ANNALES
POLONICI MATHEMATICI
94.1 (2008)

Existence and uniqueness of periodic solutions for a kind of nonlinear nth order differential equations with delays

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Abstract. By applying the continuation theorem of coincidence degree theory, we establish new results on the existence and uniqueness of 2π-periodic solutions for a class of nonlinear nth order differential equations with delays.

1. Introduction. In this paper, we study the existence and uniqueness of 2π-periodic solutions of the nonlinear nth order delay differential equation

$$x^{(n)} + \sum_{j=1}^{n-1} a_j x^{(j)} + g(t, x(t-\tau(t))) = p(t), \tag{1.1}$$

where $\tau, p : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\tau(t)$ and $p(t)$ are 2π-periodic with respect to t, g is 2π-periodic in the first variable, $n \geq 2$ is an integer, and a_j $(j = 1, \ldots, n-1)$ are constants.

During the past thirty years, there has been a great amount of work on the existence of periodic solutions for the higher-order Duffing equation

$$x^{(2k)} + \sum_{j=1}^{k-1} a_j x^{(2j)} + (-1)^{k+1} g(t, x) = 0, \tag{1.2}$$

or

$$x^{(2k+1)} + \sum_{j=1}^{k-1} a_j x^{(2j+1)} + g(t, x) = 0. \tag{1.3}$$

Many of these results can be found in [1, 5, 6, 12–14, 16] and the references cited therein. However, to the best of our knowledge, there exist few results on the existence and uniqueness of 2π-periodic solutions of (1.1).

2000 *Mathematics Subject Classification*: 34K15, 34C25.

Key words and phrases: nth order differential equation, periodic solution, existence, uniqueness, coincidence degree, delay.

This work was supported by the Scientific Research Fund of Hunan Provincial Natural Science Foundation of China (07JJ46001).

The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of 2π-periodic solutions of (1.1). Our results are new and they complement previously known results. An illustrative example is given in Section 4.

If n is even, let $n = 2k$; then equation (1.1) becomes

$$x^{(2k)} + \sum_{j=1}^{2k-1} a_j x^{(j)} + g(t, x(t-\tau(t))) = p(t). \tag{1.4}$$

If n is odd, let $n = 2k+1$; then (1.1) becomes

$$x^{(2k+1)} + \sum_{j=1}^{2k} a_j x^{(j)} + g(t, x(t-\tau(t))) = p(t). \tag{1.5}$$

For ease of exposition, throughout this paper we will adopt the following notations:

$$|x|_p = \Big(\int_0^{2\pi} |x(t)|^p\,dt\Big)^{1/p}, \qquad |x|_\infty = \max_{t\in[0,2\pi]} |x(t)|, \qquad a^+ = \max\{0, a\},$$

$$\|x\| = \sum_{j=0}^{n-1} |x^{(j)}|_\infty, \qquad x^{(0)} = x,$$

$$A_1 = 1 - a^+_{2(k-1)} - |a_{2(k-2)}| - \cdots - |a_4| - a^+_2,$$

$$A_2 = a_{2k-1} - a^+_{2k-3} - |a_{2k-5}| - \cdots - |a_3| - a^+_1,$$

$$\overline{A}_1 = 1 - a^+_{2(k-1)} - |a_{2(k-2)}| - \cdots - a^+_4 - |a_2|,$$

$$\overline{A}_2 = a_{2k-1} - a^+_{2k-3} - |a_{2k-5}| - \cdots - a^+_3 - |a_1|,$$

$$A_3 = 1 - a^+_{2k-1} - |a_{2k-3}| - \cdots - a^+_3 - |a_1|,$$

$$A_4 = a_{2k} - a^+_{2k-2} - |a_{2k-4}| - \cdots - |a_4| - a^+_2,$$

$$\overline{A}_3 = 1 - a^+_{2k-1} - |a_{2k-3}| - \cdots - |a_3| - a^+_1,$$

$$\overline{A}_4 = a_{2k} - a^+_{2k-2} - |a_{2k-4}| - \cdots - \cdots - a^+_4 - |a_2|.$$

It is convenient to introduce the following assumptions:

(H_1) There exists a constant $d_1 > 0$ such that

$$x[g(t,x) - p(t)] > 0 \quad \text{for all } t \in \mathbb{R},\ |x| \ge d_1.$$

(H_2) There exists a constant $d_2 > 0$ such that

$$x[g(t,x) - p(t)] < 0 \quad \text{for all } t \in \mathbb{R},\ |x| \ge d_2.$$

2. Several lemmas. Let us introduce the auxiliary equation

$$x^{(n)} + \lambda\Big[\sum_{j=1}^{n-1} a_j x^{(j)} + g(t, x(t-\tau(t)))\Big] = \lambda p(t), \qquad \lambda \in (0,1). \tag{2.1$_\lambda$}$$

Let

$$X = \{x \in C^{n-1}(\mathbb{R},\mathbb{R}) \mid x(t+2\pi) = x(t) \text{ for all } t \in \mathbb{R}\}$$

and

$$Y = \{x \in C(\mathbb{R},\mathbb{R}) \mid x(t+2\pi) = x(t) \text{ for all } t \in \mathbb{R}\}$$

be Banach spaces with the norms

$$\|x\|_X = \|x\| = \sum_{j=0}^{n-1} |x^{(j)}|_\infty \quad \text{and} \quad \|x\|_Y = |x|_\infty = \max_{t\in[0,2\pi]} |x(t)|.$$

Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \{x \in X \mid x^{(n)} \in C(\mathbb{R},\mathbb{R})\}$$

and for $x \in D(L)$,

$$Lx = x^{(n)}. \tag{2.2}$$

We also define a nonlinear operator $N : X \to Y$ by setting

$$Nx(t) = -\Big[\sum_{j=1}^{n-1} a_j x^{(j)} + g(t, x(t-\tau(t)))\Big] + p(t). \tag{2.2$'$}$$

It is easy to see that

$$\operatorname{Ker} L = \mathbb{R} \quad \text{and} \quad \operatorname{Im} L = \Big\{x \in Y \;\Big|\; \int_0^{2\pi} x(s)\,ds = 0\Big\}.$$

Thus L is a Fredholm operator with index zero.

Define the continuous projectors $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y/\operatorname{Im} L$ by setting

$$Px(t) = \frac{1}{2\pi}\int_0^{2\pi} x(s)\,ds$$

and

$$Qx(t) = \frac{1}{2\pi}\int_0^{2\pi} x(s)\,ds.$$

Hence, $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Denoting by $L_P^{-1} : \operatorname{Im} L \to D(L) \cap \operatorname{Ker} P$ the inverse of $L|_{D(L)\cap \operatorname{Ker} P}$, one can observe that L_P^{-1} is a compact operator. Therefore, N is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X.

In view of (2.2) and $(2.2)'$, the operator equation

$$Lx = \lambda Nx, \qquad \lambda \in (0,1),$$

is equivalent to the auxiliary equation $(2.1)_\lambda$.

We now recall the continuation theorem of [8].

LEMMA 2.1. *Let X and Y be Banach spaces. Suppose that $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero, and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that the following conditions are satisfied.*

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;
(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;
(3) *The Brouwer degree*

$$\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

Then the equation $Lx = Nx$ has a solution on $\overline{\Omega} \cap D(L)$.

The following lemmas will be useful to prove our main results in Section 3.

LEMMA 2.2. *If $x \in C^2(\mathbb{R},\mathbb{R})$ and $x(t+2\pi) = x(t)$, then*

$$|x'(t)|_2^2 \le |x''(t)|_2^2. \tag{2.3}$$

Lemma 2.2 is known as the Wirtinger inequality; for the proof, see [10, 19, 20].

LEMMA 2.3. *Let (H_1) or (H_2) hold. If $x(t)$ is a 2π-periodic solution of $(2.1)_\lambda$, then*

$$|x|_\infty \le d + \sqrt{2\pi}\,|x'|_2, \tag{2.4}$$

where $d = d_1$ or d_2 according to the case.

Proof. Let $x(t)$ be a 2π-periodic solution of $(2.1)_\lambda$. Integrating $(2.1)_\lambda$ from 0 to 2π, we see that

$$\int_0^{2\pi} [g(t, x(t-\tau(t))) - p(t)]\,dt = 0. \tag{2.5}$$

Thus, there exists a $\xi \in [0,2\pi]$ such that

$$g(\xi, x(\xi - \tau(\xi))) - p(\xi) = 0.$$

In view of (H_1) or (H_2), we obtain

$$|x(\xi - \tau(\xi))| \le d.$$

Let $\xi = 2m\pi + \bar{\xi}$, where $\bar{\xi} \in [0, 2\pi]$ and m is an integer. Then, using the Schwarz inequality and the relation

$$|x(t)| = \left| x(\bar{\xi}) + \int_{\bar{\xi}}^{t} x'(s)\, ds \right| \le d + \int_{0}^{2\pi} |x'(s)|\, ds, \qquad t \in [0, 2\pi], \tag{2.6}$$

we have

$$|x|_\infty = \max_{t\in[0,2\pi]} |x(t)| \le d + \sqrt{2\pi}\, |x'|_2, \tag{2.7}$$

which implies that (2.4) is satisfied.

Lemma 2.4. *Assume that k is even, and one of the following conditions is satisfied:*

(H_3) *$g(t,x)$ is strictly monotone in x and there exists a constant b such that*

$$0 \le b < \frac{A_1}{2\pi}, \qquad |g(t,x_1) - g(t,x_2)| \le b|x_1 - x_2| \text{ for all } t, x_1, x_2 \in \mathbb{R};$$

(H_4) *$g(t,x)$ is strictly monotone in x and there exists a constant b such that*

$$0 \le b < \frac{A_2}{2\pi}, \qquad |g(t,x_1) - g(t,x_2)| \le b|x_1 - x_2| \text{ for all } t, x_1, x_2 \in \mathbb{R}.$$

Then (1.4) has at most one 2π-periodic solution.

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two 2π-periodic solutions of (1.4). Then

$$\begin{aligned}(x_1(t) - x_2(t))^{(2k)} + \sum_{j=1}^{2k-1} a_j (x_1(t) - x_2(t))^{(j)} \\ + [g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t)))] = 0.\end{aligned} \tag{2.8}$$

Set $Z(t) = x_1(t) - x_2(t)$. Then (2.8) reads

$$Z^{(2k)}(t) + \sum_{j=1}^{2k-1} a_j Z^{(j)}(t) + [g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t)))] = 0. \tag{2.9}$$

Integrating (2.9) from 0 to 2π, we have

$$\int_0^{2\pi} [g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t)))]\, dt = 0.$$

Thus, in view of the integral mean value theorem, there exists a constant $\gamma \in [0, 2\pi]$ such that

$$g(\gamma, x_1(\gamma - \tau(\gamma))) - g(\gamma, x_2(\gamma - \tau(\gamma))) = 0. \tag{2.10}$$

Let $\gamma - \tau(\gamma) = m_1 2\pi + \widetilde{\gamma}$, where $\widetilde{\gamma} \in [0, 2\pi]$ and m_1 is an integer. Then (2.10), together with (H_3) (or (H_4)), implies that

$$Z(\widetilde{\gamma}) = x_1(\widetilde{\gamma}) - x_2(\widetilde{\gamma}) = x_1(\gamma - \tau(\gamma)) - x_2(\gamma - \tau(\gamma)) = 0. \tag{2.11}$$

Hence,

$$|Z(t)| = \Big| Z(\widetilde{\gamma}) + \int_{\widetilde{\gamma}}^{t} Z'(s)\, ds \Big| \le \int_0^{2\pi} |Z'(s)|\, ds, \qquad t \in [0, 2\pi],$$

and

$$|Z|_\infty \le \sqrt{2\pi}\, |Z'|_2. \tag{2.12}$$

Now we consider two cases.

Case (i): (H_3) *holds.* Multiplying (2.9) by $Z^{(2k)}(t)$ and then integrating from 0 to 2π, in view of (2.3), (2.9) and the Schwarz inequality, we have

$$\begin{aligned} A_1 |Z^{(2k)}|_2^2 &= A_1 \int_0^{2\pi} |Z^{(2k)}(t)|^2 dt \\ &= (1 - a_{2(k-1)}^+ - |a_{2(k-2)}| - \cdots - |a_4| - a_2^+) \int_0^{2\pi} |Z^{(2k)}(t)|^2\, dt \\ &\le \int_0^{2\pi} |Z^{(2k)}(t)|^2\, dt + \int_0^{2\pi} [-a_{2(k-1)}^+ |Z^{(2k-1)}(t)|^2 - |a_{2(k-2)}|\, |Z^{(2k-2)}(t)|^2 \\ &\quad - \cdots - |a_4|\, |Z^{(k+2)}(t)|^2 - a_2^+ |Z^{(k+1)}(t)|^2]\, dt \\ &\le \int_0^{2\pi} |Z^{(2k)}(t)|^2\, dt + \int_0^{2\pi} \sum_{j=1}^{2k-1} a_j Z^{(j)}(t) Z^{(2k)}(t)\, dt \\ &= -\int_0^{2\pi} [g(t, x_1(t - \tau(t))) - g(t, x_2(t - \tau(t)))] Z^{(2k)}(t)\, dt \\ &\le b \int_0^{2\pi} |x_1(t - \tau(t)) - x_2(t - \tau(t))|\, |Z^{(2k)}(t)|\, dt. \end{aligned} \tag{2.13}$$

From (2.3), (2.12) and the Schwarz inequality, (2.13) implies that

$$\begin{aligned} A_1 |Z^{(2k)}|_2^2 &\le b |Z|_\infty \sqrt{2\pi}\, |Z^{(2k)}|_2 \le b \sqrt{2\pi}\, |Z'|_2 \sqrt{2\pi}\, |Z^{(2k)}|_2 \\ &\le 2\pi b |Z^{(2k)}|_2^2. \end{aligned} \tag{2.14}$$

Since $Z(t), Z'(t), \ldots, Z^{(2k)}(t)$ are 2π-periodic and continuous functions, in view of (H_3), (2.11) and (2.14), we have

$$Z(t) \equiv Z'(t) \equiv \cdots \equiv Z^{(2k)}(t) \equiv 0 \quad \text{for all } t \in \mathbb{R}.$$

Thus, $x_1(t) \equiv x_2(t)$ for all $t \in \mathbb{R}$.

Case (ii): (H_4) *holds*. Multiplying (2.9) by $Z^{(2k-1)}(t)$ and then integrating from 0 to 2π, in view of (2.3), (2.9), (2.12) and the Schwarz inequality, we get

$$
\begin{aligned}
(2.15)\qquad A_2|Z^{(2k-1)}|_2^2 &= A_2\int_0^{2\pi}|Z^{(2k-1)}(t)|^2\,dt \\
&= (a_{2k-1}-a_{2k-3}^+-|a_{2k-5}|-\cdots-|a_3|-a_1^+)\int_0^{2\pi}|Z^{(2k-1)}(t)|^2\,dt \\
&\le a_{2k-1}\int_0^{2\pi}|Z^{(2k-1)}(t)|^2\,dt+\int_0^{2\pi}[-a_{2k-3}^+|Z^{(2k-2)}(t)|^2-|a_{2k-5}|\,|Z^{(2k-3)}(t)|^2 \\
&\quad -\cdots-|a_3|\,|Z^{(k+1)}(t)|^2-a_1^+|Z^{(k)}(t)|^2]\,dt \\
&\le a_{2k-1}\int_0^{2\pi}|Z^{(2k-1)}(t)|^2\,dt+\int_0^{2\pi}\sum_{j=1}^{2k-2}a_jZ^{(j)}(t)Z^{(2k-1)}(t)\,dt \\
&= -\int_0^{2\pi}[g(t,x_1(t-\tau(t)))-g(t,x_2(t-\tau(t)))]Z^{(2k-1)}(t)\,dt \\
&\le b\int_0^{2\pi}|x_1(t-\tau(t))-x_2(t-\tau(t))|\,|Z^{(2k-1)}(t)|\,dt\le 2\pi b|Z^{(2k-1)}|_2^2.
\end{aligned}
$$

From (2.11) and (H_4), (2.15) implies that

$$Z(t)\equiv Z'(t)\equiv\cdots\equiv Z^{(2k-1)}(t)\equiv 0\quad\text{for all } t\in\mathbb{R}.$$

Hence, $x_1(t)\equiv x_2(t)$ for all $t\in\mathbb{R}$. The proof of Lemma 2.4 is now complete.

In a similar fashion we can show the following:

Lemma 2.5. *Assume that k is odd, and one of the following conditions is satisfied*:

$(\widetilde{H}_3)$ *$g(t,x)$ is strictly monotone in x and there exists a constant b such that*

$$0\le b<\frac{\overline{A}_1}{2\pi},\qquad |g(t,x_1)-g(t,x_2)|\le b|x_1-x_2| \textit{ for all } t,x_1,x_2\in\mathbb{R};$$

$(\widetilde{H}_4)$ *$g(t,x)$ is strictly monotone in x and there exists a constant b such that*

$$0\le b<\frac{\overline{A}_2}{2\pi},\qquad |g(t,x_1)-g(t,x_2)|\le b|x_1-x_2| \textit{ for all } t,x_1,x_2\in\mathbb{R}.$$

Then (1.4) *has at most one 2π-periodic solution.*

3. Main results

THEOREM 3.1. *Let* (H_1) *or* (H_2) *hold. Assume that* k *is even, and either* (H_3) *or* (H_4) *is satisfied. Then* (1.4) *has a unique* 2π*-periodic solution.*

Proof. By Lemma 2.4, we only have to prove the existence. To do this, we shall apply Lemma 2.1. First, we claim that all 2π-periodic solutions of $(2.1)_\lambda$ are bounded. We consider two cases.

CASE (1): (H_3) *holds.* Let $x(t)$ be a 2π-periodic solution of $(2.1)_\lambda$. Multiplying $(2.1)_\lambda$ by $x^{(2k)}(t)$ and then integrating from 0 to 2π, in view of (2.3), (2.4), (H_3) and the Schwarz inequality, we have

$$
\begin{aligned}
(3.1)\qquad A_1|x^{(2k)}|_2^2 &= (1 - a_{2(k-1)}^+ - |a_{2(k-2)}| - \cdots - |a_4| - a_2^+) \int_0^{2\pi} |x^{(2k)}(t)|^2\,dt \\
&\le \int_0^{2\pi} |x^{(2k)}(t)|^2\,dt + \int_0^{2\pi} \lambda[-a_{2(k-1)}^+|x^{(2k-1)}(t)|^2 - |a_{2(k-2)}|\,|x^{(2k-2)}(t)|^2 \\
&\quad - \cdots - |a_4|\,|x^{(k+2)}(t)|^2 - a_2^+|x^{(k+1)}(t)|^2]\,dt \\
&\le \int_0^{2\pi} |x^{(2k)}(t)|^2\,dt + \lambda \int_0^{2\pi} \sum_{j=1}^{2k-1} a_j x^{(j)}(t) x^{(2k)}(t)\,dt \\
&= -\int_0^{2\pi} g(t, x(t-\tau(t)))x^{(2k)}(t)\,dt + \int_0^{2\pi} p(t) x^{(2k)}(t)\,dt \\
&\le \int_0^{2\pi} [|g(t, x(t-\tau(t))) - g(t,0)| + |g(t,0)|]\,|x^{(2k)}(t)|\,dt \\
&\quad + \int_0^{2\pi} |p(t)|\,|x^{(2k)}(t)|\,dt \\
&\le b\int_0^{2\pi} |x(t-\tau(t))|\,|x^{(2k)}(t)|\,dt + \int_0^{2\pi} |g(t,0)|\,|x^{(2k)}(t)|\,dt \\
&\quad + \int_0^{2\pi} |p(t)|\,|x^{(2k)}(t)|\,dt \\
&\le b\sqrt{2\pi}\,|x'|_2\sqrt{2\pi}\,|x^{(2k)}|_2 + [bd + \max_{0\le t\le 2\pi} |g(t,0)| + |p|_\infty]\sqrt{2\pi}\,|x^{(2k)}|_2 \\
&\le 2\pi b|x^{(2k)}|_2^2 + [bd + \max_{0\le t\le 2\pi} |g(t,0)| + |p|_\infty]\sqrt{2\pi}\,|x^{(2k)}|_2.
\end{aligned}
$$

Since $b < A_1/2\pi$, (2.3), (2.4) and (3.1) imply that there exists a constant

$D_1 > 0$ such that

$$
\begin{aligned}
(3.2)\qquad & |x^{(j)}|_2 \le |x^{(2k)}|_2 \le D_1, \quad j = 1,\dots,2k-1,\\
& |x|_\infty \le d + \sqrt{2\pi}\,|x'|_2 \le D_1.
\end{aligned}
$$

For $j = 1,\dots,2k-1$, noting that $x^{(j)}(t)$ are 2π-periodic, there exists a $T_j \in (0,2\pi)$ such that $x^{(j+1)}(T_j) = 0$. Therefore,

$$
\begin{aligned}
(3.3)\qquad |x^{(j)}(t)| &= \Big|\int_{T_j}^{t} x^{(j+1)}(s)\,ds\Big| \le \sqrt{2\pi}\Big(\int_0^{2\pi} |x^{(j+1)}(s)|^2\,ds\Big)^{1/2}\\
&\le \sqrt{2\pi}\,|x^{(j+1)}|_2 \le \sqrt{2\pi}\,D_1.
\end{aligned}
$$

Therefore, for all possible 2π-periodic solutions $x(t)$ of $(2.1)_\lambda$, there exists a constant M_1 such that

$$
(3.4)\qquad \|x\| = \sum_{j=0}^{2k-1} |x^{(j)}|_\infty < M_1,
$$

with $M_1 > 0$ independent of λ.

CASE (2): (H_4) *holds.* Let $x(t)$ be a 2π-periodic solution of $(2.1)_\lambda$. Multiplying $(2.1)_\lambda$ by $x^{(2k-1)}(t)$ and then integrating from 0 to 2π, by (H_4), (2.3), (2.4) and the Schwarz inequality, we have

$$
\begin{aligned}
(3.5)\qquad & A_2|x^{(2k-1)}|_2^2\\
&= (a_{2k-1} - a_{2k-3}^+ - |a_{2k-5}| - \dots - |a_3| - a_1^+)\int_0^{2\pi} |x^{(2k-1)}(t)|^2\,dt\\
&\le a_{2k-1}\int_0^{2\pi} |x^{(2k-1)}(t)|^2\,dt + \int_0^{2\pi} [-a_{2k-3}^+|x^{(2(k-1))}(t)|^2 - |a_{2k-5}|\,|x^{(2k-3)}(t)|^2\\
&\qquad - \dots - |a_3|\,|x^{(k+1)}(t)|^2 - a_1^+|x^{(k)}(t)|^2]\,dt\\
&\le a_{2k-1}\int_0^{2\pi} |x^{(2k-1)}(t)|^2\,dt + \int_0^{2\pi}\sum_{j=1}^{2k-1} a_j x^{(j)}(t)x^{(2k-1)}(t)\,dt\\
&= -\int_0^{2\pi} g(t,x(t-\tau(t)))x^{(2k-1)}(t)\,dt + \int_0^{2\pi} p(t)x^{(2k-1)}(t)\,dt\\
&\le b\int_0^{2\pi} |x(t-\tau(t))|\,|x^{(2k-1)}(t)|\,dt + \int_0^{2\pi} |g(t,0)|\,|x^{(2k-1)}(t)|\,dt\\
&\qquad + \int_0^{2\pi} |p(t)|\,|x^{(2k-1)}(t)|\,dt\\
&\le 2\pi b|x'|_2|x^{(2k-1)}|_2 + [bd + \max_{0\le t\le 2\pi} |g(t,0)| + |p|_\infty]\sqrt{2\pi}\,|x^{(2k-1)}|_2\\
&\le 2\pi b|x^{(2k-1)}|_2^2 + [bd + \max_{0\le t\le 2\pi} |g(t,0)| + |p|_\infty]\sqrt{2\pi}\,|x^{(2k-1)}|_2.
\end{aligned}
$$

Since $b < A_2/2\pi$, (2.3), (2.4) and (3.5) imply that there exists a constant $D_2 > 0$ such that

$$
\begin{aligned}
&|x^{(j)}|_2 \le |x^{(2k-1)}|_2 \le D_2, \qquad j = 1, 2, \ldots, 2k-2,\\
&|x|_\infty \le d + \sqrt{2\pi}\,|x'|_2 \le D_2.
\end{aligned}
\tag{3.6}
$$

From $(2.1)_\lambda$, (3.3) and (3.6), we obtain

$$
\begin{aligned}
|x^{(2k-1)}(t)| &= \Big|\int\limits_{T_{2k-1}}^{t} x^{(2k)}(s)\,ds\Big|\\
&\le \int\limits_0^{2\pi} \Big|-\Big[\sum_{j=1}^{2k-1} a_j x^{(j)} + g(t, x(t-\tau(t)))\Big] + p(t)\Big|\,ds\\
&\le \sum_{j=1}^{n-1} |a_j|\sqrt{2\pi}\,D_2 + 2\pi[\max_{t\in R,\,|x|\le D_2} |g(t,x)| + |p|_\infty]\\
&=: \overline{D}_1,
\end{aligned}
\tag{3.7}
$$

which, together with (3.6), implies that (3.4) also holds.

If $x \in \Omega_1 = \{x \in \operatorname{Ker} L \cap X \mid Nx \in \operatorname{Im} L\}$, then there exists a constant M_2 such that

$$
x(t) \equiv M_2, \qquad \int\limits_0^{2\pi} [g(t, M_2) - p(t)]\,dt = 0.
\tag{3.8}
$$

Thus,

$$
|x(t)| \equiv |M_2| < d \qquad \text{for all } x \in \Omega_1.
\tag{3.9}
$$

Let $M = M_1 + d$. Set

$$
\Omega = \Big\{x \in X \;\Big|\; \|x\| = \sum_{j=0}^{2k-1} |x^{(j)}|_\infty < M\Big\}.
$$

Since N is L-compact on $\overline{\Omega}$, it is easy to see from (3.4), (3.8) and (3.9) that the conditions (1) and (2) in Lemma 2.1 hold.

Furthermore, define continuous functions $\Psi_1(x,\mu)$ and $\Psi_2(x,\mu)$ by setting, for $x \in \mathbb{R}$ and $\mu \in [0,1]$,

$$
\Psi_1(x,\mu) = -(1-\mu)x - \mu\cdot\frac{1}{2\pi}\int\limits_0^{2\pi}[g(t,x) - p(t)]\,dt,
$$

$$
\Psi_2(x,\mu) = (1-\mu)x - \mu\cdot\frac{1}{2\pi}\int\limits_0^{2\pi}[g(t,x) - p(t)]\,dt.
$$

If (H_1) holds, then

$$x\Psi_1(x,\mu) \neq 0 \quad \text{for all } x \in \partial\Omega \cap \operatorname{Ker} L.$$

Hence, using the homotopy invariance theorem, we have

$$\begin{aligned}\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\left\{-\frac{1}{2\pi}\int_0^{2\pi}[g(t,x)-p(t)]\,dt, \Omega \cap \operatorname{Ker} L, 0\right\} \\ &= \deg\{-x, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.\end{aligned}$$

If (H_2) holds, then

$$x\Psi_2(x,\mu) \neq 0 \quad \text{for all } x \in \partial\Omega \cap \operatorname{Ker} L.$$

Hence, using the homotopy invariance theorem, we obtain

$$\begin{aligned}\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\left\{-\frac{1}{2\pi}\int_0^{2\pi}[g(t,x)-p(t)]\,dt,\ \Omega \cap \operatorname{Ker} L, 0\right\} \\ &= \deg\{x, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.\end{aligned}$$

In view of the above discussion, we conclude from Lemma 2.1 that Theorem 3.1 is proved.

In view of Lemma 2.5, a similar argument leads to

Theorem 3.2. *Let* (H_1) *or* (H_2) *hold. Assume that* k *is odd, and either* $(\widetilde{H}_3)$ *or* $(\widetilde{H}_4)$ *is satisfied. Then* (1.4) *has a unique* 2π*-periodic solution.*

We are now in a position to establish the existence and uniqueness of 2π-periodic solutions of equation (1.5). Similarly to the proof of Theorems 3.1 and 3.2, one can prove the following results.

Theorem 3.3. *Let* (H_1) or (H_2) *hold. Assume that* k *is even, and one of the following conditions is satisfied*:

(H_5) $g(t,x)$ *is strictly monotone in* x *and there exists a constant* b *such that*

$$0 \le b < \frac{A_3}{2\pi}, \qquad |g(t,x_1)-g(t,x_2)| \le b|x_1-x_2| \text{ for all } t, x_1, x_2 \in \mathbb{R};$$

(H_6) $g(t,x)$ *is strictly monotone in* x *and there exists a constant* b *such that*

$$0 \le b < \frac{A_4}{2\pi}, \qquad |g(t,x_1)-g(t,x_2)| \le b|x_1-x_2| \text{ for all } t, x_1, x_2 \in \mathbb{R}.$$

Then (1.5) *has a unique* 2π*-periodic solution.*

Theorem 3.4. *Let* (H_1) or (H_2) *hold. Assume that* k *is odd, and one of the following conditions is satisfied*:

($\widetilde{H}_5$) *$g(t,x)$ is strictly monotone in x and there exists a constant b such that*

$$0 \le b < \frac{\overline{A}_3}{2\pi}, \quad |g(t,x_1) - g(t,x_2)| \le b|x_1 - x_2| \text{ for all } t, x_1, x_2 \in \mathbb{R};$$

($\widetilde{H}_6$) *$g(t,x)$ is strictly monotone in x and there exists a constant b such that*

$$0 \le b < \frac{\overline{A}_4}{2\pi}, \quad |g(t,x_1) - g(t,x_2)| \le b|x_1 - x_2| \text{ for all } t, x_1, x_2 \in \mathbb{R}.$$

Then (1.5) *has a unique 2π-periodic solution.*

4. Example and remark

EXAMPLE 4.1. Let $g(t, x(t-\tau(t))) = -\frac{1}{3}x(t - 30e^{\sin t})e^{\sin t}$ and $p(t) = 2\cos t$. Then the equation

$$x^{(6)} + 100x^{(5)} + x^{(4)} - 10x^{(3)} + 20x'' - 6x' + g(t, x(t-\tau(t))) = e(t) \tag{4.1}$$

has a unique 2π-periodic solution.

Proof. It is straightforward to check that the assumptions (H_2) and ($\widetilde{H}_4$) are satisfied. Therefore, by Theorem 3.2, equation (4.1) has a unique 2π-periodic solution.

REMARK 4.1. As in [1, 2, 5, 6, 12–14], the papers [16, 17] only study the existence of periodic solutions. Therefore, the results in [1–6, 7, 9, 11–21] and the references therein cannot be applied to show the uniqueness of 2π-periodic solutions of equation (4.1). This implies that the results of this paper are essentially new.

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Received 11.3.2007
and in final form 19.3.2008 (1771)