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Zero-set property of o-minimal indefinitely Peano differentiable functions

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Abstract. Given an o-minimal expansion $\mathcal{M}$ of a real closed field R which is not polynomially bounded. Let $\mathcal{P}^\infty$ denote the definable indefinitely Peano differentiable functions. If we further assume that $\mathcal{M}$ admits $\mathcal{P}^\infty$ cell decomposition, each definable closed subset A of R^n is the zero-set of a $\mathcal{P}^\infty$ function $f : R^n \to R$. This implies $\mathcal{P}^\infty$ approximation of definable continuous functions and gluing of $\mathcal{P}^\infty$ functions defined on closed definable sets.

1. Introduction. In the present paper we discuss the zero-set property of indefinitely Peano differentiable functions in connection with o-minimality. Let R be a fixed real closed field and $\mathcal{M}$ be an o-minimal expansion of R. In the following, "definable" always means "definable with parameters in $\mathcal{M}$". We further assume the reader to be familiar with the basic properties of o-minimal structures (see for example [3]). For examples of o-minimal structures, see [5], [7], [8], [14], [20] and [23].

Let R^n be endowed with the Euclidean R-norm $\|\cdot\|$ and the corresponding topology (an *R-norm* has the same definition as the norm just taking its values in R). Peano differentiability can be seen as the natural generalisation of Fréchet differentiability to higher—in our case infinite—order. *Indefinitely Peano differentiable functions* are functions which have an infinite Taylor series at every point of their domain. More precisely:

Definition 1.1. Let $U \subset R^n$ be open. A function $f : U \to R$ is called *indefinitely Peano differentiable* at $x \in U$ if for each positive integer m there is a polynomial $p_m \in R[X_1, \ldots, X_n]$ with $\deg(p_m) \le m$ and $p_m(0) = 0$ such that

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$$\lim_{y\to x}\frac{f(y)-f(x)-p_m(y-x)}{\|y-x\|^m}=0. \tag{1.1}$$

The function $f : U \to V$ is *indefinitely Peano differentiable* in U if f is indefinitely Peano differentiable at each point of U. By $\mathcal{P}^\infty(U,V)$ we denote the set of definable indefinitely Peano differentiable functions from U to V.

For $x \in U$ the sequence of polynomials $(p_m)_{m\in\mathbb{N}}$ is uniquely determined. If $\mathfrak{m} \subset R[X_1,\ldots,X_n]$ denotes the ideal generated by $X_1,\ldots,X_n$, then $p_m \equiv p_{m+1} \bmod \mathfrak{m}^{m+1}$ for any positive integer m.

The notion of indefinite Peano differentiability differs in a crucial way from that of indefinite continuous differentiability even if we claim definability in some o-minimal expansion of R (cf. Example 2.1). But on the other hand, we also deal with an infinite Taylor series, and indefinite Peano differentiability is not a first order concept (cf. [24]).

For $p \in \mathbb{N} \cup \{\infty\}$, the symbol $\mathcal{C}^p$ is used to abbreviate "p times continuously differentiable". For every o-minimal structure $\mathcal{M}$ the zero-set property is valid for definable $\mathcal{C}^p$ functions if p is finite (cf. [6]). The results of [6] are stated and proved for the real numbers, but the proof of the zero-set property works for every real closed field. A crucial tool used in the proof of the zero-set property of $\mathcal{C}^p$ functions is $\mathcal{C}^p$ cell decomposition. For our aim we require the concept of $\mathcal{P}^\infty$ cell decomposition which we will describe in the third section.

An o-minimal structure is called *polynomially bounded* provided that every definable function $f : (0,\infty) \to R$ is ultimately bounded by some polynomial.

The main result of the present paper is the following theorem.

THEOREM 1.2. *Let $\mathcal{M}$ be an o-minimal structure expanding a real closed field R which is not polynomially bounded and which possesses $\mathcal{P}^\infty$ cell decomposition. Then each closed definable subset of R^n is the zero-set of a definable indefinitely Peano differentiable function whose domain is R^n.*

Examples of o-minimal structures satisfying the conditions of the previous theorem are given in [5] and [7]. Further, the Pfaffian closure of an o-minimal structure is o-minimal and preserves analytic cell decomposition (cf. [21] and [15]), so that the Pfaffian closures of the structures generated in [5], [7] and [14] admit $\mathcal{P}^\infty$ cell decomposition, as also do the o-minimal structures generated by certain quasi-analytic Denjoy–Carleman classes and expanded by the exponential function (cf. [20]).

Theorem 1.2 is related to the zero-set property for definable $\mathcal{C}^\infty$-functions which was proved for so-called locally polynomially bounded o-minimal structures (cf. [13, Chapter 8]).

If we assume $\mathcal{M}$ to be polynomially bounded, the zero-set property does not hold for $\mathcal{P}^\infty$ functions. In fact, for every $\mathcal{P}^\infty$ function $f : R \to R$ with

$f^{-1}(\{0\}) = (-\infty, 0]$ the mapping $g : (1,\infty) \to R$ defined by $g(t) := 1/f(1/t)$ cannot be polynomially bounded, and g is definable if f is definable.

In what follows we assume that $\mathcal{M}$ is *not polynomially bounded.*

In Section 4 we prove a smoothing tool for $\mathcal{P}^\infty$ functions (cf. Proposition 4.2). In Section 5 we reduce the proof of the zero-set property to complements of open $\mathcal{P}^\infty$ cells for which the zero-set property can be directly proved. In the final Section 6 we discuss several consequences of the zero-set property of $\mathcal{P}^\infty$ functions such as separation of disjoint closed definable sets, $\mathcal{P}^\infty$ partition of unity, approximation of definable continuous functions by $\mathcal{P}^\infty$ functions and gluing properties for $\mathcal{P}^\infty$ functions defined on open and on closed definable sets.

2. Example. For definable functions of one variable, the notion of $\mathcal{P}^\infty$ is equipollent to that of $\mathcal{C}^\infty$ (cf. [10, Proposition 7.2]). For functions of several variables the notions of $\mathcal{P}^\infty$ and $\mathcal{C}^\infty$ differ, at least if the o-minimal structure is not polynomially bounded and we only consider such o-minimal structures. For polynomially bounded o-minimal structures the inequality of these differentiability concepts is not known.

By [17] (see also [18] for the reals), there exists a definable $\mathcal{C}^\infty$ function $f : R \to (0,\infty)$ which satisfies the differential equation $f' = f$. If $R = \mathbb{R}$, this function is of course the ordinary exponential function, but also if R is distinct from $\mathbb{R}$ such functions increase faster than any polynomial. We will refer to this function as the exponential function exp.

The following example demonstrates the difference between indefinite Peano and continuous differentiability.

Example 2.1. Let $f : R^2 \to R$ be defined by

$$f(x,y) := \begin{cases} \exp\left(\frac{-1}{y^2}\right) g\left(x\exp\left(\frac{1}{y^2}\right) - 2\right), & y \neq 0, \\ 0, & y = 0, \end{cases} \tag{2.1}$$

where $g : R \to R$ is the function

$$g(t) := \begin{cases} \exp\left(\frac{1}{t^2-1}\right), & |t| < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

The function f is indefinitely Peano differentiable, but f is not continuously differentiable.

Proof. Definability is evident since the exp function is definable in $\mathcal{M}$. Obviously, f is $\mathcal{C}^\infty$ smooth outside the x-axis. If $y \neq 0$, then $|f(x,y)|$ is bounded by $\exp(-y^{-2})$ so that f is indefinitely Peano differentiable at each point of $R \times \{0\}$.

The partial derivative of f with respect to the first variable is

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} g'\left(x\exp\left(\frac{1}{y^2}\right) - 2\right), & y \neq 0, \\ 0, & y = 0. \end{cases} \tag{2.3}$$

For $t \neq 0$ this implies that $\left(\frac{3}{2}\exp(-t^{-2}), t\right) \to (0,0)$ as $t \to 0$. Therefore

$$\frac{\partial f}{\partial x}\left(\frac{3}{2}\exp\left(\frac{-1}{t^2}\right), t\right) = g'\left(\frac{-1}{2}\right) = \frac{16}{9}\exp\left(\frac{-4}{3}\right) \neq 0,\ t \neq 0. \tag{2.4}$$

Hence, the function f cannot be continuously differentiable at $(0,0)$. ∎

3. $\mathcal{P}^\infty$ cell decomposition. A definable map $f : A \to R^n$ where $A \subset R^q$ is not necessarily open is called a $\mathcal{P}^\infty$ *map* if there are a definable open neighbourhood $U \subset R^q$ containing A and a $\mathcal{P}^\infty$ map $F : U \to R^n$ such that $F|_A = f$.

We regard $\pm\infty$ as constant functions defined on arbitrary sets. For definable $f, g \in \mathcal{C}^0(X, R) \cup \{\pm\infty\}$ we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and in this case we set

$$(f,g)_X := \{(x,y) : x \in X,\ f(x) < y < g(x)\}. \tag{3.1}$$

If $h \in \mathcal{C}^0(X, R)$ we let

$$(h)_X := \{(x,y) : x \in X,\ y = h(x)\}. \tag{3.2}$$

DEFINITION 3.1. A $\mathcal{P}^\infty$ *cell* in R is either a singleton or an open interval. Suppose $\mathcal{P}^\infty$ cells in R^n are already defined. Then a $\mathcal{P}^\infty$ cell in R^{n+1} is a definable set of the form either $(h)_X$ where $h \in \mathcal{P}^\infty(X, R)$ and $X \subset R^n$ is a $\mathcal{P}^\infty$ cell, or $(f,g)_X$ where $X \subset R^n$ is a $\mathcal{P}^\infty$ cell and $f, g \in \mathcal{P}^\infty(X, R) \cup \{\pm\infty\}$ satisfy $f < g$.

A cell decomposition is a special kind of partition. Let $\pi : R^{n+1} \to R^n$ denote the projection onto the first n coordinates.

DEFINITION 3.2. A $\mathcal{P}^\infty$ *cell decomposition* of R is a finite partition of R into open intervals and singletons. A $\mathcal{P}^\infty$ *cell decomposition* of R^{n+1} is a finite partition of R^{n+1} into $\mathcal{P}^\infty$ cells $A_1, \ldots, A_r$ such that the set of projections $\pi(A_i)$ is a $\mathcal{P}^\infty$ cell decomposition of R^n.

$\mathcal{C}^p$ and $\mathcal{C}^\infty$ cells and the corresponding cell decompositions are defined analogously. For the case $p = 0$ we just speak of cells.

We assume $\mathcal{M}$ to be provided with $\mathcal{P}^\infty$ cell decomposition; that is, $\mathcal{M}$ possesses the following property.

For any definable sets $A_1, \ldots, A_k \subset R^n$ there is a $\mathcal{P}^\infty$ cell decomposition of R^n partitioning each of the A_i.

All examples of *o*-minimal structures known so far even possess the stronger $\mathcal{C}^\infty$ cell decomposition, but only $\mathcal{C}^p$ cell decomposition was verified for all o-minimal structures in [6, Theorem 11 in Appendix C], and our proofs only need $\mathcal{P}^\infty$ cell decomposition. $\mathcal{P}^\infty$ cell decomposition implies for every definable function $f : A \to R$ a $\mathcal{P}^\infty$ cell decomposition partitioning A such that for each cell $C \subset A$ the restriction $f|_C$ is $\mathcal{P}^\infty$.

From now on we assume $\mathcal{M}$ to be an o-minimal expansion of the real closed field R which is *not polynomially bounded* and which is additionally endowed with $\mathcal{P}^\infty$ *cell decomposition*.

A map $\theta : R^n \to R^k$, $(x_1, \ldots, x_n) \mapsto (x_{i(1)}, \ldots, x_{i(k)})$, with $1 \le i(1) < \cdots < i(k) \le n$ is called a *special projection*. For each $\mathcal{P}^\infty$ cell $C \subset R^n$ there exists a $d \ge 0$ and a special projection $\theta : R^n \to R^d$ which maps C $\mathcal{P}^\infty$ diffeomorphically onto an open $\mathcal{P}^\infty$ cell $B \subset R^d$ (cf. [1, Lemma 2.6] where one only has to replace $\mathcal{C}^m$ by $\mathcal{P}^\infty$).

We call d the *dimension* of the cell C. The *dimension* of an arbitrary definable set $X \subset R^n$ is

$$\dim(X) := \max\{d : X \text{ contains a cell of dimension } d\}. \tag{3.3}$$

For a detailed discussion on dimension we refer the reader to [3, Chapter 4.1]. Note that by [3, Chapter 4, Theorem 1.8] o-minimality always implies that $\dim(\partial X) < \dim(X)$ where $\partial X := \mathrm{cl}(X) \setminus X$ denotes the frontier of X.

4. Smoothing. We denote by Φ^p the set of definable, odd, strictly increasing $\mathcal{C}^p$ functions $f : R \to R$ which are p-*flat* at 0, i.e., $f^{(l)}(0) = 0$ for $l = 0, \ldots, p$, if $p < \infty$, and $l \in \mathbb{N}$ if $p = \infty$. The next lemma is [6, Lemma 8 in Appendix C]. The proof given in [6] is performed for the reals, but it also works for arbitrary real closed fields.

For a function f, we let $Z(f)$ denote its zero-set. Let p be a positive integer.

LEMMA 4.1. *Let $g : U \to R$ and $f_1, \ldots, f_l : U \setminus Z(g) \to R$ be definable, continuous functions where $U \subset R^n$ is locally closed. Then there is a $\phi \in \Phi^p$ such that*

$$\phi(g(x))f_i(x) \to 0 \text{ as } x \to y,\ x \in U \setminus Z(g),\ y \in Z(g),\ i = 1, \ldots, l. \tag{4.1}$$

PROPOSITION 4.2. *Let $U \subset R^n$ be locally closed and let $f, g : U \to R$ be definable, continuous and of class $\mathcal{P}^\infty$ on $U \setminus Z(g)$. Moreover, assume that $Z(f) \subset Z(g)$. Then there is a $\phi \in \Phi^\infty$ and a definable $h \in \mathcal{P}^\infty(U, R)$ such that*

$$\phi \circ g = hf. \tag{4.2}$$

Proof. STEP 1. We show that we can replace Φ^p by Φ^∞ in Lemma 4.1. Let $\phi_1 \in \Phi^p$ for some integer $p > 0$ and set

$$\phi_2(t) := \begin{cases} t\exp\left(\dfrac{-1}{\phi_1(t)}\right), & t > 0, \\ 0, & t = 0, \\ t\exp\left(\dfrac{1}{\phi_1(t)}\right), & t < 0. \end{cases} \tag{4.3}$$

Then ϕ_2 belongs to Φ^p, and, since $\phi_1'(0) = 0$, the function ϕ_2 is indefinitely Peano differentiable at 0 with $p_m = 0$ for all $m > 0$. By $\mathcal{P}^\infty$ cell decomposition we obtain a pointed neighbourhood V of 0, say $V = (-\delta, \delta) \setminus \{0\}$ for some $0 < \delta < 1$, such that ϕ_2 restricted to V is a $\mathcal{P}^\infty$ function. Since $0 < \exp(-1/t) < t$ for $t > 0$, we obtain the inequality

$$|\phi_2(t)| \le |\phi_1(t)| \quad \text{for } t \in V. \tag{4.4}$$

We define $\phi : R \to R$ by

$$\phi(t) = t\phi_2\left(\frac{\delta t^2}{1 + t^2}\right). \tag{4.5}$$

Obviously $\phi \in \Phi^\infty$ and, in addition, $|\phi(t)| \le |\phi_2(t)|$ near 0.

STEP 2. Both f and g are indefinitely Peano differentiable on $U \setminus Z(g)$. According to Lemma 4.1 and Step 1 we may assume that there is a $\phi \in \Phi^\infty$ such that for $f_1 : U \setminus Z(g) \to R$ with

$$f_1(x) = \frac{\exp(\operatorname{dist}(x, Z(g))^{-1})}{f(x)}, \tag{4.6}$$

we have

$$\lim_{x \to y} \phi(g(x)) f_1(x) = 0, \quad x \in U \setminus Z(g),\ y \in Z(g). \tag{4.7}$$

Let $h : U \to R$ be defined by

$$h(x) := \begin{cases} \dfrac{\phi(g(x))}{f(x)}, & x \in U \setminus Z(g), \\ 0, & \text{otherwise.} \end{cases} \tag{4.8}$$

Obviously h is indefinitely Peano differentiable on $U \setminus Z(g)$. For $x \in U \setminus Z(g)$ and $y \in Z(g)$ we obtain for $m \in \mathbb{N}$ the inequality

$$\begin{aligned} \left|\frac{h(x)}{\|x - y\|^m}\right| &\le \left|\frac{\phi \circ g(x) f_1(x) \exp(-\operatorname{dist}(x, Z(g))^{-1})}{\|x - y\|^m}\right| \\ &\le \left|\phi \circ g(x) f_1(x) \frac{\exp(-1/\|x - y\|)}{\|x - y\|^m}\right|. \end{aligned} \tag{4.9}$$

Since $t^{-m}\exp(-1/t) \to 0$ as $t \to 0$, $m \in \mathbb{N}$, by using (4.7) we obtain

$$\lim_{x \to y} \frac{h(x)}{\|x - y\|^m} = 0, \quad m \in \mathbb{N}. \tag{4.10}$$

This proves that $h \in \mathcal{P}^\infty(U, R)$. ∎

As a direct consequence we obtain the following two corollaries.

COROLLARY 4.3. *Let* $U \subset R^n$ *be locally closed, and let* $g : U \to R$ *be definable, continuous and* $\mathcal{P}^\infty$ *on* $U \setminus Z(g)$. *Then there is a* $\phi \in \Phi^\infty$ *such that* $\phi \circ g \in \mathcal{P}^\infty(U, R)$.

Proof. Apply the previous proposition to g and $f := 1$. ∎

The next corollary is a generalised Łojasiewicz inequality for $\mathcal{P}^\infty$ functions.

COROLLARY 4.4. *Let* $U \subset R^n$ *be closed and bounded, and let* $f, g : U \to R$ *be* $\mathcal{P}^\infty$ *functions with* $Z(f) \subset Z(g)$. *Then there is a* $\phi \in \Phi^\infty$ *such that*

$$|\phi(g(x))| \le |f(x)|, \quad x \in U. \tag{4.11}$$

Proof. Apply Proposition 4.2 to f and g, set $C = 1 + \sup_{x\in U} |h(x)|$ and use ϕ/C instead of ϕ. ∎

REMARK 4.5. Consider the situation described in Proposition 4.2. If the support of g is a bounded set and if we do not assume that f is $\mathcal{P}^\infty$ on $U \setminus Z(g)$, we obtain a definable continuous function h with bounded support such that $\phi(g) = hf$. Still, $\phi(g)$ is a $\mathcal{P}^\infty$ function and, in addition, we may assume that $|h| < 1$.

REMARK 4.6. If the structure is assumed to be *exponentially bounded*, that is, every continuous definable function $f : (0, \infty) \to R$ is ultimately bounded by a finite composition of exponential functions, Proposition 4.2 is valid without the assumption of $\mathcal{P}^\infty$ cell decomposition.

5. Proof of Theorem 1.2

LEMMA 5.1. *Let* $X \subset R^n$ *be a bounded open* $\mathcal{P}^\infty$ *cell. Then there is a definable* $\mathcal{P}^\infty$ *function* $F : R^n \to [0, \infty)$ *such that* $Z(F) = R^n \setminus X$.

Proof. We prove that for any bounded open $\mathcal{P}^\infty$ cell $X \subset R^n$ there is a definable continuous function $\phi : R^n \to [0, \infty)$ which is $\mathcal{P}^\infty$ on X such that $Z(\phi) = R^n \setminus X$.

$n = 1$: Then $X = (a, b)$ and we let ϕ vanish outside X and $\phi(x) = (x - a)(b - x)$ in $x \in X$.

$n - 1 \rightsquigarrow n$: Here $X = (f, g)_Y$ where $Y \subset R^{n-1}$ is an open $\mathcal{P}^\infty$ cell and $f, g : Y \to R$ are $\mathcal{P}^\infty$ functions. By the induction hypothesis there is a definable and continuous function $\phi_1 : R^{n-1} \to [0, \infty)$ which is $\mathcal{P}^\infty$ on Y and such that $Z(\phi_1) = R^{n-1} \setminus Y$. For $x = (x_1, \ldots, x_n) \in R^n$ we let $x' = (x_1, \ldots, x_{n-1})$. We define $\phi : R^n \to [0, \infty)$ by

$$\phi(x) = \begin{cases} \phi_1(x')(x_n - f(x'))(g(x') - x_n) & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

The properties of ϕ_1, f and g imply the desired properties of ϕ.

Now, by applying Corollary 4.3 to ϕ we obtain a function which satisfies the conclusion of the lemma. ∎

LEMMA 5.2. *Every bounded open definable set $U \subset R^n$ is the union of finitely many sets which are open $\mathcal{P}^\infty$ cells after some permutation of coordinates.*

Proof. After applying $\mathcal{P}^\infty$ cell decomposition to U it remains to show that if $C \subset U$ is a $\mathcal{P}^\infty$ cell of dimension $n-d$, then there is, after some permutation of coordinates, an open $\mathcal{P}^\infty$ cell $X \subset U$ such that $C \subset X$.

We prove this by induction on d.

The case $d = 0$ is evident. Let $d > 0$.

By a suitable permutation of coordinates we may assume that C is the graph of some $\mathcal{P}^\infty$ function $h = (h_{n-d+1}, \ldots, h_n) : D \to R^d$ where $D \subset R^{n-d}$ is an open $\mathcal{P}^\infty$ cell (cf. end of Section 3). Let $\Delta : D \to (0, \infty)$ be given by $\Delta(x) = \operatorname{dist}((x, h(x)), \partial U)$ where dist is the Euclidean distance function whose second argument is a set. Then, by Lemma 5.1 and Remark 4.5, there exists a $\mathcal{P}^\infty$ function $\varphi : D \to (0, \infty)$ such that $\varphi(x) < \Delta(x)$ for $x \in D$. We set $f(x) = h_{n-d+1}(x) - \varphi(x)$ and $g(x) = h_{n-d+1}(x) + \varphi(x)$, and we define $h_0 : (f, g)_D \to R$ by $h_0(x) = (h_{n-d+2}(x), \ldots, h_n(x))$ for $x \in D$. Then $Y = (f, g)_D$ is an $n-d+1$-dimensional $\mathcal{P}^\infty$ cell and $C \subset (h_0)_Y \subset U$. We obtain by the induction hypothesis an open $\mathcal{P}^\infty$ cell $X \subset U$ which contains $(h_0)_Y$, and therefore $C \subset X$. ∎

Proof of Theorem 1.2. Using the map $\tau : R^n \to (-1, 1)^n$ which is defined by

$$(x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 + x_1^2}}, \ldots, \frac{x_n}{\sqrt{1 + x_n^2}} \right) \tag{5.2}$$

we can reduce our considerations to definable closed subsets (relative to $(-1, 1)^n$) of $(-1, 1)^n$, so that Theorem 1.2 follows directly from Lemmas 5.1 and 5.2. ∎

REMARK 5.3. The statement of Lemma 5.2 remains true without the assumption that U is bounded. This is due to the fact that $\tau^{-1}(C)$ is a $\mathcal{P}^\infty$ cell after some permutation of coordinates if C is a $\mathcal{P}^\infty$ cell.

6. Consequences

6.1. *Standard consequences.* As a first consequence of Theorem 1.2 we obtain $\mathcal{P}^\infty$ separation of disjoint definable closed sets.

COROLLARY 6.1. *Let $A, B \subset R^n$ be definable, closed and disjoint. There is a definable $\phi \in \mathcal{P}^\infty(R^n, [0, 1])$ with $Z(\phi) = A$ and $\phi^{-1}(\{1\}) = B$.*

Proof. Let $g, h \in \mathcal{P}^\infty(R^n, [0, 1])$ with $Z(g) = A$ and $Z(h) = B$. Such functions exist. If a definable $\mathcal{P}^\infty$ function $f : R^n \to R$ satisfies $Z(f) = A$, then $g := f^2/(1+f^2)$ is in $\mathcal{P}^\infty(R^n,[0,1])$ with $Z(g) = A$. The function h is constructed similarly. Define $\phi : R^n \to R$ by

$$\phi := \frac{g(1+2h)}{g+2h}. \tag{6.1}$$

Then ϕ has the desired properties. ∎

The next corollary concerns $\mathcal{P}^\infty$ partitions of unity.

Corollary 6.2. *Let $U_1,\ldots,U_k \subset R^n$ be a definable open cover of R^n. Then there are functions $\phi_i \in \mathcal{P}^\infty(R^n,[0,1])$, $i = 1,\ldots,k$, such that $\sum_{i=1}^k \phi_i = 1$ and $Z(\phi_i) = R^n \setminus U_i$ for $i = 1,\ldots,k$.*

Proof. Let $\psi_i \in \mathcal{P}^\infty(R^n,[0,1])$ with $Z(\psi_i) = R^n \setminus U_i$ for $i = 1,\ldots,k$. Then the functions ϕ_i defined by

$$\phi_i = \frac{\psi_i}{\sum_{j=1}^k \psi_j}, \qquad i = 1,\ldots,k, \tag{6.2}$$

have the desired properties. ∎

By application of Corollary 6.1 we obtain a weak extension property for $\mathcal{P}^\infty$ functions.

Corollary 6.3. *Let $A \subset R^n$ be a definable closed set and let $f : A \to R$ be a $\mathcal{P}^\infty$ function. Then there is a $\mathcal{P}^\infty$ function $F : R^n \to R$ whose restriction to A equals f.*

Proof. By assumption on f there is a definable open neighbourhood U of A and a $\mathcal{P}^\infty$ function $g : U \to R$ such that $g|_A = f$. Let V be a definable open neighbourhood of A whose closure is contained in U. Take a function ϕ as in Corollary 6.1 with $B = R^n \setminus V$ and set $F := g(1-\phi)$. ∎

6.2. *$\mathcal{P}^\infty$ approximation.* Our next aim is to show that definable continuous functions can be $\mathcal{C}^0$ fine approximated by definable $\mathcal{P}^\infty$ functions. This requires the concept of $\mathcal{P}^\infty$ stratification.

A $\mathcal{P}^\infty$ *stratification* which is *compatible* with the definable sets $A_1,..,A_k \subset R^n$ is a finite partition of R^n into $\mathcal{P}^\infty$ cells $S_1,\ldots,S_r$, called *strata*, such that each A_i and ∂S_j, $i = 1,\ldots,k$ and $j = 1,\ldots,r$, is the union of some of the strata.

The o-minimal structures we consider imply $\mathcal{P}^\infty$ stratification. The proof is the same as for $\mathcal{C}^0$ cells (cf. [3, Chapter 3]), one only has to replace "cell" by "$\mathcal{P}^\infty$ cell".

The following proposition is motivated by Efroymson's theorem on approximation of continuous semialgebraic functions by Nash functions (see [9] and [19], and see [11] and [12] for corresponding results for $\mathcal{C}^\infty$ functions for o-minimal expansions of $\mathbb{R}$ with exponential function and $\mathcal{C}^\infty$ cell decomposition).

Proposition 6.4. *Let $U \subset R^n$ be open and let $f : U \to R$ be a definable continuous function. For every definable continuous function $\varepsilon : U \to (0,\infty)$ there is a definable $\mathcal{P}^\infty$ function $F : U \to R$ such that*

$$|F(u) - f(u)| < \varepsilon(u), \qquad u \in U. \tag{6.3}$$

Proof. By applying a $\mathcal{P}^\infty$ cell decomposition there are finitely many disjoint definable subsets $A_1, \ldots, A_s$ of U which cover U such that $f|_{A_i}$ is a $\mathcal{P}^\infty$ function, $i = 1, \ldots, s$. Using a $\mathcal{P}^\infty$ stratification compatible with $A_1, \ldots, A_s$ we obtain disjoint $\mathcal{P}^\infty$ cells $X_1, \ldots, X_N$ which cover U such that $f|_{X_i}$ is a $\mathcal{P}^\infty$ function, $i = 1, \ldots, N$.

We put the sets X_i in such order that $\dim(X_i) \geq \dim(X_{i+1})$ for $i = 1, \ldots, N-1$. By the properties of stratification there is for each X_i a definable open neighbourhood U_i which has empty intersection with any X_j, $j \neq i$, of dimension less than or equal to $\dim(X_i)$.

We construct definable continuous functions $F_1, \ldots, F_N : U \to R$ which are $\mathcal{P}^\infty$ smooth on $U \setminus \bigcup_{j>l} X_j$ such that

$$|F_l(u) - f(u)| < \frac{l\varepsilon(u)}{N}, \qquad u \in U, \tag{6.4}$$

and

$$F_l(u) = f(u) \quad \text{ for } u \in U \setminus \bigcup_{k=1}^{l} U_k. \tag{6.5}$$

Let k be the largest integer such that $\dim(X_k) = n$. We set $F_1 = \cdots = F_k = f$.

Suppose that $F_1, \ldots, F_{l-1}$ with the desired properties are already constructed. Since $F_{l-1}|_{X_l}$ is $\mathcal{P}^\infty$ there exist a definable open neighbourhood $V_l \supset X_l$ and a $\mathcal{P}^\infty$ function $g_l : V_l \to R$ with $f|_{X_l} = F_{l-1}|_{X_l} = g_l|_{X_l}$. The continuity of g_l yields a definable open set $W_l \subset U_l \cap V_l$ containing X_l such that

$$|g_l(u) - F_{l-1}(u)| < \varepsilon(u)/N, \qquad u \in W_l. \tag{6.6}$$

Set

$$\begin{aligned} W_l' &:= \{w : \operatorname{dist}(w, X_l) < \operatorname{dist}(w, U \setminus W_l)\}, \\ W_l'' &:= \left\{w : \operatorname{dist}(w, X_l) < \frac{1}{2}\operatorname{dist}(w, U \setminus W_l)\right\}, \end{aligned} \tag{6.7}$$

and select definable $\mathcal{P}^\infty$ functions $\phi, \psi : W_l \to [0,\infty)$ such that $Z(\phi) = W_l \setminus W_l'$ and $Z(\psi) = \operatorname{cl}(W_l'') \cap W_l$. The function $F_l : U \to R$ is defined by

$$F_l(u) := \begin{cases} F_{l-1}(u) & \text{if } u \notin W_l, \\ \dfrac{F_{l-1}(u)\psi(u) + g_l(u)\phi(u)}{\psi(u) + \phi(u)} & \text{if } u \in W_l. \end{cases} \tag{6.8}$$

The function F_l is definable and continuous. The continuity is seen as follows. The function g_l is continuous and locally bounded at every point of

$\mathrm{cl}(W_l') \cap U$. As ϕ vanishes outside of W_l', the product ϕg_l is continuous in W_l and extends continuously to U by setting the value 0 outside of W_l.

In addition, the function F_l is $\mathcal{P}^\infty$ smooth in W_l and in $U \setminus \bigcup_{j>l-1} X_j$, and therefore F_l is $\mathcal{P}^\infty$ smooth in $U \setminus \bigcup_{j>l} X_j$. By the choice of W_l we obtain the estimate

$$(6.9) \qquad |F_l(u) - f(u)| < |F_l(u) - F_{l-1}(u)| + |F_{l-1}(u) - f(u)| < l\varepsilon(u)/N$$

for $u \in U$. Set $F = F_N$. ■

REMARK 6.5. A careful study of the previous proof strengthens the statement of Proposition 6.4. We take the situation described in Proposition 6.4 and let $P \subset U$ be a definable set which contains all points at which f is not $\mathcal{P}^\infty$ smooth. If $V \subset U$ is any open definable neighbourhood of $\mathrm{cl}(P) \cap U$, we may assume that $F(u) = f(u)$ for $u \notin V$.

6.3. *Gluing properties.* As a direct consequence of the definition of $\mathcal{P}^\infty$ functions we derive the following statement.

REMARK 6.6. Let $U_1, \ldots, U_r \subset R^n$ be definable open sets, and, for $i = 1, \ldots, r$, let $f_i : U_i \to R$ be $\mathcal{P}^\infty$ functions such that $f_i(x) = f_j(x)$ for all $x \in U_i \cap U_j$. Then there is a $\mathcal{P}^\infty$ function $F : \bigcup_{i=1}^r U_i \to R$ such that $F|_{U_i} = f_i$ for $i = 1, \ldots, r$.

We now consider the gluing property on closed sets. Assume that A is a closed definable set, and recall that a definable function $f : A \to R$ is a definable $\mathcal{P}^\infty$ function if there is a definable open neighbourhood U of A and a definable $\mathcal{P}^\infty$ function $F : U \to R$ such that $F|_A = f$. By Corollary 6.3 we may assume that $U = R^n$.

Two $\mathcal{P}^\infty$ functions $f, g : R^n \to R$ are called $\mathcal{P}^\infty$ *equal* in $X \subset R^n$ if $f(x) = g(x)$ for $x \in X$, and the corresponding sequences of approximation polynomials are equal.

LEMMA 6.7. *Let C be a $\mathcal{P}^\infty$ cell and U a definable open neighbourhood of C. Let $f \in \mathcal{P}^\infty(R^n, R)$ be such that f and the zero-function are $\mathcal{P}^\infty$ equal in ∂C. Then there is an $F \in \mathcal{P}^\infty(R^n, R)$ vanishing outside of U such that F and f are $\mathcal{P}^\infty$ equal in* $\mathrm{cl}(C)$.

Proof. Using the function τ from the proof of Theorem 1.2 we may reduce our considerations to cells contained in $E = (-1, 1)^n$. After permuting coordinates we may assume that $C = (h)_X$ where $X \subset R^d$ is an open $\mathcal{P}^\infty$ cell and $h \in \mathcal{P}^\infty(X, R^{n-d})$.

Similarly to the proof of Lemma 5.2, there is a function $\varphi \in \mathcal{P}^\infty(R^d, R)$ such that φ vanishes outside X and $0 < \varphi(x) < \mathrm{dist}((x, h(x)), R^n \setminus U)$ for $x \in X$.

Let $Z = \{(x, y) : x \in X,\ \|y - h(x)\| < \varphi(x)\}$. We choose a $\varrho \in \mathcal{P}^\infty(R, R)$ with $\varrho(t) = 1$ if $t \in (-1/2, 1/2)$, and $\varrho(t) = 0$ if $t \notin (-1, 1)$, and define

$F : R^n \to R$ by

$$F(x,y) = \begin{cases} f(x,y)\varrho\left(\dfrac{\|y-h(x)\|}{\varphi(x)}\right) & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases} \tag{6.10}$$

Obviously, $\operatorname{supp}(F) \subset \operatorname{cl}(U)$. Moreover, $F \in \mathcal{P}^\infty(R^n \setminus \operatorname{cl}(Z), R)$ and $F \in \mathcal{P}^\infty(X \times R^{n-d}, R)$, hence $F \in \mathcal{P}^\infty(R^n \setminus \partial C)$. Note that f is $\mathcal{P}^\infty$ equal to the zero-function in ∂C and ϱ is a bounded function. So $F \in \mathcal{P}^\infty(R^n, R)$. Finally, for $x \in X$ and $y = h(x)$, we have $F(x,y) = f(x,y)\varrho(0) = f(x,y)$. ∎

Unlike continuously differentiable functions (see [16, p. 14, Theorem 5.5 and Remark 5.6] or [22, p. 83 Proposition 4.7 and Remark 4.8], $\mathcal{P}^\infty$ functions defined on closed sets have the gluing property without any restrictions.

PROPOSITION 6.8. *Let $A_1, \ldots, A_k \subset R^n$ be definable closed sets and $f_i : R^n \to R$ be $\mathcal{P}^\infty$ functions, $i = 1, \ldots, k$. If, for $1 \le i, j \le k$, the functions f_i and f_j are $\mathcal{P}^\infty$ equal in $A_j \cap A_i$, then there is a $\mathcal{P}^\infty$ function $F : R^n \to R$ such that F and f_i are $\mathcal{P}^\infty$ equal in A_i for $i = 1, \ldots, k$.*

Proof. We select a $\mathcal{P}^\infty$ stratification compatible with the sets $A_1, \ldots, A_k$ and denote by $S_1, \ldots, S_r$ the strata which are contained in at least one of the A_i. Without loss of generality we may assume that the strata are ordered in such a way that $\dim(S_i) \le \dim(S_{i+1})$ for $i = 1, \ldots, r-1$. Note that by the properties of stratification, the set $T_l = \bigcup_{i=1}^{l} S_i$ is closed and there is a definable open neighbourhood U_{l+1} of S_{l+1} such that $T_l \cap U_{l+1} = \emptyset$ for $l = 1, \ldots, r-1$.

We prove by induction on l that there is an $F_l \in \mathcal{P}^\infty(R^n, R)$ such that F_l and f_i are $\mathcal{P}^\infty$ equal in $A_i \cap T_l$ for $i = 1, \ldots, k$.

The case $l = 1$ is evident.

$l \rightsquigarrow l+1$: For $i = 1, \ldots, k$ let $h_i = F_l - f_i$. Then h_i and the zero-function are $\mathcal{P}^\infty$ equal in T_l. Note that ∂S_{l+1} is contained in T_l. By Lemma 6.7 there is a $g_l \in \mathcal{P}^\infty(R^n, R)$ which is $\mathcal{P}^\infty$ equal to h_i in $A_i \cap \operatorname{cl}(S_{l+1})$ for $i = 1, \ldots, k$, and which vanishes outside U_{l+1}. Now $F_{l+1} = F_l - g_l$ has the desired properties.

Set $F = F_r$. ∎

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