Nullstellensatz and cycles
of zeroes of holomorphic mappings

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Abstract. The local Nullstellensatz exponent for holomorphic mappings via intersection theory for the cases of isolated and quasi-complete intersection is considered.

1. Introduction. One of the natural problems in local intersection theory concerns the Nullstellensatz exponent. It is already a classical result that in the case of a set-theoretic complete intersection the degree of the cycle of zeroes of a holomorphic mapping (see [PT], [TW]) is a good exponent for an ideal generated by their components. The general local case seems to be much more difficult; moreover the definition of the cycles of zeroes in this case is quite new. Improper intersections in complex-analytic geometry were first studied by P. Tworzewski [T]. Some interesting applications of the index of intersection for analytic cycles proposed in his paper were found in [Cy], [CKT].

Obviously the natural question arises: does this index always give an exponent for Nullstellensatz? This question is also important in algebraic geometry (cf. [Br], [JKS], [K₁], [K₂], [Te]). In this paper we consider two special local cases: the so-called quasi-complete intersection and isolated intersection. The main tool we use to describe the first one is Theorem 3.5 which gives a kind of generalization of Schumacher's Theorem ([Ł]). The latter case is treated using only the basic properties of improper isolated intersection presented in [ATW]. One can find another approach to this result in [S].

2. Intersection multiplicity. For the convenience of the reader we compile in this section some basic notions of local intersection theory (see [Ch], [D], [T] for more details).

I. *Analytic cycles, their multiplicities and proper intersection.* In this paper *analytic* means complex-analytic, and *manifold* means a complex manifold satisfying the second axiom of countability. Let M be a manifold of dimension m. An *analytic cycle* on M is a formal sum

$$A = \sum_{j \in J} \alpha_j Z_j$$

where $\alpha_j \neq 0$ for $j \in J$ are integers and $\{Z_j\}_{j \in J}$ is a locally finite family of distinct irreducible analytic subsets of M.

The analytic set $\bigcup_{j \in J} Z_j$ is called the *support* of the cycle A and is denoted by $|A|$. If all the components of A have the same dimension k, then A is called a *k-cycle*. We say that A is *positive* if $\alpha_j > 0$ for all $j \in J$.

For the cycle A, we consider the natural extension of the local multiplicity of analytic sets. Namely, if $a \in M$ and $\nu(Z_j, a)$ denotes the multiplicity of Z_j at the point a (see [D, p. 194]), then the sum

$$\nu(A, a) = \sum_{j \in J} \alpha_j \nu(Z_j, a)$$

is well defined and called the *multiplicity* of A at a.

There exists a unique decomposition

$$A = T_{(m)} + T_{(m-1)} + \ldots + T_{(0)},$$

where $T_{(j)}$ is a j-cycle for $j = 0, \ldots, m$. For our purpose it will be useful to introduce the notion of the *extended multiplicity* of A at a defined by

$$\widetilde{\nu}(A, a) = (\nu(T_{(m)}, a), \ldots, \nu(T_{(0)}, a)) \in \mathbb{Z}^{m+1}.$$

Denote by $\nu(A)$ and $\widetilde{\nu}(A)$ the functions

$$\nu(A) : M \ni x \mapsto \nu(A, x) \in \mathbb{Z}, \quad \widetilde{\nu}(A) : M \ni x \mapsto \widetilde{\nu}(A, x) \in \mathbb{Z}^{m+1}.$$

Observe that $\nu(A, x) = \widehat{\widetilde{\nu}(A, x)}$, where $\widehat{\nu}$ denotes the sum of the coordinates of $\nu \in \mathbb{Z}^{m+1}$.

Let now X and Y be pure dimensional analytic subsets of M. We say that X and Y *meet properly* on M if $\dim(X \cap Y) = \dim X + \dim Y - m$. Then we have the *intersection product* $X \cdot Y$ of X and Y, which is an analytic cycle on M defined by the formula

$$X \cdot Y = \sum_{Z} i(X \cdot Y, Z)Z,$$

where the summation extends over all analytic components Z of $X \cap Y$ and $i(X \cdot Y, Z)$ denotes the intersection multiplicity along Z in the sense of Draper ([D, Def. 4.5]; cf. [Ch]). Such multiplicities are positive integers.

II. *Intersections of analytic sets with submanifolds.* Let M be an m-dimensional manifold. Fix a closed s-dimensional submanifold S of M and an open subset U of M such that $U \cap S \neq \emptyset$.

For a given cycle $A = \sum_{j \in J} \alpha_j Z_j$ analytic on M, by its *part supported by* S we mean the cycle $A^S = \sum_{j \in J,\, Z_j \subset S} \alpha_j Z_j$. Denote by $\mathcal{H}(U)$ the set of all $\mathcal{H} := (H_1, \ldots, H_{m-s})$ satisfying the following conditions:

(1) H_j is a smooth hypersurface of U containing $U \cap S$ if $j = 1, \ldots, m-s$,
(2) $\bigcap_{j=1}^{m-s} T_x(H_j) = T_x S$ for each $x \in U \cap S$.

For a given analytic subset Z of M of pure dimension k we denote by $\mathcal{H}(U, Z)$ the set of all $\mathcal{H} \in \mathcal{H}(U)$ such that $((U \setminus S) \cap Z) \cap H_1 \cap \ldots \cap H_j$ is an analytic subset of $U \setminus S$ of pure dimension $k - j$ (or empty) for $j = 1, \ldots, k$. Each system $\mathcal{H} \in \mathcal{H}(U, Z)$ will be called an *admissible system* for the set Z and submanifold S.

Following [T] we present an algorithm which allows us to produce for every $\mathcal{H} \in \mathcal{H}(U, Z)$ an analytic cycle $Z \cdot \mathcal{H}$ in $S \cap U$. In each step of the algorithm we get a cycle $Z_i = Z_i^S + (Z_i - Z_i^S)$. Denote by $i_{\mathcal{H}} \in \{0, \ldots, m-s\}$ the first index i for which $|Z_i - Z_i^S| = \emptyset$.

ALGORITHM 2.1.

Step 0. Let $Z_0 = Z \cap U$. Then $Z_0 = Z_0^S + (Z_0 - Z_0^S)$, where Z_0^S is the part of Z_0 supported by $S \cap U$.

Step 1. Let $Z_1 = (Z_0 - Z_0^S) \cdot H_1$. Then $Z_1 = Z_1^S + (Z_1 - Z_1^S)$, where Z_1^S is the part of Z_1 supported by $S \cap U$.

Step 2. Let $Z_2 = (Z_1 - Z_1^S) \cdot H_2$. Then $Z_2 = Z_2^S + (Z_2 - Z_2^S)$, where Z_2^S is the part of Z_2 supported by $S \cap U$.

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Step $i_{\mathcal{H}}$. Let $Z_{i_{\mathcal{H}}} = (Z_{i_{\mathcal{H}}-1} - Z_{i_{\mathcal{H}}-1}^S) \cdot H_{i_{\mathcal{H}}}$. Now we have the decomposition

$$Z_{i_{\mathcal{H}}} = Z_{i_{\mathcal{H}}}^S + (Z_{i_{\mathcal{H}}} - Z_{i_{\mathcal{H}}}^S), \quad \text{and} \quad |Z_{i_{\mathcal{H}}} - Z_{i_{\mathcal{H}}}^S| \cap S = \emptyset.$$

We call the positive analytic cycle $Z \cdot \mathcal{H} = Z_0^S + Z_1^S + \ldots + Z_{i_{\mathcal{H}}}^S$ in $S \cap U$ the *result* of the above algorithm.

We will say that Z and an admissible system $\mathcal{H}$ realize a *quasi-complete intersection* at a if every irreducible component of the support of the cycle $Z_i - Z_i^S$ is a complete ideal intersection at a for $i \in \{0, \ldots, i_{\mathcal{H}}\}$. Recall that an analytic subset X of a manifold M is a *complete ideal intersection* at a if the ideal of its germ $I(X_a)$ has a system of $p = \operatorname{codim} X_a$ generators.

At an arbitrary point $a \in S \cap Z$ we define $g(a)$ as follows. Let

$$\widetilde{g}(a) = \widetilde{g}(Z, S)(a) := \min_{\text{lex}}\{\widetilde{\nu}(Z \cdot \mathcal{H}, a) : \mathcal{H} \in \mathcal{H}(U, Z),\, a \in U\} \in \mathbb{N}^{s+1}$$

and $g(a) = g(Z, S)(a) = \widehat{\widetilde{g}(a)}$. The number $g(a)$ is called the *index of intersection* of Z and S at the point a (see [T, Def. 4.2]).

Let now M be an open subset of a normed linear space N. We are interested in more convenient systems where hypersurfaces have the form of parts of linear hyperplanes from the grassmannian manifold $G^{m-1}(N)$.

Denote by $A(Z,S)(a) \subset (G^{m-1}(N))^{m-s}$ the set of all admissible linear systems $\mathcal{H}$ for Z and S at the point a such that $\widetilde{g}(Z,S)(a) = \widetilde{\nu}(Z \cdot \mathcal{H}, a)$.

We will say that Z and S realize a *quasi-complete intersection* at $a \in Z \cap S$ if there exists a system $\mathcal{H} \in A(Z,S)(a)$ such that Z and $\mathcal{H}$ realize a quasi-complete intersection at a.

REMARK 2.2 ([AR], [N]). If S is a linear subspace of a normed linear space M then to obtain the result of the algorithm it suffices to consider the admissible systems $\mathcal{H}$ which consist of parts of linear hyperplanes. Moreover in such a case the index of intersection of Z and S at a is realized for generic systems of linear hyperplanes in M.

III. *Intersection of analytic sets—general case.* Let X and Y be irreducible analytic subsets of an m-dimensional manifold M and let $a \in M$. By standard diagonal construction the *multiplicity of intersection* of the sets X and Y at a is defined to be

$$d(a) = d(X,Y)(a) = g(X \times Y, \Delta_M, (a,a)).$$

The *intersection product* of the irreducible analytic sets X and Y is the unique analytic cycle $X \bullet Y$ in M such that $\nu(X \bullet Y) = d(X,Y)$ (see [T, Def. 6.3]).

The above definition can be naturally extended to the case of the intersection of a finite number of irreducible analytic subsets and next to arbitrary analytic cycles by multilinearity.

REMARK 2.3 ([AR], [N]). The index and multiplicity of intersection of an analytic set X with a submanifold S at the point a, and of the analytic set $X \times S$ with the diagonal Δ_M at the point (a,a), coincide.

Roughly speaking we are not obliged to pass by the diagonal construction for the case of intersection with a submanifold. This remark will be very useful in our considerations of the cycles of zeroes of holomorphic mappings.

3. Cycles of zeroes of holomorphic mappings and Nullstellensatz. Let U be a neighbourhood of zero in $\mathbb{C}^m$ and $(f_1, \ldots, f_r) : U \to \mathbb{C}^r$ be a holomorphic mapping. Consider two analytic subsets of $U \times \mathbb{C}^r$: the graph of f, $G_f = \{(x,y) \in U \times \mathbb{C}^r : y_i = f_i(x)\}$, and $Y_f = U \times \{0\}^r$.

DEFINITION 3.1. In this situation we define the cycle Z_f on U given by

$$Z_f \times \{0\} = G_f \bullet Y_f$$

to be the *cycle of zeroes* of f.

Obviously Z_f is an analytic cycle in U but is not necessarily pure dimensional. Note that if the mapping realizes a set-theoretic complete intersection (which means in particular that $\dim f^{-1}(0) = m - r$), then this definition coincides with [TW].

Let us start by recalling the following two results that gave the motivation for this paper. We denote by $I_z(f_1, \ldots, f_r)$ the ideal of $\mathcal{O}_z$ generated by the germs of f_i at $z \in U$.

The first theorem is just a special case of [PT, Theorem 1.2].

THEOREM 3.2. *Suppose that the mapping* $(f_1, \ldots, f_r) : U \to \mathbb{C}^r$ *is holomorphic, realizes a set-theoretic complete intersection and* $g : U \to \mathbb{C}$ *is a holomorphic function vanishing on the set* $f^{-1}(0)$. *Let* $a \in U$. *Then there exist an open neighbourhood* $\Omega \subset U$ *of* a *and* $h_1, \ldots, h_r \in \mathcal{O}(\Omega)$ *such that*

$$g(z)^\mu = f_1(z)h_1(z) + \ldots + f_r(z)h_r(z) \quad \text{for } z \in \Omega$$

where $\mu = \nu(Z_f, a)$ *is the degree of the cycle of zeroes of* f *at* a.

The second one is a reformulation of [CKM, Theorem 1].

THEOREM 3.3. *Let* $f = (f_1, \ldots, f_r)$ *be a holomorphic mapping in a neighbourhood of* $a \in \mathbb{C}^m$ *realizing a set-theoretic complete intersection at the point* a. *Then the following conditions are equivalent*:

(i) $I_a(f_1, \ldots, f_r) = \operatorname{Rad} I_a(f_1, \ldots, f_r)$.
(ii) *If* $\operatorname{Rad} I_a(f_1, \ldots, f_r) = I_1 \cap \ldots \cap I_s$, *where* I_j *are prime ideals in* $\mathcal{O}_a$ *and* $I_i \not\subset I_j$ *for* $i \neq j$ *then* $I_a(f_1, \ldots, f_r)\mathcal{O}_{I_j} = I_j\mathcal{O}_{I_j}$ *for all* $j \in \{1, \ldots, s\}$.

In particular, as a direct consequence of Theorem 3.3 (cf. [CKM, Remark 1]) we get the following:

COROLLARY 3.4. *Suppose that* X *is an* $(m - r)$-*dimensional analytic subset of a neighbourhood of* $a \in \mathbb{C}^m$ *with the irreducible germ* X_a. *If* $g = (g_1, \ldots, g_r)$ *where* $g_1, \ldots, g_r$ *are generators of the ideal* $I_a(X)$ *then* $\nu(X, a) = \nu(Z_g, a)$.

We start with the following generalization of Schumacher's Theorem (see [Ł]).

THEOREM 3.5. *Let* X *be the zero set of the mapping* $(f_1, \ldots, f_r) : U \to \mathbb{C}^r$ *which realizes a set-theoretic complete intersection in* U, g *be a holomorphic function in* U *and* $a \in X$. *Suppose that there exists a dense subset* $\widetilde{X}$ *of* X *such that* $g_z \in I_z(f_1, \ldots, f_r)$ *for* $z \in \widetilde{X}$. *Then* $g_a \in I_a(f_1, \ldots, f_r)$.

Proof. First recall that the family $I_z(f_1, \ldots, f_r) \subset \mathcal{O}_z$ is a coherent family of ideals.

As $(f_1, \ldots, f_r)$ realizes a set-theoretic complete intersection in U, there exists a neighbourhood Ω of a such that the primary decomposition of the

ideal $I_a(f_1, \ldots, f_r) = J_a^1 \cap \ldots \cap J_a^s$ corresponds to the irreducible components of the set $X \cap \Omega$, so in Ω we have:

(1) $X \cap \Omega = X_1 \cup \ldots \cup X_s$,
(2) $I_z(f_1, \ldots, f_r) = J_z^1 \cap \ldots \cap J_z^s$ and J_z^i are primary ideals for $z \in \Omega$,
(3) $I(X_z^i) = \operatorname{Rad} J_z^i$ for $z \in \Omega$ and $i = 1, \ldots, s$.

Consider a new coherent family of ideals $J_z := \{h_z \in \mathcal{O}_z : h_z g_z \in I_z(f_1 \ldots, f_r)\}$, $z \in \Omega$. The coherence implies that the zero set $Z = V(\{J_z\})$ of this family is an analytic subset in Ω.

Fix now $i \in \{1, \ldots, s\}$; we will show that $g_a \in J_a^i$. As $\widetilde{X}$ is a dense subset of X, by assumption we obtain $X_a^i \not\subset Z_a$ and hence $I(Z_a) \not\subset I(X_a^i)$. So there exists $h_a \in I(Z_a)$ such that $h_a \notin I(X_a^i)$. By the Nullstellensatz, $h_a^l \in J_a$ for some $l \in \mathbb{N}$ and as $I(X_i)_a$ is a prime ideal, $h_a^l \notin \operatorname{Rad} J_a^i = I((X_i)_a)$.

By definition of J_a we have $h_a^l g_a \in I_a$ and so $h_a^l g_a \in J_a^i$. As J_a^i is a primary ideal we get $g_a \in J_a^i$ and $I_a(f_1, \ldots, f_r) = J_a^1 \cap \ldots \cap J_a^s$. ∎

COROLLARY 3.6. *Suppose that* $f = (f_1, \ldots, f_r)$ *realizes a set-theoretic complete intersection in* U *and* $Z_f = \sum_i k_i X_i$. *Let* $g_i : U \to \mathbb{C}$ *be a holomorphic function such that* $g_i|_{X_i} = 0$ *for* $i = 1, \ldots, s$. *Then there exists a neighbourhood* $W \subset U$ *of zero in* $\mathbb{C}^m$ *such that* $\prod_{i=1}^s g_i^{k_i}(z) = \sum_{j=1}^r h_j(z) f_j(z)$ *for some functions* h_j *holomorphic in* W.

Proof. Take $h := g_1^{k_1} \ldots g_s^{k_s}$ and note that for each $i = 1, \ldots, s$ there exists a dense subset $\widetilde{X}_i$ of X_i such that $h_z \in I_z(f_1, \ldots, f_r)$ for $z \in \widetilde{X}_i$. This is a direct consequence of Theorem 3.2 and the fact that at regular points of X_i we have $\nu(Z_f, z) = k_i$.

Now Theorem 3.5 implies the result. ∎

From the above corollary we get immediately the following separation type inequality ([Cy], [CKT]):

REMARK 3.7. (a) In the situation of Theorem 3.5 there exists a neighbourhood of zero in $\mathbb{C}^m$ in which

$$|g_1(z)|^{k_1} \ldots |g_s(z)|^{k_s} \le C|(f_1, \ldots, f_r)(z)| \le cd(z, X)$$

for some constants $C, c > 0$, where $d(\cdot, X)$ denotes the distance to the set $X \subset \mathbb{C}^m$.

(b) Working with separation problems one can naturally wonder if a more general inequality with the distance functions could hold:

$$d(z, X_1)^{k_1} \ldots d(z, X_s)^{k_s} \le c|(f_1, \ldots, f_r)(z)|$$

for some constant $c > 0$. Unfortunately, this inequality is false as the example of the function $f(x, y) = x^3 - y^2$ shows.

THEOREM 3.8. *Let* $f = (f_1, \ldots, f_p) : U \to \mathbb{C}^p$ *be a holomorphic mapping where* U *is an open neighbourhood of* $a \in \mathbb{C}^n$. *Let* $\mathcal{H} = (H_1, \ldots, H_p)$ *be an*

admissible system of linear hyperplanes for $S = Y_f$ at a and suppose that G_f and $\mathcal{H}$ realize a quasi-complete intersection at $b = (a, f(a))$. If $g : U \to \mathbb{C}$ is a holomorphic function vanishing on the zero set of $(f_1, \ldots, f_p)$, then $g_a^d \in I_a(f_1, \ldots, f_p)$ for $d = \nu(G_f \cdot \mathcal{H}, b)$.

Proof. First note that without loss of generality we can assume that in the neighbourhood U we have $H_i = \{y_i = 0\}$.

Step (1). By assumption, every component of the cycle $Z_1 - Z_1^S = k_1 Y_1 + \ldots + k_{r_1} Y_{r_1}$ is a complete ideal intersection. Let $I_a(Y_i) = (g_{1,i})$. As g vanishes in a neighbourhood of a on the support of Z_1^S, by Corollary 3.6 we obtain

$$g_a^{\nu(Z_1^S, a)} \prod_{i=1}^{r_1} g_{1,i}^{k_i} \in I_a(f_1).$$

Step (2). Fix now $i \in \{1, \ldots, r_1\}$ and consider the intersection

$$Y_i \cdot H_2 = (Y_i \cdot H_2)^S + Y_i \cdot H_2 - (Y_i \cdot H_2)^S = \sum_{m=1}^{p_{i,2}} l_{i,m} X_{i,m} + \sum_{m=1}^{r_{i,2}} k_{i,m} Y_{i,m}.$$

Once more we know that each $Y_{i,m}$ is a complete ideal intersection at a, so we can write $I_a(Y_{i,m}) = (g_{i,m}^{2,1}, g_{i,m}^{2,2})$ for some germs from $\mathcal{O}_a$. Applying again Corollary 3.6 and the fact that g vanishes in a neighbourhood of a on the support of the first part of the sum, we get

$$g_a^{\sum_{m=1}^{p_{i,2}} l_{i,m}} \prod_{m=1}^{r_{i,2}} (g_{i,m}^{2,s})^{k_{i,m}} \in I_a(g_{1,i}, f_2),$$

and as

$$\sum_{i=1}^{r_1} k_i \Big(\sum_{m=1}^{p_{i,2}} l_{i,m} \Big) \leq \nu(Z_2^S, a)$$

(Corollary 3.4), we have

$$g_a^{\nu(Z_1^S, a) + \nu(Z_2^S, a)} \prod_{i=1}^{r_1} \prod_{m=1}^{r_{i,2}} (g_{i,m}^{2,s})^{k_{i,m} k_i} \in I_a(f_1, f_2)$$

for $s = 1, 2$.

It is important to remember that thanks to Corollary 3.4 the generators $g_{1,i}$ do not create any "outer" multiplicity if we take their cycle.

Step (t). By assumption, every component of the cycle $Z_{t-1} - Z_{t-1}^S = k_1^{t-1} Y_1^{t-1} + \ldots + k_{r_{t-1}}^{t-1} Y_{r_{t-1}}^{t-1}$ is a complete ideal intersection. Fix now $i \in \{1, \ldots, r_{t-1}\}$ and let $I_a(Y_i^{t-1}) = (g_{i,m}^{t-1,1}, g_{i,m}^{t-1,2}, \ldots, g_{i,m}^{t-1,t-1})$. Consider the

intersection
$$Y_i^{t-1} \cdot H_t = (Y_i^{t-1} \cdot H_t)^S + Y_i^{t-1} \cdot H_t - (Y_i^{t-1} \cdot H_t)^S$$
$$= \sum_{m=1}^{p_{i,t}} l_{i,m}^t X_{i,m}^t + \sum_{m=1}^{r_{i,t}} k_{i,m}^t Y_{i,m}^t.$$

Once more we know that each $Y_{i,m}^t$ is a complete ideal intersection at a, so we can write $I_a(Y_{i,m}^t) = (g_{i,m}^{t,1}, g_{i,m}^{t,2}, \ldots, g_{i,m}^{t,t})$ for some germs from $\mathcal{O}_a$. Applying again Corollary 3.6 and the fact that g vanishes in a neighbourhood of a on the support of the first part of the sum, we get

$$g_a^{\sum_{m=1}^{p_{i,t}} l_{i,m}^t} \prod_{m=1}^{r_{i,t}} (g_{i,m}^{t,s})^{k_{i,m}^t} \in I_a(g_{i,m}^{t-1,1}, g_{i,m}^{t-1,2}, \ldots, g_{i,m}^{t-1,t-1}, f_t),$$

and as

$$\sum_{i=1}^{r_{t-1}} k_i^{t-1} \Big(\sum_{m=1}^{p_{i,t}} l_{i,m}^t \Big) \le \nu(Z_t^S, a)$$

(Corollary 3.4), for $s = 1, \ldots, t$ we have

$$g_a^{\nu(Z_1^S, a) + \ldots + \nu(Z_t^S, a)} \prod_{i=1}^{r_t} \prod_{m=1}^{r_{i,t}} (g_{i,m}^{t,s})^{k_{i,m}^t k_i^{t-1}} \in I_a(f_1, \ldots, f_t).$$

By assumption we can repeat the same considerations at each step of the algorithm and finally in the last step we obtain a cycle totally supported by S, so g vanishes on this support in a neighbourhood of a and we get

$$g_a^{\nu(Z_1^S, a) + \ldots + \nu(Z_p^S, a)} \in I_a(f_1, \ldots, f_p).$$

As $\nu(Z_1^S, a) + \ldots + \nu(Z_p^S, a) = \nu(Z \cdot \mathcal{H}, a)$ this ends the proof. ∎

As a direct consequence of the definition of the cycle of zeroes for a holomorphic mapping we get the following corollary.

COROLLARY 3.9. *Let $f = (f_1, \ldots, f_p) : U \to \mathbb{C}^p$ be a holomorphic mapping where U is an open neighbourhood of $a \in \mathbb{C}^n$. Suppose that G_f and Y_f realize a quasi-complete intersection at the point $b = (a, f(a))$. If $g : U \to \mathbb{C}$ is a holomorphic function vanishing on the zero set of $(f_1, \ldots, f_p)$, then $g_a^d \in I_a(f_1, \ldots, f_p)$ for $d = \nu(Z_f, a)$.*

4. Multiplicity of isolated intersection. In the case of an isolated intersection we can use another, equivalent approach to the definition of isolated intersection multiplicity. As in this section we apply the definitions from [ATW], we first recall basic facts on isolated intersections of analytic sets. For the equivalence of this definition with the general one, see [T, Th. 6.6].

Let Z be a pure k-dimensional locally analytic subset of a complex manifold M of dimension m. Let N be a submanifold of M of dimension n such that N intersects Z at an isolated point $a \in M$. We denote by $\mathcal{F}_a(Z, N)$ the set of all locally analytic subsets V of M satisfying:

(1) V has pure dimension $m - k$,
(2) $N_a \subset V_a$,
(3) a is an isolated point of $V \cap Z$,

where N_a, V_a denote the germs of N and V at a.

Observe that for $V \in \mathcal{F}_a(Z, N)$ the intersection of Z and V is proper at a and we can consider the classical intersection multiplicity $i(Z \cdot V; a)$ in the sense of Draper [D] (cf. [W]). We define

$$\widetilde{i}(Z \cdot N; a) = \min\{i(Z \cdot V) : V \in \mathcal{F}_a(Z, N)\},$$
$$\mathcal{P}_a(Z, N) = \{V \in \mathcal{F}_a(Z, N) : i(Z \cdot V; a) = \widetilde{i}(Z \cdot N; a)\}.$$

Note that [ATW, Th. 4.4] gives a full characterization of the family $\mathcal{P}_a(Z, N)$.

Having disposed of this preliminary step we can now turn to the general case. Let X, Y be pure dimensional locally analytic subsets of a complex manifold M such that a is an isolated point of $X \cap Y$. The positive integer

$$i(X \cdot Y; a) = \widetilde{i}((X \times Y) \cdot \Delta_M; (a, a))$$

is defined to be the *multiplicity of intersection* of X and Y at a. It is the same multiplicity as defined in Section 2 [T, Th. 6.6].

If Y is a submanifold the definition of $i(X \cdot Y; a)$ presented above coincides with that of $\widetilde{i}(X \cdot Y; a)$ introduced earlier.

Let $f = (f_1, \ldots, f_k) : U \to \mathbb{C}^k$ be a holomorphic mapping on $U \subset \mathbb{C}^n$, $a \in U$ and assume that $f^{-1}(0) = \{a\}$. Define $Z = G_f$ to be the graph of f in $M = U \times \mathbb{C}^k$ and $N = Y_f = U \times \{0\}^k$. Suppose that g is a holomorphic function on U such that $g(a) = 0$.

THEOREM 4.1 (cf. [S]). *In the above situation there exists a neighbourhood $W \subset U$ of the point a in $\mathbb{C}^n$ and a system of functions $h_1, \ldots, h_k$ holomorphic on W such that $g^d = f_1 h_1 + \ldots + f_k h_k$ on W where $d = i(Z \cdot N, (a, 0)) = \nu(Z_f, a)$.*

Proof. Write $c = (a, 0)$ and let $V \in \mathcal{P}_c(Z; N)$. We know that $i(Z \cdot N; c) = i(Z \cdot V; c)$, and [ATW, Th. 4.4] implies that V_c is a germ of a manifold. We also have $\dim Z = \dim N = n$. As the intersection is isolated it is obvious that $k \geq n$.

Using the standard properties of intersection theory we can assume to have the following situation:

(1) $c = 0$ and $M = U \times B \times D \subset \mathbb{C}^n_x \times \mathbb{C}^n_z \times \mathbb{C}^{k-n}_w$, where B and D are the unit balls in $\mathbb{C}^n$, $\mathbb{C}^{k-n}$ respectively,

(2) $N = U \times \{0\}^k$,

(3) $V = U \times \{0\}^n \times D$,

(4) $Z \cap V = \{0\}$,

(5) $\pi|_Z : Z \to B$ is an analytic covering of multiplicity $d = i(Z \cdot N; 0)$, where $\pi : M \to B$ is the natural projection.

We will now define on $M \times \mathbb{C} \subset \mathbb{C}^n_x \times \mathbb{C}^n_z \times \mathbb{C}^{k-n}_w \times \mathbb{C}_t$ the following function:

$$P(x, z, w, t) = \prod_{i=1}^{d} (t - g(x_i))$$

for $(\pi|_Z)^{-1}(\pi(x, z, w)) = \{(x_1, z, w_1), \ldots, (x_d, z, w_d)\}$, where $(x_i, z, w_i) \in Z$ are counted with their multiplicities. Using standard arguments about symmetric polynomials and the Riemann theorem it is easy to see that $P(x, z, w, t)$ is a holomorphic function.

Observe now that as $N \subset V$ and $(z, w) = (y_1, \ldots, y_n, y_{n+1}, \ldots, y_k)$, the above function has the following properties:

(1) If $(x, z, w) = (x, y_1, \ldots, y_k) = (x, 0)$ then $(\pi|_Z)^{-1}(z) = \{0\}$, so we have $P(x, z, w, t) = t^d$,

(2) $P(x, f(x), g(x)) = 0$.

The first property implies that $P(x, z, w, t) - t^d$ lies in the ideal generated by $y_1, \ldots, y_k$, so there exist holomorphic functions $h_1, \ldots, h_k$ in a neighbourhood W of zero in $M \times \mathbb{C}$ such that $P(x, z, w, t) = t^d + y_1 h_1(x, z, w, t) + \ldots + y_k h_k(x, z, w, t)$ on W. Application of the second property ends the proof. ∎

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