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Borel methods of summability and ergodic theorems

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Abstract. Passing from Cesàro means to Borel-type methods of summability we prove some ergodic theorem for operators (acting in a Banach space) with spectrum contained in $\mathbb{C} \setminus (1, \infty)$.

1. Introduction. Let X be a Banach space. Denote by $B(X)$ the algebra of bounded linear operators acting in X. Take $u \in B(X)$. If the Cesàro averages

$$n^{-1} \sum_{k=0}^{n-1} u^k$$

converge, say, weakly then the spectrum of u is necessarily contained in the unit disc $\{|z| \le 1\}$. Passing from the Cesàro means to the Borel-type methods of summability [4], [5] one can extend the ergodic theorems to the case of operators u with the spectrum $\sigma(u)$ contained in the Mittag-Leffler star for $z \mapsto (1-z)^{-1}$, i.e. with $\sigma(u) \subset \mathbb{C} \setminus (1, \infty)$. A discussion of such possibilities is the main goal of the paper.

Let us begin with some notation and definitions. For $\alpha > 0$ and a sequence $x = (\xi_n)$ of numbers (or vectors), put

$$B_\alpha(t, x) = \alpha e^{-t} \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \xi_n, \qquad t > 0. \tag{1}$$

The function $B_\alpha(t, x)$ is called the B_α-*transform* of the sequence $x = (\xi_n)$. If $\lim_{t\to\infty} B_\alpha(t, x) = \xi$, then we say that (ξ_n) is *summable to* ξ *by the method* B_α, and write $\xi_n \to \xi(B_\alpha)$, or $B_\alpha\text{-}\lim_{n\to\infty} \xi_n = \xi$. The family of methods $\{B_\alpha : \alpha > 0\}$ is *consistent*, i.e. for every $\alpha', \alpha'' > 0$, $B_{\alpha'}\text{-}\lim \xi_n = \xi$ and $B_{\alpha''}\text{-}\lim \xi_n = \eta$ implies $\xi = \eta$ (cf. [5]). For our purposes it will be enough to take $\alpha = 2^{-k}$, $k = 0, 1, \ldots$, so in what follows we consider only the family $\mathcal{B} = \{B_{2^{-k}} : k = 0, 1, \ldots\}$. By the consistency just mentioned the family $\mathcal{B}$

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may be treated as a $\mathcal{B}$-method of summability: a sequence x is $\mathcal{B}$-summable when it is $B_{2^{-k}}$-summable for some $k \in \mathbb{N}$. Borel methods of summability are *right-translative*, i.e. the B_α-summability of $x = (\xi_0, \xi_1, \ldots)$ implies the B_α-summability of $x^- = (0, \xi_0, \xi_1, \ldots)$. Notice that the B_α-method is not *left-translative*, i.e. the B_α-summability of x does not imply, in general, the B_α-summability of $x^+ = (\xi_1, \xi_2, \ldots)$ (cf. [5]).

Before formulating the main results let us start with the following lemma.

2. LEMMA. *Fix $\alpha = 2^{-k}$ and $0 < d < 1$. Put*

$$D_{\alpha,d} = \{z \in \mathbb{C} : \operatorname{Re} z \le 0\} \cup \{z \in \mathbb{C} : \operatorname{Re} z^{1/\alpha} \le 1 - d\}.$$

Let Δ be a bounded Borel subset of $D_{\alpha,d}$. Here and elsewhere let $\zeta = (z^n)$. Then, for $t > 0$ and $z \in \Delta$,

$$|B_\alpha(t, \zeta)| \le C e^{-dt/2}, \tag{2}$$

for some constant C depending only on Δ.

Proof. A crucial point in the proof is a suitable representation of the Mittag-Leffler function

$$E_\alpha(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(n\alpha + 1)}, \quad \alpha > 0, \quad \text{for } w = t^\alpha z.$$

We follow Włodarski [5]. Let us remark that, for a fixed $\alpha = 2^{-k}$, the function

$$f(t) = E_\alpha(t^\alpha z) = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} z^n \tag{3}$$

(as a function of $t > 0$) satisfies the differential equation

$$f'(t) = g(t) + z^{2^k} f(t) \quad \text{with} \quad g(t) = \sum_{\nu=1}^{2^k - 1} \frac{t^{\nu 2^{-k} - 1}}{\Gamma(\nu 2^{-k})} z^\nu.$$

This is easy to check. Consequently, we have

$$f(t) = \exp(z^{2^k} t) \Bigg[1 + \int_0^t \exp(-z^{2^k} u) \sum_{\nu=1}^{2^k - 1} \frac{u^{\nu 2^{-k} - 1}}{\Gamma(\nu 2^{-k})} z^\nu \, du \Bigg].$$

The substitution $z^{2^k} = vt$ leads to the formula

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^{n 2^{-k}}}{\Gamma(n 2^{-k} + 1)} z^n \\ &\qquad = \exp(t z^{2^k}) \Bigg[1 + \sum_{\nu=1}^{2^k - 1} \alpha_\nu^{(k)}(z) \int\limits_{[0, z^{2^k} t]} \frac{e^{-v} v^{\nu 2^{-k} - 1}}{\Gamma(\nu 2^{-k})} \, dv \Bigg], \end{aligned} \tag{4}$$

where

$$\alpha_\nu^{(k)}(z) = \frac{z^\nu}{[x^{2^k\nu}]^{1/2^k}} = e^{i\theta_\nu^{(k)}(z)}.$$

The functions $\alpha_\nu^{(k)}(z)$ are determined by fixing the rational power $w \mapsto w^{1/2^k}$ as taking its values in the angle $\{z = re^{i\theta} : r \geq 0, -\pi/2^{k-1} < \theta \leq \pi/2^{k-1}\}$. In particular, $\alpha_1^{(1)}(z) = -1$ for $\operatorname{Re} z < 0$, and $\alpha_\nu^{(k)}(1) = 1$ for $1 \leq \nu \leq 2^k - 1$, $k = 1, 2, \ldots$

For $z = 1$, the formula (4) gives

$$\sum_{n=0}^{\infty} \frac{t^{n2^{-k}}}{\Gamma(n2^{-k}+1)} = e^t \Big[1 + \sum_{\nu=1}^{2^k-1} \frac{1}{\Gamma(\nu 2^{-k})} \int\limits_{[0,t]} e^{-u} u^{\nu 2^{-k}-1}\, du\Big] \tag{5}$$

(cf. [5], p. 144).

Put $Q = \{z \in \mathbb{C} : \operatorname{Re} z^{2^k} \leq 1 - d\}$.

For $\operatorname{Re} z < 1$, $t > 1$ and $\beta > -1$, we have the inequality

$$\Big| e^{t(z-1)} \int\limits_{[0,zt]} u^\beta e^{-u}\, du \Big| \leq C|zt|^{\beta+1} \max(e^{-t}, e^{-t(1-\operatorname{Re} z)}). \tag{6}$$

We omit a rather standard proof.

Let $z \in \Delta \cap Q$, where Δ is a fixed bounded set. Then by (4) and (6), we get (2).

Now assume that $\operatorname{Re} z \leq 0$. Then, clearly,

$$B_1(t, \zeta) = e^{-t(1-z)}, \tag{7}$$

so $|B_1(t,\zeta)| \leq e^{-t}$.

Consider the following transformation W:

$$W(f)(t) = \frac{e^{-t}}{2\sqrt{\pi t}} \int\limits_0^\infty \exp\Big(-\frac{u^2}{4t} + u\Big) f(u)\, du, \tag{8}$$

defined for continuous functions $f : (0,\infty) \to \mathbb{R}$ (cf. [5], p. 140). The transformation W is *regular* in the sense that $\lim_{n\to\infty} f(u) = \beta$ implies $\lim_{t\to\infty} W(f)(t) = \beta$. Moreover, we have

$$W(B_{2^{-k}}(\cdot, x))(t) = B_{2^{-(k+1)}}(t), \qquad t > 0, \tag{9}$$

Applying to both sides of (7) the kth iteration of the transformation W defined in (8) and taking into account the positivity of W and (9) we easily get

$$|B_{2^{-k}}(t,\zeta)| \leq Ce^{-t} \quad \text{for } \operatorname{Re} z \leq 0. \ \blacksquare$$

As an easy consequence of Lemma 2, we get the following result.

3. Theorem (Uniform ergodic theorem). *Let $u \in B(X)$ with $\sigma(u) \subset \mathbb{C} \setminus (1,\infty)$. If $1 \notin \sigma(u)$ then there exists a $k \in \mathbb{N}$ such that*

$$B_{2^{-k}}\text{-}\lim_{n\to\infty} u^n = 0 \quad \textit{in the uniform operator topology.}$$

If $1 \in \sigma(u) \subset \mathbb{C} \setminus (1,\infty)$ and 1 is a pole of u of order one, then

$$B_{2^{-k}}\text{-}\lim_{n\to\infty} u^n = \mathbb{E}\{1\} \quad \textit{uniformly,}$$

where $\mathbb{E}\{1\}$ denotes the spectral projection of u at $\{1\}$.

Proof. Suppose $1 \notin \sigma(u)$. Since $\sigma(u)$ is compact and $\sigma(u) \subset \mathbb{C} \setminus [1,\infty)$, there exist $0 < d < 1$ and $k \in \mathbb{N}$ such that $\sigma(u) \subset D_{2^{-k},d} = \{\operatorname{Re} z \le 0\} \cup \{\operatorname{Re} z^{2^k} \le 1-d\}$. Let $R(\cdot,u)$ be the resolvent of u and, for $x = (u^n)$, $\zeta = (z^n)$, let

$$B_{2^{-k}}(t,x) = \frac{1}{2\pi i}\int_K B_{2^{-k}}(t,\zeta)R(z,u)\,dz$$

be a representation of the Borel transform $B_{2^{-k}}(t,x)$ as a Cauchy integral, i.e. K is the (oriented) boundary of an open set $V \supset \sigma(u)$; K consists of a finite number of rectifiable Jordan curves (cf. [3], p. 568). By Lemma 2 we easily get

$$\|B_{2^{-k}}(t,x)\| \le Ce^{-dt/2}, \quad t > 1.$$

Now, let $1 \in \sigma(u) \subset \mathbb{C} \setminus (1,\infty)$. Then, putting

$$f_t(u) = B_{2^{-k}}(t,x), \quad x = (u^n),$$

we can write

$$f_t(u) = f_t(u)\mathbb{E}(\sigma(u)\setminus\{1\}) + B_{2^{-k}}(t,\mathbf{1})\mathbb{E}\{1\},$$

with $\mathbf{1} = (1,1,\ldots)$, where, for a spectral set A of u, $\mathbb{E}(A)$ denotes the corresponding projection operator (cf. [3], p. 573). To conclude the proof it is enough to pass with t to infinity. ∎

Taking *discrete* Borel methods, i.e. considering the transforms $B_\alpha(m,x)$ only for $m = 1,2,\ldots$, we can easily prove the following theorem.

4. Theorem (Individual ergodic theorem). *Let $X = \mathbb{L}_p(\mu)$, $p \ge 1$, and let $u \in B(X)$ with $\sigma(u) \subset \mathbb{C} \setminus (1,\infty)$. If $1 \notin \sigma(u)$ then there exists a $k \in \mathbb{N}$ such that*

$$B_{2^{-k}}\text{-}\lim_{n\to\infty} u^n f = 0 \quad \mu\textit{-almost everywhere, for every } f \in X.$$

If $1 \in \sigma(u) \subset \mathbb{C} \setminus (1,\infty)$ and 1 is a pole of u of order one then, for every $f \in X$,

$$B_{2^{-k}}\text{-}\lim_{n\to\infty} u^n f = \mathbb{E}\{1\}f \quad \mu\textit{-almost everywhere.}$$

Proof. The proof can be obtained as an easy modification of the previous argument. Namely, using the above estimates we get easily

$$\sum_{m=1}^{\infty} \|B_{2^{-k}}(m,x)f\|_p^p < \infty,$$

for every $f \in X$. The rest is trivial. ∎

In the case $1 \in \sigma(u) \subset \mathbb{C} \setminus (1,\infty)$ and when 1 is not a simple pole one cannot expect the results as clear as the above theorems. The asymptotic behaviour of u heavily depends on its spectral properties near the value 1. The sequence (z^n) with z close to 1 is rather slowly divergent and Borel summability methods are efficient for rapidly divergent sequences (cf. [4]). It is worth noting here that for a sequence (X_n) of independent identically distributed random variables the limit $B_1\text{-}\lim X_n = \mathcal{E}X_1$ (expectation of X_1) exists almost everywhere if and only if $\mathcal{E}(X_1^2) < \infty$, so in the classical context of the Strong Law of Large Numbers, the Borel methods are *less* efficient than the Cesàro means (cf. [1], [2]).

Let X be again an arbitrary Banach space. For $u \in B(X)$, we say that (u^n) is *strongly B_α-summable* to P when $B_\alpha\text{-}\lim_{n\to\infty} u^n\xi = P\xi$ for every $\xi \in X$. By the right-translativity of B_α, we then also have $B_\alpha\text{-}\lim u^{n-1}\xi = P\xi$. By the continuity of u, we get $uP\xi = P\xi$. Consequently, $B_\alpha\text{-}\lim u^{n+1}\xi = P\xi$ (left-translativity of B_α for sequences of the form $(u^n\xi)$), and also $P^2 = P$, $uP = Pu$.

For $x = (u^n)_{n=0}^{\infty}$, let $x^+ = (u^{n+1})_{n=0}^{\infty}$.

5. Theorem (Mean ergodic theorem). *Let $u \in B(X)$, where X is a Banach space. Then the sequence $x = (u^n)_{n=0}^{\infty}$ is strongly B_α-summable to a projection Q if and only if the following conditions are satisfied*:

(i) $\sup_{0<t<\infty} \|B_\alpha(t,x)\| < \infty$,
(ii) $B_\alpha(t, x^+ - x) \to 0$ *strongly as* $t \to \infty$,
(iii) *the family* $\{B_\alpha(t,x) : t > 0\}$ *is weakly sequentially compact.*

Proof. Necessity. (i) is a consequence of the Banach–Steinhaus theorem. (ii) follows from the translativity of B_α for the sequence $(u^n\xi)$. (iii) is obvious.

Sufficiency. Put

$$X_0 = \{\xi \in X : u\xi = \xi\}, \qquad X_1 = \{u\xi - \xi : \xi \in X\}^-.$$

Obviously, $B_\alpha\text{-}\lim u^n\xi = \xi = Q\xi$ for $\xi \in X_0$. Put $Y = \{u\xi - \xi : \xi \in X\}$. If $\eta \in Y$ then $\eta = u\xi - x$ for some $\xi \in X$, and we have, for $x = (u^n)$,

$$\alpha e^{-t} \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} u^n\eta = B_\alpha(t, x^+ - x) \to 0$$

strongly as $t \to \infty$, by (ii).

It is enough to show that $X = X_0 + X_1$, because in this case the proof can be completed by a standard approximation. Fix $\xi \in X$. By (iii), we find a vector $\overline{\xi}$ such that

$$\overline{\xi} = w\text{-}\lim_{k\to\infty} B_\alpha(t_k, x)\xi,$$

for some $t_k \nearrow \infty$ (here w-lim denotes the weak limit).

We have

$$\begin{aligned} u\overline{\xi} &= w\text{-}\lim_{k\to\infty} u(B_\alpha(t_k, x)\xi) \\ &= w\text{-}\lim_{k\to\infty} B_\alpha(t_k, x^+ - x)\xi + w\text{-}\lim_{k\to\infty} B_\alpha(t_k, x)\xi = \overline{\xi}, \end{aligned}$$

by (ii). We have just proved that $\overline{\xi} \in X_0$, and we shall show that $\xi - \overline{\xi} \in X_1$. By the Hahn–Banach theorem it is enough to check that, for every $\phi \in X^*$ which disappears on X_1, we have $\phi(\xi - \overline{\xi}) = 0$. But if $\phi = 0$ on X_1 then, in particular, $\phi(u\xi) = \phi(\xi)$ for every $\xi \in X$, so $\phi(\xi) = \phi(u^n\xi)$, $n = 1, 2, \ldots$ Consequently,

$$\phi(B_\alpha(t_k, x)\xi) = \alpha e^{-t_k} \sum_{n=0}^{\infty} \frac{t_k^{n\alpha}}{\Gamma(n\alpha + 1)}\, \phi(u^n\xi) = \phi(\xi) B_\alpha(t_k, \mathbf{1}).$$

Passing with k to infinity we get $\phi(\overline{\xi}) = \phi(\xi)$, i.e. $\phi(\xi - \overline{\xi}) = 0$. ∎

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