,A,B)$ of analytic functions involving a linear operator is introduced. The objective is to investigate various properties and characteristics of this class. Several applications of the results (obtained here) to a class of fractional calculus operators are also considered. The results contain some of the earlier work in univalent function theory.

1. Introduction. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in the open unit disc $E = \{z : |z| < 1\}$. Let $\mathcal{S}$, $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ $(0 \le \alpha < 1)$ be the usual subclasses of functions in $\mathcal{A}$ that are univalent, starlike of order α and convex of order α, respectively. We note that

$$f(z) \in \mathcal{K}(\alpha) \;\Leftrightarrow\; zf'(z) \in \mathcal{S}^*(\alpha).$$

For arbitrary fixed real numbers A, B $(-1 \le B < A \le 1)$, let $\mathcal{P}(A,B)$ denote the class of functions of the form

$$\phi(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

which are analytic in E and satisfy the condition

$$\phi(z) \prec \frac{1+Az}{1+Bz} \quad (z \in E)$$

where the symbol $\prec$ stands for subordination. The class $\mathcal{P}(A,B)$ was introduced and studied by Janowski [5].

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For a function $f \in \mathcal{A}$ given by (1.1), the *generalized Bernardi integral operator* $\mathcal{F}_\delta$ is

$$(1.2)\qquad \mathcal{F}_\delta(z) = \frac{\delta+1}{z^\delta}\int_0^z t^{\delta-1} f(t)\,dt = z + \sum_{n=2}^{\infty} \frac{\delta+1}{\delta+n}\, a_n z^n \qquad (\delta > -1,\ z \in E).$$

It readily follows from (1.2) that

$$f(z) \in \mathcal{A} \ \Leftrightarrow\ \mathcal{F}_\delta(z) \in \mathcal{A}.$$

Several essentially equivalent definitions of fractional calculus have been given in the literature (cf., e.g., [12], [13], [14]). We state the following definitions due to Owa [10] which have been used rather frequently in the theory of analytic functions.

Definition 1. The *fractional integral of order* λ is defined, for a function $f(z)$, by

$$(1.3)\qquad D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(\zeta - z)^{1-\lambda}}\, d\zeta \qquad (\lambda > 0),$$

and the *fractional derivative of order* λ is defined by

$$(1.4)\qquad D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(\zeta - z)^{\lambda}}\, d\zeta \qquad (0 \le \lambda < 1),$$

where $f(z)$ is analytic in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ involved in (1.3) (and that of $z-\zeta$ involved in (1.4)) is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2. Under the hypotheses of Definition 1, the *fractional derivative of order* $n+\lambda$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n}(D_z^\lambda f(z)) \qquad (0 \le \lambda < 1,\ n \in \mathbb{N}_0 = \{0,1,\ldots\}).$$

Let

$$\phi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\, z^{n+1} \qquad (c \ne 0, -1, -2, \ldots)$$

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)(x+2)\ldots(x+n-1), & n \in \mathbb{N}_0, \\ 1, & n = 0. \end{cases}$$

We note that $\phi(a,1;z) = z/(1-z)^a$ and $\phi(2,1;z)$ is the well known Koebe function.

Corresponding to the function $\phi(a,c;z)$ and for an analytic function $f(z)$ given by (1.1), Carlson and Shaffer [4] defined a linear operator $\mathcal{L}(a,c)$ by

$$\mathcal{L}(a,c)f(z)=\phi(a,c;z)*f(z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}\,a_nz^n \tag{1.5}$$

where the symbol $*$ stands for the Hadamard product (or convolution). We see that if $a=0,-1,-2,\ldots,$ then $\mathcal{L}(a,c)f(z)$ is a polynomial. For $a\neq 0,-1,-2,\ldots,$ an application of the root test shows that the infinite series for $\mathcal{L}(a,c)f(z)$ has the same radius of convergence as that of $f(z)$ because $\lim_{n\to\infty}|(a)_n/(c)_n|^{1/n}=1$. Hence, $\mathcal{L}(a,c)$ maps $\mathcal{A}$ into itself. The Ruscheweyh derivatives [11] of $f(z)$ are $\mathcal{L}(n+1,1)f(z), n\in\mathbb{N}_0$.

We further observe that

$$\mathcal{L}(a,a)f(z)=f(z),\quad \mathcal{L}(2,1)f(z)=zf'(z),\quad \mathcal{L}(\delta+1,\delta+2)f(z)=F_\delta(z)$$

and

$$\mathcal{L}(2,2-\lambda)f(z)=\Gamma(2-\lambda)z^\lambda D_z^\lambda f(z)=J_z^\lambda f(z)\quad (0\le\lambda<1).$$

Making use of the operator $\mathcal{L}(a,c)$, we now introduce a subclass of $\mathcal{A}$ as follows:

Definition 3. A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{S}(a,c,A,B)$ if it satisfies

$$\frac{\mathcal{L}(a,c-1)f(z)}{\mathcal{L}(a,c)f(z)}\prec\frac{1+Az}{1+Bz}\quad (z\in E) \tag{1.6}$$

for some $a>0,\ c>1$, and $-1\le B<A\le 1$. By the definition of subordination, it follows that

$$\left|\frac{\mathcal{L}(a,c-1)f(z)-\mathcal{L}(a,c)f(z)}{A\,\mathcal{L}(a,c)f(z)-B\,\mathcal{L}(a,c-1)f(z)}\right|<1\quad (z\in E). \tag{1.7}$$

For convenience, we put

$$\mathcal{S}(2,2-\lambda,\beta(1-2\alpha),-\beta)=\mathcal{S}(\lambda,\alpha,\beta)\quad (0\le\lambda<1,\ 0\le\alpha<1,\ 0<\beta\le 1),$$

the class consisting of functions in $\mathcal{A}$ satisfying the condition

$$\left|\frac{J_z^{\lambda+1}f(z)-J_z^\lambda f(z)}{J_z^\lambda f(z)+(1-2\alpha)J_z^{\lambda+1}f(z)}\right|<\beta\quad (z\in E).$$

The following observations are obvious:

(i) $\mathcal{S}(0,\alpha,1)=\mathcal{S}^*(\alpha)$ is the class of starlike functions of order α;

(ii) $\mathcal{S}(\lambda,\gamma,1)=\mathcal{A}(\lambda+1,\lambda,\gamma)$ $(0\le\lambda<1,\ -\lambda/(1-\lambda)\le\gamma<1)$, the class studied by Kim and Srivastava [7].

In the present paper, we derive various properties and characteristics of the class $\mathcal{S}(a,c,A,B)$ by using the techniques of Briot–Bouquet differential subordination. We also obtain a sufficient condition, coefficient estimates and distortion theorems for this class. Further, we give some applications of our results to a class of fractional calculus operators. Many of our results improve and generalize the corresponding ones in [2], [3], and [7].

2. Preliminaries. In order to establish our results, we need the following lemmas.

LEMMA 1 [8]. *If* $-1 \le B < A \le 1$, $\beta > 0$ *and the complex number* γ *satisfy* $\mathrm{Re}(\gamma) \ge -\beta(1-A)/(1-B)$, *then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = \frac{1+Az}{1+Bz}$$

has a univalent solution in E *given by*

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}\,dt} - \dfrac{\gamma}{\beta}, & B \ne 0, \\ \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta\int_0^z t^{\beta+\gamma-1}\exp(\beta At)\,dt} - \dfrac{\gamma}{\beta}, & B = 0. \end{cases} \tag{2.1}$$

If $p(z)$ *is analytic in* E *and satisfies*

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec \frac{1+Az}{1+Bz}$$

then

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz}$$

and $q(z)$ *is the best dominant.*

LEMMA 2 [16]. *Let* μ *be a positive measure on the unit interval* $[0,1]$. *Let* $g(t,z)$ *be an analytic function in* E *for each* $t \in [0,1]$, *and integrable in* t *for each* $z \in E$ *and for almost all* $t \in [0,1]$, *and suppose that* $\mathrm{Re}\{g(t,z)\} > 0$ *on* E, $g(t,-r)$ *is real and* $\mathrm{Re}\{1/g(t,z)\} \ge 1/g(t,-r)$ *for* $|z| \le r$ *and* $t \in [0,1]$. *If* $g(z) = \int_0^z g(t,z)\,d\mu(t)$, *then* $\mathrm{Re}\{1/g(z)\} \ge 1/g(-r)$ *for* $|z| \le r$.

For real or complex numbers α_1, α_2 and β_1 ($\beta_1 \ne 0, -1, -2, \ldots$), the hypergeometric function ${}_2F_1(z)$ is defined by

$${}_2F_1(z) = {}_2F_1(\alpha_1, \alpha_2; \beta_1; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n}{(\beta_1)_n}\frac{z^n}{n!}. \tag{2.2}$$

We note that the series in (2.2) converges absolutely in E (cf. [15]). The following identities are well known [15].

LEMMA 3. *For real or complex α_1, α_2 and β_1 ($\beta_1 \neq 0, -1, -2, \ldots$), we have*

$$\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}(1-tz)^{-\alpha_1}\,dt = \frac{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}{\Gamma(\beta_1)}\,{}_2F_1(\alpha_1,\alpha_2;\beta_1;z) \quad (\operatorname{Re}\alpha_1 > \operatorname{Re}\alpha_2 > 0), \tag{2.3}$$

$$ {}_2F_1(\alpha_1,\alpha_2;\beta_1;z) = {}_2F_1(\alpha_2,\alpha_1;\beta_1;z), \tag{2.4}$$

$$ {}_2F_1(\alpha_1,\alpha_2;\beta_1;z) = (1-z)^{-\alpha_1}{}_2F_1(\alpha_1,\beta_1-\alpha_2;\beta_1;z). \tag{2.5}$$

LEMMA 4. *Let $p(z)$ be analytic in E with $p(0) = 1$ and $p(z) \neq 0$ for $0 < |z| < 1$, and let $-1 \le B < A \le 1$.*

(i) *Let $B \neq 0$ and μ be a complex number with $\mu \neq 0$. Let A, B and μ satisfy either*

$$\left|\mu\,\frac{A-B}{B} - 1\right| \le 1 \quad \text{or} \quad \left|\mu\,\frac{A-B}{B} + 1\right| \le 1. \tag{2.6}$$

If $p(z)$ satisfies

$$1 + \frac{zp'(z)}{\mu p(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in E)$$

then

$$p(z) \prec q(z) = (1+Bz)^{\mu(A-B)/B}$$

and $q(z)$ is the best dominant.

(ii) *Let $B = 0$, μ be a complex number with $\mu \neq 0$, and $|\mu A| < \pi$. If $p(z)$ satisfies*

$$1 + \frac{zp'(z)}{\mu p(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in E)$$

then

$$p(z) \prec \exp(\mu Az)$$

and this is the best dominant.

To avoid repetition we lay down, once for all, that $a > 0$, $c > 1$, $0 \le \lambda < 1$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le B < A \le 1$.

3. Main results

THEOREM 1. *If $B < A \le (c-B)/(c-A)$, then:*

(i) *$\mathcal{S}(a,c,A,B) \subset \mathcal{S}(a,c+1,A^*,B)$ where $A^* = \{(c-1)A + B\}/c$. Furthermore, if $f \in \mathcal{S}(a,c,A,B)$ then*

$$\frac{\mathcal{L}(a,c)f(z)}{\mathcal{L}(a,c+1)f(z)} \prec \frac{1}{cQ(z)} = \widetilde{q}(z) \qquad (z \in E)$$

where

$$Q(z) = \begin{cases} \int_0^1 t^{c-1}\left(\dfrac{1+Btz}{1+Bz}\right)^{(c-1)(A-B)/B} dt, & B \neq 0, \\ \int_0^1 t^{c-1} \exp\{(c-1)(t-1)Az\}\, dt, & B = 0, \end{cases} \tag{3.1}$$

and $\widetilde{q}(z)$ *is the best dominant.*

(ii) *If* $B < 0$, $A \leq \min\{(c-B)/(c-1), -2B/(c-1)\}$ *then for* $f \in \mathcal{S}(a, c, A, B)$,

$$\operatorname{Re}\left\{\frac{\mathcal{L}(a,c)f(z)}{\mathcal{L}(a,c+1)f(z)}\right\} > \left\{{}_2F_1\left(1, \frac{(c-1)(A-B)}{B}; c+1; \frac{B}{B-1}\right)\right\}^{-1} \quad (z \in E). \tag{3.2}$$

The result is best possible.

Proof. From (1.5), it follows that

$$z(\mathcal{L}(a,c)f(z))' = (c-1)\mathcal{L}(a,c-1)f(z) + (2-c)\mathcal{L}(a,c)f(z). \tag{3.3}$$

Let $f \in \mathcal{S}(a, c, A, B)$. Setting

$$p(z) = \frac{\mathcal{L}(a,c)f(z)}{\mathcal{L}(a,c+1)f(z)}, \tag{3.4}$$

we see that $p(z)$ is analytic in E and $p(0) = 1$. Making use of logarithmic differentiation in (3.4) and using the identity (3.3) in the resulting equation, we get

$$P(z) + \frac{zP'(z)}{(c-1)P(z)+1} \prec \frac{1+Az}{1+Bz} \quad (z \in E) \tag{3.5}$$

where $P(z) = \{c\,p(z) - 1\}/(c-1)$. Using Lemma 1, we deduce that

$$P(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in E) \tag{3.6}$$

where $q(z)$ is the best dominant of (3.5) and is given by (2.1) for $\beta = c-1$ and $\gamma = 1$. Again by (3.6), we obtain

$$p(z) \prec \frac{1}{c\,Q(z)} = \widetilde{q}(z) \quad (z \in E),$$

where $Q(z)$ is given by (3.1). This proves the first part of the theorem.

Now we prove (ii). We show that

$$\inf_{|z|<1} \{\operatorname{Re}(\widetilde{q}(z))\} = \widetilde{q}(-1). \tag{3.7}$$

If we set $\alpha_1 = \{(c-1)(B-A)\}/B$, $\alpha_2 = c$, $\beta_1 = c+1$, then $\beta_1 > \alpha_2 > 0$. From (3.1), by using (2.3)–(2.5) we see that for $B \neq 0$,

$$Q(z) = (1+Bz)^{\alpha_1}\int_0^1 t^{\alpha_2-1}(1+Btz)^{-\alpha_1}\,dt \tag{3.8}$$
$$= \frac{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}{\Gamma(\beta_1)}\,{}_2F_1\left(1,\alpha_1;\beta_1;\frac{Bz}{1+Bz}\right).$$

To prove (3.7), we show that $\operatorname{Re}\{1/Q(z)\} \geq 1/Q(-1)$, $z \in E$. Again, by (3.8) for $B < 0$, $A < -2B/(c-1)$ (so that $\beta_1 > \alpha_1 > 0$), (3.1) can be written as

$$Q(z) = \int_0^1 g(t,z)\,d\mu(t),$$

where

$$g(t,z) = \frac{1+Bz}{1+(1-t)Bz},$$

and

$$d\mu(t) = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_1)}t^{\alpha_1-1}(1-t)^{\beta_1-\alpha_1-1}\,dt$$

is a positive measure on $[0,1]$.

For $-1 \leq B < 0$, it may be noted that $\operatorname{Re}\{g(t,z)\} > 0$, $g(t,-r)$ is real for $0 \leq r < 1$, $t \in [0,1]$ and

$$\operatorname{Re}\left\{\frac{1}{g(t,z)}\right\} \geq \frac{1-(1-t)Br}{1-Br} = \frac{1}{g(t,-r)} \qquad (|z| \leq r < 1,\ t \in [0,1]).$$

Therefore, by using Lemma 2, we deduce that $\operatorname{Re}\{1/Q(z)\} \geq 1/Q(-r)$, $|z| \leq r < 1$ and by taking $r \to 1^-$, we obtain $\operatorname{Re}\{1/Q(z)\} \geq 1/Q(-1)$, $z \in E$. In the case $A = -2B/(c-1)$, we obtain the required assertion by letting $A \to (-2B/(c-1))^+$. This proves (3.7).

The result is best possible because of the best dominant property of $\widetilde{q}(z)$.

Putting $a = 2$, $c = 2-\lambda$, $A = \beta(1-2\alpha)$ and $B = -\beta$ in Theorem 1, we get

Corollary 1. *If $f \in \mathcal{S}(\lambda,\alpha,\beta)$, then*

$$\operatorname{Re}\left\{\frac{J_z^{\lambda}f(z)}{J_z^{\lambda-1}f(z)}\right\} > \left\{{}_2F_1\left(1,2(1-\lambda)(1-\alpha);3-\lambda;\frac{\beta}{\beta+1}\right)\right\}^{-1} \qquad (z \in E).$$

The result is best possible.

For $\lambda = 0$ and $\beta = 1$, Corollary 1 yields

Corollary 2. *If* $f \in \mathcal{S}^*(\alpha)$, *then*

$$\mathrm{Re}\left\{zf(z)\Big(\int_0^z f(t)\,dt\Big)^{-1}\right\} > 2({}_2F_1(1, 2(1-\alpha); 3; 1/2))^{-1} \qquad (z \in E).$$

The result is best possible.

Theorem 2. *Let* $f \in \mathcal{S}(a, c, A, B)$, *where* $-1 \le B < A \le 1$ $(B \neq 0)$. *If either*

$$\left|(c-1)\,\frac{A-B}{B} - 1\right| \le 1 \quad or \quad \left|(c-1)\,\frac{A-B}{B} + 1\right| \le 1$$

then

$$\frac{\mathcal{L}(a,c)f(z)}{z} \prec (1+Bz)^{(c-1)(A-B)/B} \qquad (z \in E). \tag{3.9}$$

In case $B = 0$, *i.e., for* $f \in \mathcal{S}(a, c, A, 0)$ $(0 < A \le 1)$, *we have*

$$\frac{\mathcal{L}(a,c)f(z)}{z} \prec \exp((c-1)Az) \qquad (z \in E), \tag{3.10}$$

where $|A| < \pi/(c-1)$. *The result is best possible.*

Proof. Setting $p(z) = (\mathcal{L}(a,c)f(z))/z$, we note that $p(z)$ is analytic in E, $p(0) = 1$ and $p(z) \neq 0$ for $z \in E$. Logarithmic differentiation $p(z)$ followed by the use of the identity (3.3) yields

$$1 + \frac{zp'(z)}{(c-1)p(z)} = \frac{\mathcal{L}(a,c-1)f(z)}{\mathcal{L}(a,c)f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in E).$$

For such $p(z)$, from (i) and (ii) of Lemma 4, we get the relations (3.9) and (3.10) of the theorem.

Corollary 3. *Under the hypotheses of Theorem* 2, *we have*, *for* $|z| = r < 1$,

$$|\mathcal{L}(a,c)f(z)| \le \begin{cases} r(1+Br)^{(c-1)(A-B)/B}, & B \neq 0, \\ r\exp((c-1)Ar), & B = 0, \end{cases} \tag{3.11}$$

and

$$|\mathcal{L}(a,c)f(z)| \ge \begin{cases} r(1-Br)^{(c-1)(A-B)/B}, & B \neq 0, \\ r\exp(-(c-1)Ar), & B = 0. \end{cases} \tag{3.12}$$

All the above estimates are sharp.

Proof. For $B \neq 0$, we deduce from (3.9) that

$$\frac{\mathcal{L}(a,c)f(z)}{z} = (1+B\omega(z))^{(c-1)(A-B)/B},$$

where $\omega(z)$ is analytic in E satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| \le |z|$ for $z \in E$.

(i) When $B > 0$,

$$\begin{aligned}\left|\frac{\mathcal{L}(a,c)f(z)}{z}\right| &= \left|(1+B\omega(z))^{(c-1)(A-B)/B}\right| \\ &= \left|\exp\left[\frac{(c-1)(A-B)}{B}\log(1+B\omega(z))\right]\right| \\ &= \exp\left[\operatorname{Re}\left\{\frac{(c-1)(A-B)}{B}\log(1+B\omega(z))\right\}\right] \\ &= \exp\left[\frac{(c-1)(A-B)}{B}\log|(1+B\omega(z))|\right] \\ &\le |(1+B\omega(z)|^{(c-1)(A-B)/B} \le (1+Br)^{(c-1)(A-B)/B}.\end{aligned}$$

(ii) When $B < 0$, we put $B = -D,\ D > 0$, so that

$$\begin{aligned}\left|\frac{\mathcal{L}(a,c)f(z)}{z}\right| &= |(1+B\omega(z))^{(c-1)(A-B)/B}| = |\{(1-D\omega(z))^{-1}\}^{(c-1)(A-B)/B}| \\ &\le |(1-D\omega(z))^{-1}|^{(c-1)(A-B)/B} \le \left(\frac{1}{1-Dr}\right)^{(c-1)(A-B)/B} \\ &\le (1+Br)^{(c-1)(A-B)/B}.\end{aligned}$$

In case $B = 0$ and $|A| < \pi/(c-1)$, we have

$$\left|\frac{\mathcal{L}(a,c)f(z)}{z}\right| = \exp\{(c-1)A\operatorname{Re}(\omega(z))\} \le \exp\{(c-1)Ar\}.$$

This proves the assertion (3.11). Similarly, we can prove (3.12).

The bounds are sharp, being attained by the function $f(z)$ defined by

$$\mathcal{L}(a,c)f(z) = \begin{cases} z(1+B\delta_1 z)^{(c-1)(A-B)/B}, & B \ne 0, \\ z\exp((c-1)Az), & B = 0. \end{cases}$$

For $a = 2$, $c = 2-\lambda$, $A = \beta(1-2\alpha)$ and $B = -\beta$, Corollary 3 yields

Corollary 4. *If* $f \in \mathcal{S}(\lambda,\alpha,\beta)$, *then for* $|z| = r < 1$,

$$\frac{r}{(1+\beta r)^{2(1-\lambda)(1-\alpha)}} \le |J_z^\lambda(z)| \le \frac{r}{(1-\beta r)^{2(1-\lambda)(1-\alpha)}}.$$

The bounds are sharp.

Corollary 5. *If* $f \in \mathcal{S}(\lambda,\alpha,\beta)$, *then*

$$\operatorname{Re}\left\{\frac{J_z^\lambda(z)}{z}\right\} > (1+\beta)^{-2(1-\lambda)(1-\alpha)} \qquad (z \in E).$$

The result is sharp.

Theorem 3. *Let* δ *be a real number satisfying*

(3.13) $$B < A \le B + \frac{(1-B)(\delta+1)}{c-1}.$$

(i) *If* $f \in \mathcal{S}(a,c,A,B)$, *then the function* $\mathcal{F}_\delta$ *defined by* (1.2) *belongs to the class* $\mathcal{S}(a,c,A,B)$. *Furthermore,*

$$\frac{\mathcal{L}(a,c-1)\mathcal{F}_\delta(z)}{\mathcal{L}(a,c)\mathcal{F}_\delta(z)} \prec \frac{1}{c-1}\left[\frac{1}{Q(z)} - (2+\delta-c)\right] \equiv \widetilde{q}(z) \quad (z \in E) \tag{3.14}$$

where

$$Q(z) = \begin{cases} \int_0^1 t^\delta \left(\frac{1+Btz}{1+Bz}\right)^{(c-1)(A-B)/B} dt, & B \neq 0, \\ \int_0^1 t^\delta \exp\{(c-1)(t-1)Az\}\, dt, & B = 0, \end{cases} \tag{3.15}$$

and $\widetilde{q}(z)$ *is the best dominant.*

(ii) *If* $B < 0$ *and*

$$A \le \min\left\{\frac{(1-B)(\delta+1)}{c-1} + B, -\frac{(\delta+3-c)B}{c-1}\right\},$$

then for $f \in \mathcal{S}(a,c,A,B)$, *we have*

$$\operatorname{Re}\left\{\frac{\mathcal{L}(a,c-1)\mathcal{F}_\delta(z)}{\mathcal{L}(a,c)\mathcal{F}_\delta(z)}\right\} > \frac{1}{c-1}\left[(\delta+1)\left\{{}_2F_1\left(1, \frac{(c-1)(A-B)}{B}; \delta+2; \frac{B}{B-1}\right)\right\}^{-1} - (2+\delta-c)\right] \quad (z \in E).$$

The result is best possible.

Proof. Since $\mathcal{F}_\delta(z) = z + \sum_{n=2}^{\infty}\{(\delta+1)/(\delta+n)\}a_n z^n$, it follows from (1.5) that

$$z(\mathcal{L}(a,c)\mathcal{F}_\delta(z))' = (\delta+1)\mathcal{L}(a,c)f(z) - \delta\mathcal{L}(a,c)\mathcal{F}_\delta(z). \tag{3.16}$$

Putting

$$p(z) = \frac{\mathcal{L}(a,c-1)\mathcal{F}_\delta(z)}{\mathcal{L}(a,c)\mathcal{F}_\delta(z)}, \tag{3.17}$$

we see that $p(z)$ is analytic in E with $p(0) = 1$. Since $f \in \mathcal{S}(a,c,A,B)$, it is clear that $\mathcal{L}(a,c)f(z) \neq 0$ in $0 < |z| < 1$ so that (3.3) and (3.16) give

$$\frac{\mathcal{L}(a,c)\mathcal{F}_\delta(z)}{\mathcal{L}(a,c)f(z)} = \frac{\delta+1}{(c-1)p(z) + (2+\delta-c)} \quad (z \in E). \tag{3.18}$$

Making use of the logarithmic differentiation in (3.18) and using (3.17), we deduce that

$$p(z) + \frac{zp'(z)}{(c-1)p(z) + (2+\delta-c)} = \frac{\mathcal{L}(a,c-1)f(z)}{\mathcal{L}(a,c)f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in E).$$

Using Lemma 1, we obtain

$$p(z) \prec \frac{1}{c-1}\left[\frac{1}{Q(z)} - (2+\delta-c)\right] \equiv \widetilde{q}(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in E),$$

where $Q(z)$ is given by (3.15), and $\widetilde{q}(z)$ is the best dominant. This proves the first part of the theorem.

Proceeding as in Theorem 1 we get the second part.

Taking $a = 2$, $c = 2-\lambda$, $A = \beta(1-2\alpha)$ and $B = -\beta$ in Theorem 3, we obtain

Corollary 6. *Let δ be a real number satisfying $\delta \geq \{(1-\lambda)(1-2\alpha) - (1+\beta\lambda)\}/(1+\beta)$.*

(i) *If $f \in \mathcal{S}(\lambda,\alpha,\beta)$, then the function $\mathcal{F}_\delta$ defined by* (1.2) *belongs to the class $\mathcal{S}(\lambda,\alpha,\beta)$. Furthermore,*

$$\frac{J_z^{1+\lambda}\mathcal{F}_\delta(z)}{J_z^{\lambda}\mathcal{F}_\delta(z)} \prec \frac{1}{1-\lambda}\left\{\frac{1}{Q(z)} - (\delta+\lambda)\right\} \equiv \widetilde{q}(z) \qquad (z \in E)$$

where $Q(z)$ is obtained from (3.15) *for $c = 2-\lambda$, $A = \beta(1-2\alpha)$ and $B = -\beta$.*

(ii) *If*

$$\delta \geq \max\left\{\frac{(1-\lambda)(1-2\alpha)-(1+\beta\lambda)}{1+\beta}, \frac{(1-\lambda)(1-2\alpha)-\beta(1+\lambda)}{\beta}\right\}$$

and $f \in \mathcal{S}(\lambda,\alpha,\beta)$, then $\mathcal{F}_\delta \in \mathcal{S}(\lambda,\varrho,\beta)$, where

$$\varrho = \frac{1}{1-\lambda}\left[(\delta+1)\left\{{}_2F_1\left(1, 2(1-\lambda)(1-\alpha); \delta+2; \frac{\beta}{\beta+1}\right)\right\}^{-1} - (\delta+\lambda)\right].$$

The result is best possible.

Remark. Substituting $\lambda = 0$ and $\beta = 1$ in part (ii) of Corollary 6, we see that $f \in \mathcal{S}^*(\alpha)$ $(0 \leq \alpha < 1)$ implies that $\mathcal{F}_\delta \in \mathcal{S}^*(\varrho_2)$, where $\varrho_2 = (\delta+1)\{{}_2F_1(1, 2(1-\alpha); \delta+2; 1/2)\}^{-1} - \delta$, provided $\delta \geq -\alpha$. This is an improvement of a recent result of Bajpai and Srivastava [2] and Bernardi [3] for $\delta = 1, 2, \ldots$

Theorem 4. *Let $f \in \mathcal{A}$ be given by* (1.1) *and $-1 \leq B < 0$. If*

$$\sum_{n=2}^{\infty} \frac{\{(1-B)(n-1)+(A-B)(c-1)\}(a)_n}{(c-1)_n}\,|a_n| \leq A-B \tag{3.19}$$

then $f \in \mathcal{S}(a,c,A,B)$. The result is sharp.

Proof. Suppose (3.19) holds. Then for $|z| = r < 1$,

$$\begin{aligned}
&|\mathcal{L}(a,c-1)f(z)-\mathcal{L}(a,c)f(z)|-|A\mathcal{L}(a,c)f(z)-B\mathcal{L}(a,c-1)f(z)|\\
&\quad\le \sum_{n=2}^{\infty}\frac{(n-1)(a)_n}{(c-1)_n}\,|a_n|r^n\\
&\qquad -\left\{(A-B)r-\sum_{n=2}^{\infty}\frac{\{(A-B)(c-1)-(n-1)B\}(a)_n}{(c-1)_n}\,|a_n|r^n\right\}\\
&\quad< \sum_{n=2}^{\infty}\frac{(n-1)(a)_n}{(c-1)_n}\,|a_n|\\
&\qquad -\left\{(A-B)-\sum_{n=2}^{\infty}\frac{\{(A-B)(c-1)-(n-1)B\}(a)_n}{(c-1)_n}\,|a_n|\right\}\\
&\quad= \sum_{n=2}^{\infty}\frac{\{(1-B)(n-1)+(A-B)(c-1)\}(a)_n}{(c-1)_n}\,|a_n|-(A-B)\le 0.
\end{aligned}$$

Thus, it follows from (1.7) that $f\in\mathcal{S}(a,c,A,B)$.

The result is sharp for the functions

$$f_n(z)=z+\sum_{n=2}^{\infty}\frac{(A-B)(c-1)_n}{\{(1-B)(n-1)+(A-B)(c-1)\}(a)_n}\,z^n \qquad (n\ge 2),$$

because

$$\left|\frac{\mathcal{L}(a,c-1)f_n(z)-\mathcal{L}(a,c)f_n(z)}{A\mathcal{L}(a,c)f_n(z)-B\mathcal{L}(a,c-1)f_n(z)}\right|=1 \qquad \text{for } z=\exp(i\pi/n).$$

Corollary 7. *Let $f\in\mathcal{A}$ be given by* (1.1). *If*

$$\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(1-\lambda)\{(1+\beta)(n-1)+2\beta(1-\alpha)\}}{\Gamma(n+1-\lambda)}\,|a_n|\le 2\beta(1-\alpha)$$

then $f\in\mathcal{S}(\lambda,\alpha,\beta)$. The result is sharp.

Theorem 5. *If f given by* (1.1) *belongs to $\mathcal{S}(a,c,A,B)$, then*

$$|a_n|\le\frac{(A-B)(c-1)_n}{(n-1)(a)_{n-1}}\prod_{j=2}^{n-1}\left(1+\frac{(A-B)(c-1)}{j-1}\right) \qquad (n\ge 2). \tag{3.20}$$

The result is sharp.

Proof. Since $f\in\mathcal{S}(a,c,A,B)$, we have

$$\mathcal{L}(a,c-1)f(z)=p(z)\mathcal{L}(a,c)f(z) \tag{3.21}$$

where $p(z)=1+p_1z+p_2z^2+\ldots\in\mathcal{P}(A,B)$. Substituting the power series expansions of $\mathcal{L}(a,c-1)f(z)$, $\mathcal{L}(a,c)f(z)$ and $p(z)$ in (3.21) and comparing the coefficients of z^n on both sides of the resulting equation, we obtain

$$\tag{3.22} \frac{(n-1)(a)_{n-1}}{(c-1)_n}\, a_n = p_{n-1} + \frac{(a)_1}{(c)_1}\, p_{n-2}a_2 + \ldots + \frac{(a)_{n-2}}{(c)_{n-2}}\, p_1 a_{n-1}.$$

Using the fact [1] that

$$|p_n| \le A - B \quad (n \ge 1)$$

in (3.22), we get

$$\tag{3.23} \frac{(n-1)(a)_{n-1}}{(c-1)_n}\, |a_n| \le (A-B)\left\{1 + \sum_{m=2}^{n-1} \frac{(a)_{m-1}}{(c)_{m-1}}\, |a_m|\right\}.$$

We will prove by induction that the assertion (3.20) is satisfied for $n \ge 2$. If $n = 2$, then

$$|a_2| \le \frac{(c-1)_2(A-B)}{(a)_1}.$$

Now suppose that (3.20) is satisfied for $n \le k$. Then, from (3.23), we have

$$\begin{aligned} \frac{k(a)_k}{(c-1)_{k+1}}\, |a_{k+1}| &\le (A-B)\left\{1 + \sum_{m=2}^{k} \frac{(a)_{m-1}}{(c)_{m-1}}\, |a_m|\right\} \\ &\le (A-B)\left\{1 + \sum_{m=2}^{k} \frac{(A-B)(c-1)}{m-1} \prod_{j=2}^{m-1} \left(1 + \frac{(A-B)(c-1)}{j-1}\right)\right\} \\ &= (A-B) \prod_{j=2}^{k} \left(1 + \frac{(A-B)(c-1)}{j-1}\right). \end{aligned}$$

Hence

$$|a_n| \le \frac{(A-B)(c-1)_n}{(n-1)(a)_{n-1}} \prod_{j=2}^{n-1} \left(1 + \frac{(A-B)(c-1)}{j-1}\right)$$

for all $n \ge 2$.

Finally, we note that the result is sharp for the functions $f_n(z)$ given by

$$f_n(z) = z + \frac{(A-B)(c-1)_n}{(n-1)(a)_{n-1}} \prod_{j=2}^{n-1} \left(1 + \frac{(A-B)(c-1)}{j-1}\right) z^n \quad (n \ge 2).$$

Corollary 8. *If f, given by* (1.1), *belongs to the class $\mathcal{S}(\lambda, \alpha, \beta)$, then*

$$|a_n| \le \frac{2\beta(1-\alpha)\Gamma(n+1-\lambda)}{(n-1)\Gamma(n+1)\Gamma(1-\lambda)} \prod_{j=2}^{n-1} \left(1 + \frac{2\beta(1-\alpha)(1-\lambda)}{j-1}\right) \quad (n \ge 2).$$

The result is sharp.

Theorem 6. *Let f, given by* (1.1), *belong to the class $\mathcal{S}(a,c,A,B)$ and μ be any complex number. Then*

$$|a_3-\mu a_2^2| \le \frac{(A-B)(c-1)_3}{2(a)_2} \times \max\left\{1, \left|\{B-(A-B)(c-1)\}+\mu\frac{2(A-B)(a+1)(c-1)_2}{a(c+1)}\right|\right\}.$$

The result is sharp.

Proof. For $f \in \mathcal{S}(a,c,A,B)$, we have, by (1.6),

$$\sum_{n=2}^{\infty}\frac{(n-1)(a)_{n-1}}{(c-1)_n}\,a_n z^n = \left\{(A-B)z+\sum_{n=2}^{\infty}\frac{\{(A-B)(c-1)-(n-1)B\}(a)_{n-1}}{(c-1)_n}\,a_n z^n\right\}\left\{\sum_{j=1}^{\infty}\omega_j z^j\right\} \tag{3.24}$$

where $\omega(z)=\sum_{j=1}^{\infty}\omega_j z^j$ is analytic in E with $|\omega(z)|<1$ for $z\in E$. On equating the coefficients of z^2 and z^3 on both sides of (3.24), we deduce that

$$a_2=\frac{(A-B)(c-1)}{a}\,\omega_1 \tag{3.25}$$

and

$$a_3=\frac{(A-B)(c-1)}{2\,(a)_2}\,\{\omega_2+((A-B)(c-1)-B)\omega_1^2\}. \tag{3.26}$$

It is known [6] that for every complex number γ,

$$|\omega_2-\gamma\omega_1^2|\le\max\{1,|\gamma|\} \tag{3.27}$$

and the estimate is sharp. Now, by using (3.25) and (3.26), we obtain

$$|a_3-\mu a_2^2|\le\frac{(A-B)(c-1)_3}{2\,(a)_2}\,|\omega_2-\gamma\omega_1^2|, \tag{3.28}$$

where

$$\gamma=\{B-(A-B)(c-1)\}+\mu\,\frac{2(A-B)(a+1)(c-1)_2}{a(c+1)}.$$

The assertion of the theorem follows by using (3.27) in (3.28). The result is sharp as the estimate (3.27) is sharp.

Corollary 9. *If f, given by* (1.1), *belongs to the class $\mathcal{S}(\lambda,\alpha,\beta)$, then for any complex number μ*

$$|a_3-\mu a_2^2|\le\frac{\beta(1-\alpha)\Gamma(4-\lambda)}{3!\,\Gamma(1-\lambda)} \times\max\left\{1,\left|\frac{6\beta(1-\alpha)(1-\lambda)(2-\lambda)}{3-\lambda}-\beta\{2(1-\alpha)(1-\lambda)+1\}\right|\right\}.$$

The result is sharp.

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