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The sharp version of a criterion for starlikeness related to the operator of Alexander

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Abstract. The method of convolution is used to determine a sharp condition for starlikeness of analytic functions defined in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and having the property $f(0) = f'(0) - 1 = 0$. The integral version is given using the Alexander integral operator.

1. Introduction. Let $\mathcal{H}(U)$ be the class of analytic functions in the unit disc U and let $\mathcal{A} = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$. The subclass of $\mathcal{A}$ consisting of starlike functions is denoted by S^*. It is well known that $S^* = \{f \in \mathcal{A} : \mathrm{Re}(zf'(z)/f(z)) > 0,\ z \in U\}$. In [2] it was proved that if $f \in \mathcal{A}$ satisfies the condition

$$\mathrm{Re}\{f'(z) + zf''(z)\} > -\frac{\pi^2 - 6}{24 - \pi^2} = -0.273\ldots, \qquad z \in U,$$

then $f \in S^*$. This result is equivalent to the result obtained in [1]. In [3] an improvement of the results obtained in [1] and [2] is given. In [1] the author conjectured that if $f \in \mathcal{A}$, then the greatest value of c for which the inequality $\mathrm{Re}(f'(z) + zf''(z)) > -c,\ z \in U$, implies $f \in S^*$, is

$$c = \frac{2\ln 2 - 1}{2(1 - \ln 2)} = 0.629\ldots$$

In this paper this conjecture will be confirmed.

2. Preliminaries. Let $A_0 = \{f \in \mathcal{H}(U) : f(0) = 1\}$ and $\mathcal{P} = \{f \in A_0 : \mathrm{Re}(f(z)) > 0 \text{ for all } z \in U\}$. We will need the following definitions and lemmas to prove the main result. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be two analytic functions in U. The *Hadamard product* of f and g is defined

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by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

For $V \subset A_0$, *the dual set* of V is

$$V^d = \{g \in A_0 : (f * g)(z) \neq 0 \text{ for all } f \in V \text{ and } z \in U\}.$$

Let $h_T \in \mathcal{A}$ be the function defined by

$$h_T(z) = \frac{iT\frac{z}{1-z} + \frac{z}{(1-z)^2}}{1+iT}, \qquad T \in \mathbb{R}.$$

The following two lemmas are established in [4].

LEMMA 1 ([4, p. 94]). *The function $f \in \mathcal{A}$ is in the class S^* if and only if*

$$\frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0 \quad \textit{for all } z \in U \textit{ and } T \in \mathbb{R}.$$

LEMMA 2 ([4, p. 23]). *The dual set of the class $\mathcal{P} = \{f \in A_0 : \mathrm{Re}(f(z)) > 0,\ z \in U\}$ is*

$$\mathcal{P}^d = \{f \in A_0 : \mathrm{Re}(f(z)) > 1/2,\ z \in U\}.$$

LEMMA 3. *If the function $f : [0,1] \to (0,\infty)$ is increasing and continuous, then for every $a \in [0,\infty)$,*

$$\int_0^1 \frac{x^{a+1}}{f(x)}\,dx \leq \frac{a+1}{a+2}\int_0^1 \frac{x^a}{f(x)}\,dx.$$

Proof.

$$\int_0^1 \frac{x^{a+1}}{f(x)}\,dx = \frac{a+1}{a+2}\int_0^1 \frac{x^a}{f\big(x^{\frac{a+1}{a+2}}\big)}\,dx \leq \frac{a+1}{a+2}\int_0^1 \frac{x^a}{f(x)}\,dx. \ \blacksquare$$

LEMMA 4. *If $\theta \in (0,2\pi)$ and $\beta > 0$ then the following identity holds:*

$$(1)\qquad \int_0^\infty \frac{x(e^{\theta x} + e^{(2\pi-\theta)x})}{(\beta^2+x^2)(e^{2\pi x}-1)}\,dx + i\beta\int_0^\infty \frac{e^{(2\pi-\theta)x} - e^{\theta x}}{(\beta^2+x^2)(e^{2\pi x}-1)}\,dx = \frac{1}{2\beta} + \sum_{k=1}^{\infty}\frac{e^{i\theta k}}{k+\beta}.$$

Proof. Consider the function

$$f(z) = \frac{e^{i\theta z}}{(\beta+z)(e^{2\pi i z}-1)}, \qquad \beta > 0,$$

where $\theta \in (0,2\pi)$ is a fixed number. Let $\mathbb{N}$ denote the set of natural numbers. The contour $\Gamma(r,n)$ is constructed as follows: $\Gamma(r,n) = \gamma_1 \cup \gamma_3 \cup \gamma_2 \cup \gamma_4$ where $\gamma_1(t) = R_n e^{i(\pi t - \pi/2)}$, $\gamma_2(t) = re^{i(-\pi t+\pi/2)}$, $\gamma_3(t) = iR_n + t(ir - iR_n)$,

$\gamma_4(t) = -ir + t(ir - iR_n)$, $t \in [0,1]$, $r \in (0,1)$ and $R_n = n + 1/2$, $n \in \mathbb{N}$. The residue theorem implies that

$$\int_{\Gamma(r,n)} f(z)\,dz = 2\pi i \sum_{k\in\mathbb{N},\,0<k<n+1/2} \operatorname{Res}(f,k). \tag{2}$$

A computation leads to

$$\begin{aligned} &\lim_{r\to 0} \int_{\gamma_2} f(z)\,dz = -i\pi \cdot \operatorname{Res}(f,0), \\ &\operatorname{Res}(f,z_k) = \operatorname{Res}(f,k) = \frac{e^{i\theta k}}{2\pi i(k+\beta)}, \qquad k \in \mathbb{N}. \end{aligned} \tag{3}$$

We now give a detailed proof for the equality

$$\lim_{R_n\to\infty} \int_{\gamma_1} f(z)\,dz = 0. \tag{4}$$

Let $(\alpha_n)_{n\geq 2}$ be the sequence defined by

$$\alpha_n = \frac{\ln n}{4\pi^2(n+1/2)}, \qquad n \geq 2.$$

We have

$$\begin{aligned} \Big|\int_{\gamma_1} f(z)\,dz\Big| \leq &\Big|\int_0^{1/2-\alpha_n} f(\gamma_1(t))\gamma_1'(t)\,dt\Big| \\ &+ \Big|\int_{1/2-\alpha_n}^{1/2+\alpha_n} f(\gamma_1(t))\gamma_1'(t)\,dt\Big| + \Big|\int_{1/2+\alpha_n}^{1} f(\gamma_1(t))\gamma_1'(t)\,dt\Big|. \end{aligned} \tag{5}$$

If $n > \beta$, the inequalities $|e^{2\pi i\gamma_1(t)} - 1| \geq e^{-2\pi(n+1/2)\sin(\pi t-\pi/2)} - 1$ and $|\gamma_1(t) + \beta| \geq n + 1/2 - \beta$ for $t \in (0,1/2)$ imply that

$$\begin{aligned} &\Big|\int_0^{1/2-\alpha_n} f(\gamma_1(t))\gamma_1'(t)\,dt\Big| \\ &\quad \leq \int_0^{1/2-\alpha_n} \frac{e^{-\theta(n+1/2)\sin(\pi t-\pi/2)}}{e^{-2\pi(n+1/2)\sin(\pi t-\pi/2)} - 1}\,\frac{n+1/2}{n+1/2-\beta}\,dt \\ &\quad \leq \frac{1}{2}\,\frac{1}{e^{(2\pi-\theta)(n+1/2)\sin(\pi\alpha_n)} - 1}\,\frac{n+1/2}{n+1/2-\beta} \xrightarrow[n\to\infty]{} 0. \end{aligned} \tag{6}$$

Since $\lim_{t\to 1/2} e^{2\pi i\gamma_1(t)} = -1$ and $\lim_{n\to\infty} \alpha_n = 0$, there is a natural number n_0 such that $|e^{2\pi i\gamma_1(t)} - 1| > 1$ for all $n > n_0$ and all $t \in [1/2-\alpha_n, 1/2+\alpha_n]$.

Consequently, the second term on the right side of (5) can be estimated by

$$(7)\qquad \int\limits_{1/2-\alpha_n}^{1/2+\alpha_n} \frac{n+1/2}{n+1/2-\beta}\, e^{-\theta(n+1/2)\sin(\pi t-\pi/2)}\, dt$$
$$\le 2\alpha_n \frac{n+1/2}{n+1/2-\beta}\, e^{\theta(n+1/2)\sin(\pi\alpha_n)}$$
$$\le 2\alpha_n \frac{n+1/2}{n+1/2-\beta}\, e^{2\pi(n+1/2)\pi\alpha_n} = 2\alpha_n \frac{n+1/2}{n+1/2-\beta}\sqrt{n} \xrightarrow[n\to\infty]{} 0.$$

Finally, the inequalities

$$|e^{i\theta\gamma_1(t)}| \le e^{-\theta(n+1/2)\sin(\pi t-\pi/2)}, \qquad |e^{2\pi i\gamma_1(t)}-1| \ge 1-e^{-2\pi(n+1/2)\sin(\pi t-\pi/2)}$$

for $t\in[1/2+\alpha_n,1]$ imply that the third term on the right side on (5) is at most

$$(8)\qquad \int\limits_{1/2+\alpha_n}^{1} \frac{e^{-\theta(n+1/2)\sin(\pi t-\pi/2)}}{1-e^{-2\pi(n+1/2)\sin(\pi t-\pi/2)}}\, \frac{n+1/2}{n+1/2-\beta}\, dt$$
$$\le \frac{1}{2}\, \frac{e^{-\theta(n+1/2)\sin(\pi\alpha_n)}}{1-e^{-2\pi(n+1/2)\sin(\pi\alpha_n)}}\, \frac{n+1/2}{n+1/2-\beta} \xrightarrow[n\to\infty]{} 0.$$

Summarizing, we see that (4) follows from (5)–(8). From (2)–(4) we obtain

$$\lim_{\substack{R_n\to\infty\\ r\to 0}} \Big(\int\limits_{\gamma_3} f(z)\,dz + \int\limits_{\gamma_4} f(z)\,dz\Big) = \frac{1}{2\beta} + \sum_{k=1}^{\infty} \frac{e^{i\theta k}}{k+\beta},$$

which is equivalent to

$$(9)\qquad \int\limits_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x})}{(\beta^2+x^2)(e^{2\pi x}-1)}\, dx + i\beta \int\limits_0^\infty \frac{e^{(2\pi-\theta)x}-e^{\theta x}}{(\beta^2+x^2)(e^{2\pi x}-1)}\, dx$$
$$= \frac{1}{2\beta} + \sum_{k=1}^{\infty} \frac{e^{i\theta k}}{k+\beta}. \blacksquare$$

LEMMA 5. *Let* $f:[a,b]\times[0,\infty)\to\mathbb{R}$ *be a continuous function and let* $g:[0,\infty)\to\mathbb{R}$ *be an increasing non-constant function. Suppose that*:

(i) $\big|f(\beta_1,x)-f(\beta_2,x)\big| \le |\beta_1-\beta_2|q(x)$ *for all* $\beta_1,\beta_2\in[a,b]$ *and* $x\in[0,\infty)$, *where* $q:[0,\infty)\to[0,\infty)$ *is a continuous function and the improper integral* $\int_0^\infty q(x)\,dg(x)$ *is convergent*,

(ii) *the function* f *is differentiable with respect to the first variable*, *and* $\partial f/\partial\beta:(a,b)\times[0,\infty)\to\mathbb{R}$ *is continuous*,

(iii) *the parametric improper integral* $\int_0^\infty \frac{\partial f}{\partial\beta}(\beta,x)\,dg(x)$ *is uniformly convergent on every compact subset of* (a,b).

Then the function

$$h : [a,b] \to \mathbb{R}, \qquad h(\beta) = \int_0^\infty f(\beta, x)\, dg(x),$$

is differentiable at every $\beta_0 \in (a,b)$ *and*

$$h'(\beta_0) = \int_0^\infty \frac{\partial f}{\partial \beta}(\beta_0, x)\, dg(x).$$

Proof. Let $\beta_0 \in (a,b)$. There is a $\delta > 0$ for which $I = [\beta_0 - \delta, \beta_0 + \delta] \subset (a,b)$.

According to (i) and (iii) there exists an $N > 0$ so that $g(N) > g(0)$ and

$$\left| \int_M^\infty \frac{\partial f}{\partial \beta}(\beta, x)\, dg(x) \right| < \frac{\varepsilon}{3} \quad \text{and} \quad \left| \int_M^\infty q(x)\, dg(x) \right| < \frac{\varepsilon}{3} \qquad \text{for } M \in [N, \infty) \text{ and } \beta \in I. \tag{10}$$

Condition (i) also implies that

$$\int_N^\infty \left| \frac{f(\beta, x) - f(\beta_0, x)}{\beta - \beta_0} \right| dg(x) \le \int_N^\infty q(x)\, dg(x). \tag{11}$$

The (Lagrange) mean value theorem implies that there is a $\theta_{(\beta,x)} \in (0,1)$ such that

$$\frac{f(\beta, x) - f(\beta_0, x)}{\beta - \beta_0} = \frac{\partial f}{\partial \beta}(\beta + \theta_{(\beta,x)}(\beta_0 - \beta), x).$$

Since $\partial f / \partial \beta$ is uniformly continuous on the compact set $I \times [0, N]$, there is a $\delta' \in (0, \delta)$ such that if $|\beta - \beta_0| < \delta'$, then

$$\left| \frac{\partial f}{\partial \beta}(\beta, x) - \frac{\partial f}{\partial \beta}(\beta_0, x) \right| < \frac{\varepsilon}{3(g(N) - g(0))} \qquad \text{for } x \in [0, N].$$

If $|\beta - \beta_0| < \delta'$, then $|\beta + \theta_{(\beta,x)}(\beta_0 - \beta) - \beta_0| < \delta'$, and so

$$\left| \frac{f(\beta, x) - f(\beta_0, x)}{\beta - \beta_0} - \frac{\partial f}{\partial \beta}(\beta_0, x) \right| < \frac{\varepsilon}{3(g(N) - g(0))} \qquad \text{for } x \in [0, N]. \tag{12}$$

Provided that $0 < |\beta - \beta_0| < \delta'$, inequalities (10)–(12) imply

$$\left| \frac{h(\beta) - h(\beta_0)}{\beta - \beta_0} - \int_0^\infty \frac{\partial f}{\partial \beta}(\beta_0, x) dg(x) \right|$$
$$= \left| \int_0^\infty \left(\frac{f(\beta, x) - f(\beta_0, x)}{\beta - \beta_0} - \frac{\partial f}{\partial \beta}(\beta_0, x) \right) dg(x) \right|$$

$$\leq \int_0^N \left| \frac{f(\beta,x)-f(\beta_0,x)}{\beta-\beta_0} - \frac{\partial f}{\partial \beta}(\beta_0,x) \right| dg(x)$$

$$+ \int_N^\infty \left| \frac{f(\beta,x)-f(\beta_0,x)}{\beta-\beta_0} \right| dg(x) + \left| \int_N^\infty \frac{\partial f}{\partial \beta}(\beta_0,x)\, dg(x) \right|$$

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

and the proof is complete. ■

Remark 1. If we put, in Lemma 5, $g(x) = [x]$ where $[x]$ denotes the integer part of x, then the improper integral $\int_0^\infty f(\beta,x)\, dg(x)$ and the series $\sum_{n=1}^\infty f(\beta,n)$ are convergent or divergent at the same time. If they are convergent then

$$\int_0^\infty f(\beta,x)\, dg(x) = \sum_{n=1}^\infty f(\beta,n).$$

Hence we obtain an analogous lemma for the differentiability of series.

3. The main result

Theorem 1. *If $f \in \mathcal{A}$ and*

$$\mathrm{Re}(f'(z) + zf''(z)) > \frac{1-2\ln 2}{2(1-\ln 2)} \qquad \text{for } z \in U,$$

then f belongs to the class S^*, and the result is sharp.

Proof. The condition $\mathrm{Re}(f'(z) + zf''(z)) > -\alpha$, $z \in U$, is equivalent to

$$\mathrm{Re}\left(\frac{\alpha + f'(z) + zf''(z)}{1+\alpha} \right) > 0, \qquad z \in U,$$

which can be rewritten as

$$\frac{\alpha + f' + zf''}{1+\alpha} \in \mathcal{P}.$$

According to the representation theorem of Herglotz there is a probability measure μ on $[0, 2\pi]$ so that

$$\frac{\alpha + f'(z) + zf''(z)}{1+\alpha} = \int_0^{2\pi} \frac{1+e^{-it}z}{1-e^{-it}z}\, d\mu(t),$$

hence, if $f(z) = z + \sum_{n=1}^\infty a_n z^n$, then

$$1 + \frac{1}{1+\alpha} \sum_{n=2}^\infty n^2 a_n z^{n-1} = 1 + 2 \sum_{n=1}^\infty z^n \int_0^{2\pi} e^{-int}\, d\mu(t).$$

This implies $a_n = \frac{2(1+\alpha)}{n^2}\int_0^{2\pi} e^{-i(n-1)t}\,d\mu(t)$ for $n \in \mathbb{N}$, $n \geq 2$ and

$$f(z) = z + 2(1+\alpha)\sum_{n=2}^{\infty}\frac{z^n}{n^2}\int_0^{2\pi} e^{-i(n-1)t}\,d\mu(t).$$

By Lemma 1, we have to prove that

$$\text{(13)}\qquad \frac{f(z)}{z} * \frac{h_T(z)}{z} = \Big(1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int}\,d\mu(t)\Big) * \left(1 + (1+\alpha)\sum_{n=1}^{\infty}\frac{1+n+iT}{(1+iT)(n+1)^2}\,z^n\right) \neq 0, \qquad z \in U,\ T \in \mathbb{R}.$$

The function defined by the development $1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int}\,d\mu(t)$ belongs to the class $\mathcal{P}$ for every $z \in U$ and every probability measure μ. From Lemma 2 and (13) it follows that f is starlike if and only if the function

$$g(z) = 1 + (1+\alpha)\sum_{n=1}^{\infty}\frac{1+n+iT}{(1+iT)(n+1)^2}\,z^n, \qquad z \in U,$$

belongs to the dual set $\mathcal{P}^d$, that is,

$$\operatorname{Re}\left(1 + (1+\alpha)\sum_{n=1}^{\infty}\frac{1+n+iT}{(1+iT)(n+1)^2}\,z^n\right) > \frac{1}{2}, \qquad z \in U,\ T \in \mathbb{R}.$$

This inequality is equivalent to

$$\text{(14)}\qquad \operatorname{Re}\left(\frac{1}{2(1+\alpha)} + \sum_{n=1}^{\infty}\frac{1+n+iT}{(1+iT)(n+1)^2}\,z^n\right) > 0, \qquad z \in U,\ T \in \mathbb{R}.$$

We will determine

$$\inf_{\substack{\theta\in[0,2\pi]\\ T\in\mathbb{R}}} \operatorname{Re}\left(\sum_{n=1}^{\infty}\frac{1+n+iT}{(1+iT)(n+1)^2}\,e^{in\theta}\right) =: m.$$

If $1/(2(1+\alpha)) = -m$, then α will be the greatest real number for which (14) is valid. Let us introduce the notation

$$\begin{aligned} M(\theta,T) &= \operatorname{Re}\left(\sum_{n=1}^{\infty}\frac{1+n+iT}{(1+iT)(n+1)^2}\,e^{in\theta}\right) \\ &= \operatorname{Re}\Bigg(-1 + \frac{1}{1+T^2}\sum_{n=0}^{\infty}\frac{e^{in\theta}}{n+1} + \frac{T^2}{1+T^2}\sum_{n=0}^{\infty}\frac{e^{in\theta}}{(n+1)^2} \\ &\qquad - \frac{iT}{1+T^2}\sum_{n=0}^{\infty}\frac{e^{in\theta}}{n+1} + \frac{iT}{1+T^2}\sum_{n=0}^{\infty}\frac{e^{in\theta}}{(n+1)^2}\Bigg). \end{aligned}$$

The following identities hold for every $\theta \in (0, 2\pi)$:

$$\sum_{n=0}^{\infty} \frac{e^{in\theta}}{n+1} = \int_0^1 \frac{1 - t\cos\theta}{1 + t^2 - 2t\cos\theta}\,dt + i\sin\theta \int_0^1 \frac{t}{1 + t^2 - 2t\cos\theta}\,dt,$$

$$\sum_{n=0}^{\infty} \frac{e^{in\theta}}{(n+1)^2} = \int_0^1\!\!\int_0^1 \frac{1 - tx\cos\theta}{1 + t^2x^2 - 2tx\cos\theta}\,dt\,dx + i\sin\theta \int_0^1\!\!\int_0^1 \frac{tx}{1 + t^2x^2 - 2tx\cos\theta}\,dt\,dx,$$

and imply

$$\begin{aligned} M(\theta, T) = -1 &+ \frac{1}{1+T^2}\int_0^1 \frac{1 - t\cos\theta}{1 + t^2 - 2t\cos\theta}\,dt \\ &+ \frac{T^2}{1+T^2}\int_0^1\!\!\int_0^1 \frac{1 - tx\cos\theta}{1 + t^2x^2 - 2tx\cos\theta}\,dt\,dx \\ &+ \frac{T\sin\theta}{1+T^2}\int_0^1 \frac{t}{1 + t^2 - 2t\cos\theta}\,dt \\ &- \frac{T\sin\theta}{1+T^2}\int_0^1\!\!\int_0^1 \frac{tx}{1 + t^2x^2 - 2tx\cos\theta}\,dt\,dx. \end{aligned}$$

Using the identities

$$\int_0^1 \frac{1 - t\cos\theta}{1 + t^2 - 2\cos\theta}\,dt = \int_0^1 \frac{1}{1+t}\,dt + (1+\cos\theta)\int_0^1 \frac{t(1-t)}{(1+t)(1 + t^2 - 2t\cos\theta)}\,dt,$$

$$\begin{aligned} \int_0^1\!\!\int_0^1 \frac{1 - tx\cos\theta}{1 + t^2x^2 - 2tx\cos\theta}\,dt\,dx &= \int_0^1\!\!\int_0^1 \frac{1}{1+tx}\,dt\,dx \\ &+ (1+\cos\theta)\int_0^1\!\!\int_0^1 \frac{tx(1-tx)}{(1+tx)(1 + t^2x^2 - 2tx\cos\theta)}\,dt\,dx, \qquad \theta \in (0, 2\pi), \end{aligned}$$

it follows that

$$M(\theta, T) = -1 + \int_0^1 \frac{1}{1+t}\,dt + \frac{1}{1+T^2}\,L_1(\theta, T),$$

where

$$\begin{aligned} L_1(\theta, T) := (1+\cos\theta)&\int_0^1 \frac{t(1-t)}{(1+t)(1 + t^2 - 2t\cos\theta)}\,dt \\ &+ T\sin\theta \int_0^1\!\!\int_0^1 \frac{t(1-x)(1-t^2x)}{(1 + t^2 - 2t\cos\theta)(1 + t^2x^2 - 2tx\cos\theta)}\,dt\,dx \end{aligned}$$

$$+ T^2\left(\int_0^1\!\!\int_0^1 \frac{t(1-x)}{(1+t)(1+tx)}\,dt\,dx + (1+\cos\theta)\int_0^1\!\!\int_0^1 \frac{tx(1-tx)}{(1+tx)(1+t^2x^2-2tx\cos\theta)}\,dt\,dx\right).$$

If we can prove that

$$\inf_{\substack{\theta\in[0,2\pi]\\ T\in\mathbb{R}}} L_1(\theta,T)=0,$$

then we will obtain

$$\inf_{\substack{\theta\in[0,2\pi]\\ T\in\mathbb{R}}} M(\theta,T)=-1+\ln 2.$$

Since $L_1(\theta,T)$ is a polynomial of the second degree in T with a positive dominant coefficient and $L_1(\pi,0)=0$, it is sufficient to show that the discriminant of $L_1(\theta,T)$ satisfies $\varDelta_{L_1}(\theta)\le 0$ for all $\theta\in(0,2\pi)$. Now,

$$\varDelta_{L_1}(\theta)=\sin^2\theta\left(\int_0^1\!\!\int_0^1 \frac{t(1-x)(1-t^2x)}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\,dt\,dx\right)^2 - 4(1+\cos\theta)\int_0^1 \frac{t(1-t)}{(1+t)(1+t^2-2t\cos\theta)}\,dt\left(\int_0^1\!\!\int_0^1 \frac{t(1-x)}{(1+t)(1+tx)}\,dt\,dx + (1+\cos\theta)\int_0^1\!\!\int_0^1 \frac{tx(1-tx)}{(1+tx)(1+t^2x^2-2tx\cos\theta)}\,dt\,dx\right).$$

Because $\sup_{\theta\in[0,2\pi]}\varDelta_{L_1}(\theta)=\sup_{\theta\in[0,\pi]}\varDelta_{L_1}(\theta)$ we can restrict to $\theta\in[0,\pi]$. Some calculations lead to

$$\varDelta_{L_1}(\theta)=4\cos^2\frac{\theta}{2}\left[\sin^2\frac{\theta}{2}\left(\int_0^1\!\!\int_0^1 \frac{t(1-x)(1-t^2x)}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\,dt\,dx\right)^2 - 2\int_0^1 \frac{t(1-t)}{(1+t)(1+t^2-2t\cos\theta)}\,dt\left(\int_0^1\!\!\int_0^1 \frac{t(1-x)}{(1+t)(1+tx)}\,dt\,dx + (1+\cos\theta)\int_0^1\!\!\int_0^1 \frac{tx(1-tx)}{(1+tx)(1+t^2x^2-2tx\cos\theta)}\,dt\,dx\right)\right].$$

We will prove the inequality

$$\sin^2\frac{\theta}{2}\left(\int_0^1\!\!\int_0^1 \frac{t(1-x)(1-t^2x)}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\,dt\,dx\right)^2 \le 2\int_0^1 \frac{t(1-t)}{(1+t)(1+t^2-2t\cos\theta)}\,dt\int_0^1\!\!\int_0^1 \frac{t(1-x)}{(1+t)(1+tx)}\,dt\,dx \tag{15}$$

for $\theta \in [\pi/2, \pi]$, which implies the claimed condition $\Delta_{L_1}(\theta) \leq 0$ in the case $\theta \in [\pi/2, \pi]$.

If $\theta \in [\pi/2, \pi]$ the function $g : [0, 1] \to \mathbb{R}$, $g(t) = (1+t)(1+t^2-2t\cos\theta)$, is strictly increasing and so Lemma 3 implies that

$$\frac{2}{3}\int_0^1 \frac{t}{(1+t)(1+t^2-2t\cos\theta)}\,dt \geq \int_0^1 \frac{t^2}{(1+t)(1+t^2-2t\cos\theta)}\,dt,$$

which is equivalent to

$$\begin{aligned}(16)\qquad &\int_0^1 \frac{t-t^2}{(1+t)(1+t^2-2t\cos\theta)}\,dt \\ &\qquad\geq \frac{2}{3}\int_0^1\int_0^1 \frac{t(1-x)}{(1+t)(1+t^2-2t\cos\theta)}\,dt\,dx, \qquad \theta \in [\pi/2, \pi].\end{aligned}$$

This shows that it is enough to prove

$$\begin{aligned}(17)\qquad &\sin^2\frac{\theta}{2}\left(\int_0^1\int_0^1 \frac{t(1-x)(1-t^2x)}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\,dt\,dx\right)^2 \\ &\qquad\leq \frac{4}{3}\int_0^1\int_0^1 \frac{t(1-x)}{(1+t)(1+t^2-2t\cos\theta)}\,dt\,dx \int_0^1\int_0^1 \frac{t(1-x)}{(1+t)(1+tx)}\,dt\,dx.\end{aligned}$$

This will yield (15) in the case $\theta \in [\pi/2, \pi]$.

The inequality of Cauchy–Schwarz implies

$$\begin{aligned}(18)\qquad &\left(\int_0^1\int_0^1 \frac{t(1-x)}{(1+t)\sqrt{(1+t^2-2t\cos\theta)(1+tx)}}\,dt\,dx\right)^2 \\ &\qquad\leq \int_0^1\int_0^1 \frac{t(1-x)}{(1+t)(1+t^2-2t\cos\theta)}\,dt\,dx \int_0^1\int_0^1 \frac{t(1-x)}{(1+t)(1+tx)}\,dt\,dx\end{aligned}$$

for $\theta \in [0, \pi]$. If we prove

$$\begin{aligned}(19)\qquad &\left|\sin\frac{\theta}{2}\right|\int_0^1\int_0^1 \frac{t(1-x)(1-t^2x)}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\,dt\,dx \\ &\qquad\leq \frac{2}{\sqrt{3}}\int_0^1\int_0^1 \frac{t(1-x)}{(1+t)\sqrt{(1+t^2-2t\cos\theta)(1+tx)}}\,dt\,dx, \qquad \theta \in [\pi/2, \pi],\end{aligned}$$

then this inequality and (18) imply (17).

Inequality (19) can be deduced from

$$\frac{t(1-x)(1-t^2x)\sin\frac{\theta}{2}}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)} \le \frac{2}{\sqrt{3}}\,\frac{t(1-x)}{(1+t)\sqrt{(1+t^2-2t\cos\theta)(1+tx)}}$$

for $\theta \in [\pi/2,\pi]$ and $t, x \in [0,1]$. The latter inequality is valid because through direct calculations it can be shown that

$$\frac{\sin\frac{\theta}{2}}{\sqrt{1+t^2-2t\cos\theta}} \le \frac{1}{1+t} \quad\text{and}\quad \frac{1-t^2x}{1+t^2x^2-2tx\cos\theta} \le \frac{2}{\sqrt{3(1+tx)}},$$

for $\theta \in [\pi/2,\pi,\ t \in [0,1]$.

In order to finish the proof, we must also prove that $L_1(\theta,T) \ge 0$ for $\theta \in [0,\pi/2]$ and $T \in \mathbb{R}$. To do this, we will apply Lemmas 4 and 5.

We have proved in Lemma 4 the identity

$$\int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x})}{(\beta^2+x^2)(e^{2\pi x}-1)}\,dx + i\beta\int_0^\infty \frac{e^{(2\pi-\theta)x}-e^{\theta x}}{(\beta^2+x^2)(e^{2\pi x}-1)}\,dx = \frac{1}{2\beta}+\sum_{k=1}^{\infty}\frac{e^{i\theta k}}{k+\beta}. \tag{20}$$

Let f_1, f_2 be the functions defined by

$$f_1, f_2 : [1/2,3/2]\times[0,\infty) \to \mathbb{R}, \quad f_1(\beta,x) = \frac{x(e^{\theta x}+e^{(2\pi-\theta)x})}{(\beta^2+x^2)(e^{2\pi x}-1)},$$
$$f_2(\beta,x) = \frac{e^{(2\pi-\theta)x}-e^{\theta x}}{(\beta^2+x^2)(e^{2\pi x}-1)}.$$

Since

$$|f_k(\beta_1,x)-f_k(\beta_2,x)| \le |\beta_1-\beta_2|\,\frac{3x(e^{2\pi x}+1)}{((1/2)^2+x^2)^2(e^{2\pi x}-1)} =: |\beta_1-\beta_2|q(x),$$
$$x \in (0,\infty),\ \beta \in [1/2,3/2],\ k=1,2,$$

and the integral $\int_0^\infty q(x)\,dx$ is convergent, the functions f_1, f_2 satisfy condition (i) of Lemma 5. Condition (ii) is automatically satisfied. Since

$$\left|\frac{\partial f_k}{\partial\beta}(\beta,x)\right| \le \frac{3x(e^{2\pi x}+1)}{((1/2)^2+x^2)^2(e^{2\pi x}-1)},$$
$$x \in (0,\infty),\ \beta \in [1/2,3/2],\ k=1,2,$$

according to a theorem due to Weierstrass the convergence of the integral

$$\int_0^\infty \frac{x(e^{2\pi x}+1)}{((1/2)^2+x^2)^2(e^{2\pi x}-1)}\,dx$$

implies that the integral $\int_0^\infty \frac{\partial f}{\partial \beta}(\beta,x)\,dx$ is uniformly convergent with respect to β on $[1/2,3/2]$, and so condition (iii) of Lemma 5 is also satisfied.

If $\beta \in \big(1/2,3/2\big)$ then from Lemma 5 and Remark 1 we see that both sides of the equality (20) can be differentiated with respect to β:

$$(21)\qquad 2\beta\int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x})}{(\beta^2+x^2)^2(e^{2\pi x}-1)}\,dx + i\int_0^\infty \frac{(\beta^2-x^2)(e^{(2\pi-\theta)x}-e^{\theta x})}{(\beta^2+x^2)^2(e^{2\pi x}-1)}\,dx = \frac{1}{2\beta^2}+\sum_{k=1}^\infty \frac{e^{i\theta k}}{(k+\beta)^2}.$$

In particular, (20) and (21) imply that

$$\int_0^\infty \frac{2xe^{\pi x}}{(1+x^2)(e^{2\pi x}-1)}\,dx = \frac{1}{2}+\sum_{n=1}^\infty \frac{(-1)^n}{n+1} = -\frac{1}{2}+\ln 2,$$
$$\int_0^\infty \frac{4xe^{\pi x}}{(1+x^2)^2(e^{2\pi x}-1)}\,dx = \frac{1}{2}+\sum_{n=1}^\infty \frac{(-1)^n}{(n+1)^2} = \frac{\pi^2}{12}-\frac{1}{2}.$$

Using (20) and (21) for $\beta=1$ and the particular cases, we find that

$$M(\theta,T) = -1+\ln 2+\frac{1}{1+T^2}\,L_2(\theta,T)+\frac{T^2}{1+T^2}\bigg(\frac{\pi^2}{12}-\ln 2\bigg),$$

where

$$(22)\qquad L_2(\theta,T) = \int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x}-2e^{\pi x})}{(1+x^2)(e^{2\pi x}-1)}\,dx + T\int_0^\infty \frac{2x^2(e^{(2\pi-\theta)x}-e^{\theta x})}{(1+x^2)^2(e^{2\pi x}-1)}\,dx + T^2\int_0^\infty \frac{2x(e^{\theta x}+e^{(2\pi-\theta)x}-2e^{\pi x})}{(1+x^2)^2(e^{2\pi x}-1)}\,dx.$$

We have to show that $L_2(\theta,T)>0$ for $\theta\in[0,\pi/2]$ and $T\in\mathbb{R}$. $L_2(\theta,T)$ is a polynomial of the second degree with respect to T with discriminant

$$\Delta_2(\theta) = 4\bigg(\int_0^\infty \frac{x^2(e^{(2\pi-\theta)x}-e^{\theta x})}{(1+x^2)^2(e^{2\pi x}-1)}\,dx\bigg)^2 - 8\int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x}-2e^{\pi x})}{(1+x^2)^2(e^{2\pi x}-1)}\,dx\int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x}-2e^{\pi x})}{(1+x^2)(e^{2\pi x}-1)}\,dx.$$

The Cauchy–Schwarz inequality implies

$$(23)\qquad \int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x}-2e^{\pi x})}{(1+x^2)^2(e^{2\pi x}-1)}\,dx\int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x}-2e^{\pi x})}{(1+x^2)(e^{2\pi x}-1)}\,dx \ge \bigg(\int_0^\infty \frac{x\sqrt{1+x^2}(e^{\theta x}+e^{(2\pi-\theta)x}-2e^{\pi x})}{(1+x^2)^2(e^{2\pi x}-1)}\,dx\bigg)^2.$$

The inequality

$$(24)\qquad \sqrt{2}\int_0^\infty \frac{x\sqrt{1+x^2}(e^{\theta x}+e^{(2\pi-\theta)x}-2e^{\pi x})}{(1+x^2)^2(e^{2\pi x}-1)}\,dx \geq \int_0^\infty \frac{x^2(e^{(2\pi-\theta)x}-e^{\theta x})}{(1+x^2)^2(e^{2\pi x}-1)}\,dx, \qquad \theta\in[0,\pi/2],$$

and (23) imply that $\Delta_2(\theta)\leq 0$ for $\theta\in[0,\pi/2]$, which leads to $L_2(\theta,T)\geq 0$ for $\theta\in[0,\pi/2]$ and $T\in\mathbb{R}$.

On the other hand, the inequality

$$\sqrt{2(1+x^2)}(e^{\theta x}+e^{(2\pi-\theta)x}-2e^{\pi x})\geq x(e^{(2\pi-\theta)x}-e^{\theta x})$$

for $x\in(0,\infty)$ and $\theta\in[0,\pi/2]$ implies (24). This inequality is equivalent to

$$e^{(\pi-\theta)x}\geq \frac{\sqrt{2(1+x^2)}+x}{\sqrt{2(1+x^2)}-x}, \qquad x\in[0,\infty),\ \theta\in[0,\pi/2].$$

To check this last inequality, it is sufficient to consider the case $\theta=\pi/2$, and consequently only

$$e^{\frac{\pi}{2}x}\geq \frac{\sqrt{2(1+x^2)}+x}{\sqrt{2(1+x^2)}-x}, \qquad x\in[0,\infty),$$

must be proved. This is easily done using the derivative of the function

$$h:[0,\infty)\to\mathbb{R}, \qquad h(x)=e^{\frac{\pi}{2}x}-\frac{\sqrt{2(1+x^2)}+x}{\sqrt{2(1+x^2)}-x}. \ \blacksquare$$

THEOREM 2. *The largest value of c for which the condition*

$$(25)\qquad f\in\mathcal{A}, \qquad \mathrm{Re}(f'(z)+zf''(z))>-c, \qquad z\in U,$$

implies the univalence of the function f, is $c=\frac{2\ln 2-1}{2(1-\ln 2)}$.

Proof. According to the proof of Theorem 1, condition (25) implies that

$$f(z)=z+2(1+c)\sum_{n=2}^\infty \frac{z^n}{n^2}\int_0^{2\pi} e^{-i(n-1)t}\,d\mu(t)$$

and

$$f'(z)=1+2(1+c)\sum_{n=1}^\infty \frac{z^n}{n+1}\int_0^{2\pi} e^{-int}\,d\mu(t)$$

$$=\Big(1+2\sum_{n=1}^\infty z^n\int_0^{2\pi} e^{-int}\,d\mu(t)\Big)*\Big(1+(1+c)\sum_{n=1}^\infty \frac{z^n}{n+1}\Big).$$

Because $1+2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int}\, d\mu(t) \in \mathcal{P}$, Lemma 2 implies that the necessary condition of univalence, $f'(z) \neq 0$ for $z \in U$, holds if and only if

$$\mathrm{Re}\left(1+(1+c)\sum_{n=1}^{\infty} \frac{z^n}{n+1}\right) > \frac{1}{2}, \qquad z \in U. \tag{26}$$

It is simple to prove that condition (26) is equivalent to

$$\frac{1}{2(1+c)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \geq 0,$$

which can be rewritten in the form

$$c \leq \frac{2\ln 2 - 1}{2(1-\ln 2)}.$$

Thus, by Theorem 1 the proof is finished. ∎

4. Integral version of the result. *Alexander's operator* is defined by

$$A(f)(z) = \int_0^z \frac{f(t)}{t}\, dt, \qquad z \in U.$$

Let $R_c = \{f \in \mathcal{A} : \mathrm{Re}(f'(z)) > -c,\ z \in U\}$. Using this notation, Theorem 1 becomes:

THEOREM 3. *If* $c = \frac{2\ln 2 - 1}{2(1-\ln 2)}$ *then* $A(R_c) \subseteq S^*$, *and the result is sharp.*

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