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Einstein-Hermitian and anti-Hermitian 4-manifolds

by Włodzimierz Jelonek (Kraków)

Abstract. We study 4-dimensional Einstein-Hermitian non-Kähler manifolds admitting a certain anti-Hermitian structure. We also describe Einstein 4-manifolds which are of cohomogeneity 1 with respect to an at least 4-dimensional group of isometries.

0. Introduction. Einstein-Hermitian non-Kähler surfaces are recently a subject of intensive investigation (see [LeB], [G-M], [C-S-V], [P-P], [A-G-2]). LeBrun has proved that every such compact surface is a blow-up of $\mathbb{C}P^2$ in at least three points and has necessarily a positive scalar curvature. He also showed that the only Einstein-Hermitian metric on the blow-up of $\mathbb{C}P^2$ at one point is D. Page's metric. Earlier Grantcharov and Muskarov [G-M] investigated compact Hermitian surfaces which are $*$-Einstein. They proved that every such non-Kähler surface is conformal to an extremal Kähler metric with non-constant positive scalar curvature and has positive (clearly constant) scalar curvature. After the works of LeBrun [LeB], Apostolov and Gauduchon [A-G-1] and Cho, Sekigawa and Vanhecke [C-S-V] it is clear that every Einstein-Hermitian surface must be $*$-Einstein. Plebański and Przanowski [P-P] have given a local classification of Einstein-Hermitian surfaces which admit a Killing vector field.

With every Hermitian non-Kähler 4-manifold (M,g,J) there are related two natural distributions $\mathcal{D}=\{X\in TM:\nabla_X J=0\}$, $\mathcal{D}^\perp=\{Y\in TM: g(Y,X)=0 \text{ for all } X\in\mathcal{D}\}$ defined in the open set $U=\{x:|\nabla J_x|\neq 0\}$. These distributions are J-invariant and on U we can define the opposite almost Hermitian structure $\overline{J}$ by $\overline{J}X=JX$ if $X\in\mathcal{D}^\perp$ and $\overline{J}X=-JX$ if $X\in\mathcal{D}$; we call it the natural opposite almost Hermitian structure. It is not difficult to check that for the famous Einstein-Hermitian manifold $\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$ with D. Page's metric (see [P], [B], [K]) the opposite structure $\overline{J}$ is Hermitian and this structure extends to a global opposite Hermitian structure.

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Natural questions arise for general Einstein-Hermitian non-Kähler manifolds: When is $\overline{J}$ Hermitian? Under what conditions does it extend to a global opposite Hermitian structure? The first question in the case of self-dual Einstein-Hermitian 4-manifolds was recently answered by Apostolov and Gauduchon [A-G-2]. We give the answers to these questions for compact Einstein-Hermitian manifolds and partial answers for arbitrary Einstein-Hermitian surfaces. Our method is based on introducing a special orthonormal frame naturally related to the Hermitian structure J and the metric g. We show that for Einstein 4-manifolds the set U is dense and the set $\{x : |\nabla J_x| = 0\}$ is a totally geodesic submanifold of (M, g). We prove that the opposite almost Hermitian structure $\overline{J}$ of an Einstein-Hermitian non-ASD surface is Hermitian if and only if the metric g is of cohomogeneity 1. We also give a local description of non-Kähler non-locally symmetric Einstein-Hermitian surfaces admitting an opposite Hermitian structure as products $\mathbb{R} \times P_0$ where P_0 is a 3-dimensional naturally reductive manifold, and prove that they are always of cohomogeneity 1 with the group of local isometries of dimension at least 4. We show that if (M, g, J) is a compact Einstein-Hermitian non-Kähler manifold for which $\overline{J}$ is integrable then $\overline{J}$ extends to a global structure and (M, g, J) is isometrically biholomorphic to $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ with D. Page's metric. From the result of LeBrun it easily follows that every Hermitian non-Kähler Einstein manifold which admits an opposite Hermitian structure must be biholomorphic to $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ with D. Page's metric so the above result for compact surfaces is a simple consequence of [LeB].

1. Hermitian 4-manifolds. Let (M, g, J) be an *almost Hermitian manifold*, i.e. J is an almost complex structure orthogonal with respect to g, i.e. $g(X, Y) = g(JX, JY)$ for all $X, Y \in \mathfrak{X}(M)$. We say that (M, g, J) is a *Hermitian manifold* if its almost Hermitian structure J is integrable. Set $\bigwedge^2 M = \bigwedge^2 T^*M$. In what follows we identify the bundle TM with T^*M by means of g, so we also write $\bigwedge^2 M = \bigwedge^2 TM$. The Hodge star operator $*$ (which depends on the orientation of M) defines an endomorphism $* : \bigwedge^2 M \to \bigwedge^2 M$ with $*^2 = \mathrm{id}$ and we denote by $\bigwedge^+, \bigwedge^-$ its eigensubbundles corresponding to $1, -1$ respectively.

In what follows we consider 4-dimensional Hermitian manifolds (M, g, J) which we also call *Hermitian surfaces*. Such manifolds are always oriented and we choose an orientation in such a way that the Kähler form $\Omega(X, Y) = g(JX, Y)$ is self-dual (i.e. $\Omega \in \bigwedge^+ M$). The vector bundle of self-dual forms admits a decomposition

$$\bigwedge^+ M = \mathbb{R}\Omega \oplus LM, \tag{1.1}$$

where LM denotes the bundle of real J-skew invariant 2-forms (i.e. $LM =$

$\{\Phi \in \bigwedge M : \Phi(JX, JY) = -\Phi(X,Y)\}$). The bundle LM is a complex line bundle over M with the complex structure $\mathcal{J}$ defined by $(\mathcal{J}\Phi)(X,Y) = -\Phi(JX,Y)$. For a 4-dimensional Hermitian manifold the covariant derivative of the Kähler form Ω is locally expressed by

$$\nabla\Omega = a \otimes \Phi + \mathcal{J}a \otimes \mathcal{J}\Phi, \tag{1.2}$$

where $\mathcal{J}a(X) = -a(JX)$. The *Lee form* θ of (M,g,J) is defined by the equality $d\Omega = \theta \wedge \Omega$. We have $\theta = -\delta\Omega \circ J$. By ϱ we denote the Ricci tensor of a Riemannian manifold (M,g) and by τ the scalar curvature of (M,g), i.e. $\tau = \operatorname{tr}_g \varrho$. A Hermitian manifold (M,g,J) is said to have *Hermitian Ricci tensor* if $\varrho(X,Y) = \varrho(JX,JY)$ for all $X, Y \in \mathfrak{X}(M)$. An involutive distribution is called a *foliation*. A foliation D is called *minimal* if each of its leaves is a minimal submanifold of (M,g), i.e. the trace of its second fundamental form (the mean curvature) vanishes. A Hermitian 4-manifold (M,g,J) is said to have an *opposite Hermitian structure* if it admits an orthogonal Hermitian structure $\overline{J}$ with anti-self-dual Kähler form $\overline{\Omega}$. We then call (M,g,J) an *anti-Hermitian manifold* with anti-Hermitian structure $\overline{J}$. For any almost Hermitian 4-manifold the following formula holds (see [G-H]):

$$\tfrac{1}{2}(\varrho(X,Y) + \varrho(JX,JY)) - \tfrac{1}{2}(\varrho^*(X,Y) + \varrho^*(Y,X)) = \tfrac{1}{4}(\tau - \tau^*)g(X,Y), \tag{1.3}$$

where ϱ^* is the $*$-*Ricci tensor* defined by

$$\varrho^*(X,Y) = \tfrac{1}{2}\operatorname{tr}\{Z \mapsto R(X,JY)JZ\}, \tag{1.4}$$

where $R(X,Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z$ and $\tau^* = \operatorname{tr}_g \varrho^*$. By $\mathcal{D}$ we denote the *nullity distribution* of (M,g,J) defined by $\mathcal{D} = \{X \in TM : \nabla_X J = 0\}$. For a Hermitian manifold it follows from (1.2) that $\mathcal{D}$ is J-invariant. Consequently, $\dim \mathcal{D} = 2$ in $M_0 = \{x \in M : \nabla J_x \neq 0\}$. We call the nullity distribution *involutive* if $\mathcal{D}_{|M_0}$ is involutive. We denote by $\mathcal{D}^\perp$ the orthogonal complement of $\mathcal{D}$ in M_0.

The curvature tensor R of a 4-dimensional manifold (M,g) determines an endomorphism $\mathcal{R}$ of the bundle $\bigwedge^2 M$ defined by $g(\mathcal{R}(X\wedge Y), Z \wedge W) = \mathcal{R}(X \wedge Y, Z\wedge W) = -R(X,Y,Z,W) = -g(R(X,Y)Z,W)$. Note that $\varrho^* = \mathcal{J}\mathcal{R}(\Omega)$ and $\tau^* = 2\mathcal{R}(\Omega,\Omega)$. Set $\mathcal{R}_{\bigwedge^+ M} = p_{\bigwedge^+ M} \circ \mathcal{R}_{|\bigwedge^+ M}$ where $p_{\bigwedge^+ M} : \bigwedge M \to \bigwedge^+ M$ is the orthogonal projection. Then $\operatorname{tr} \mathcal{R}_{\bigwedge^+ M} = \tau/4$. We also have (see [C-S-V, p. 16])

$$\frac{\tau - \tau^*}{2} = \delta\theta + 2\alpha^2, \tag{1.5}$$

where $\alpha^2 = |\nabla J|^2/8$. The *conformal scalar curvature* κ is defined by (see

[A-G-1, p. 425])

$$\kappa = \tau - \tfrac{3}{2}(|\theta|^2 + 2\delta\theta) = \tfrac{1}{2}(3\tau^* - \tau). \tag{1.6}$$

We say that an almost Hermitian manifold (M, g, J) satisfies the *second condition* (G_2) *of A. Gray* if its curvature tensor R satisfies

$$(G_2) \quad R(X,Y,Z,W) - R(JX,JY,Z,W) = R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$. We say that it satisfies the *condition* (G_3) *of A. Gray* if

$$(G_3) \quad R(JX,JY,JZ,JW) = R(X,Y,Z,W)$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$. Define $B = \frac{1}{2}(\mathcal{R} - *\mathcal{R}*)$; $W = \frac{1}{2}(\mathcal{R} + *\mathcal{R}*)_0 = \frac{1}{2}(\mathcal{R} + *\mathcal{R}*) - \frac{\tau}{12}\mathrm{Id}$; $W^+ = \frac{1}{2}(W + *W)$; $W^- = \frac{1}{2}(W - *W)$. Then

$$\mathcal{R} = \frac{\tau}{12}\,\mathrm{Id} + B + W^+ + W^-.$$

The tensor W is called the *Weyl tensor* and its components W^+, W^- are called the *self-dual* and *anti-self-dual Weyl tensors*.

In what follows we use the following result of A. Derdziński (see [S-V, p. 219, Prop. 5] or [D-V, p. 476, Cor. 7.2]).

PROPOSITION 1. *Let (M, g) be a 4-dimensional Einstein manifold such that $W \in \mathrm{End}(\bigwedge^2 M)$ has constant eigenvalues. Then (M, g) is locally symmetric.*

2. Hermitian surfaces with Hermitian Ricci tensor.

Note that for every manifold satisfying condition (G_3) we have $\mathcal{R}(LM) \subset \bigwedge^+ M$, its Ricci tensor ϱ is J-invariant and its $*$-Ricci tensor is symmetric. Indeed, since $R(j(X \wedge Y), j(Z \wedge W)) = R(X \wedge Y, Z \wedge W)$ where $j(X \wedge Y) = JX \wedge JY$, we have $\mathcal{R}(\ker(j - \mathrm{id}), \ker(j + \mathrm{id})) = 0$. Since $\ker(j - \mathrm{id}) = \bigwedge^- M \oplus \mathbb{R}\Omega$ and $\ker(j + \mathrm{id}) = LM$ we get $g(\mathcal{R}(LM), \bigwedge^- M \oplus \mathbb{R}\Omega) = 0$. Consequently, $\mathcal{R}(LM) \subset LM \subset \bigwedge^+ M$. In fact the condition $\mathcal{R}(LM) \subset \bigwedge^+ M$ holds if and only if the Ricci tensor ϱ of (M, g) is J-invariant (see [D, p. 5, (i)]) and an almost Hermitian 4-manifold (M, g, J) with J-invariant Ricci tensor and symmetric $*$-Ricci tensor satisfies (G_3).

LEMMA A. *Let (M, g, J) be a Hermitian 4-manifold. Assume that $|\nabla J| \neq 0$ on M. Then for any local orthonormal oriented basis $\{E_1, E_2\}$ of $\mathcal{D}^\perp$ there exists a global oriented orthonormal basis $\{E_3, E_4\}$ of $\mathcal{D}$ independent of the choice of $\{E_1, E_2\}$ such that*

$$\nabla\Omega = \alpha(\theta_1 \otimes \Phi + \theta_2 \otimes \Psi), \tag{2.1}$$

where $\Phi = \theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4$, $\Psi = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3$, $\alpha = \frac{1}{2\sqrt{2}}|\nabla J|$ *and* $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ *is a cobasis dual to* $\{E_1, E_2, E_3, E_4\}$. *What is more,* $\delta\Omega = -2\alpha\theta_3$, $\theta = -2\alpha\theta_4$.

Proof. Let $\{E_1, E_2\}$ be any orthonormal basis of $\mathcal{D}^\perp$, $E_2 = JE_1$. Then (1.2) holds where $a = \alpha\theta_1$. Choose any orthonormal basis $\{E'_3, E'_4 = JE'_3\}$ in $\mathcal{D}$. Define $\Phi' = \theta_1 \wedge \theta'_3 - \theta_2 \wedge \theta'_4$, $\Psi' = \theta_1 \wedge \theta'_4 + \theta_2 \wedge \theta'_3$. Then $\{\Phi', \Psi'\}$ is an oriented orthonormal local basis in LM. Thus we have

$$\Phi = (\cos\phi)\Phi' - (\sin\phi)\Psi', \qquad \Psi = (\sin\phi)\Phi' + (\cos\phi)\Psi'$$

for some local function ϕ. Then

$$\nabla\Omega = \alpha\{\theta_1((\cos\phi)\Phi' - (\sin\phi)\Psi') + \theta_2((\sin\phi)\Phi' + (\cos\phi)\Psi')\}.$$

Define $E_3 = (\cos\phi)E'_3 - (\sin\phi)E'_4$, $E_4 = (\sin\phi)E'_3 + (\cos\phi)E'_4$. Then $\{E_3, E_4\}$ is the basis we are looking for. From (2.1) it is easy to get $\delta\Omega = -2\alpha\theta_3$, $\theta = -2\alpha\theta_4$. ■

Any frame $\{E_1, E_2, E_3, E_4\}$ constructed as above will be called *standard* (or *special*).

The following lemma is well known (it means that for a Hermitian surface the component W_3^+ of the positive Weyl tensor vanishes).

Lemma B. *Let* (M, g, J) *be a Hermitian surface. Then for any local orthonormal basis* $\{\Phi, \Psi\}$ *of* LM *we have* $\mathcal{R}(\Phi, \Phi) = \mathcal{R}(\Psi, \Psi)$ *and* $\mathcal{R}(\Phi, \Psi) = 0$.

It is known that a Hermitian manifold (M, g, J) satisfies the second condition of Gray if and only if its Ricci tensor is J-invariant, it has symmetric $*$-Ricci tensor and the component W_3^+ of the positive Weyl tensor vanishes (i.e. $\mathcal{R}_{LM} = a\,\mathrm{id}_{LM}$ where $\mathcal{R}_{LM} = p_{LM} \circ \mathcal{R}_{|LM}$ and p_{LM} is the orthogonal projection $p_{LM} : \bigwedge M \to LM$). It is well known that any almost Hermitian manifold satisfying (G_2) satisfies (G_3) and that any Hermitian manifold satisfying (G_3) satisfies (G_2) (i.e. for Hermitian manifolds these two conditions are equivalent).

Lemma C. *Let* (M, g, J) *be a Hermitian surface with* J*-invariant Ricci tensor* (*i.e.* $\mathcal{R}(LM) \subset \bigwedge^+ M$). *Let* $\{E_1, E_2, E_3, E_4\}$ *be a local orthonormal frame such that* (2.1) *holds. Then*

(a) $$\Gamma_{11}^3 = \Gamma_{22}^3 = E_3 \ln\alpha,$$

(b) $$\Gamma_{44}^3 = \Gamma_{21}^4 = -\Gamma_{12}^4 = -E_3 \ln\alpha,$$

(c) $$\Gamma_{21}^3 = -\Gamma_{12}^3, \qquad \Gamma_{11}^4 = \Gamma_{22}^4,$$

(d) $$-\Gamma_{21}^3 + \Gamma_{22}^4 = \alpha,$$

(e) $$\Gamma_{33}^4 = -E_4 \ln\alpha + \alpha,$$

where $\nabla_X E_i = \sum \omega_i^j(X) E_j$ *and* $\Gamma_{kj}^i = \omega_j^i(E_k)$.

Proof. Note that $\Gamma^i_{kj} = -\Gamma^j_{ki}$. We have

$$g(\nabla_{E_1} JX, Y) = \alpha\Phi(X,Y), \quad g(\nabla_{E_2} JX, Y) = -\alpha\Psi(X,Y), \quad \nabla_{E_3} J = 0, \quad \nabla_{E_4} J = 0. \tag{2.2}$$

Write $p(X) = \frac{1}{2} g(\nabla_X \Phi, \Psi) = \omega_1^2(X) + \omega_3^4(X)$. Then

$$\nabla_X \Omega = \alpha\theta_1(X)\Phi + \alpha\theta_2(X)\Psi,$$
$$\nabla_X \Phi = -\alpha\theta_1(X)\Omega + p(X)\Psi,$$
$$\nabla_X \Psi = -\alpha\theta_2(X)\Omega - p(X)\Phi.$$

Consequently, using (2.2) we get

$$g(R(E_1,E_3).JX,Y) = -\nabla_{[E_1,E_3]}\Omega - E_3\alpha\Phi - \alpha p(E_3)\Psi, \tag{2.3a}$$

$$g(R(E_1,E_4).JX,Y) = -\nabla_{[E_1,E_4]}\Omega - E_4\alpha\Phi - \alpha p(E_4)\Psi, \tag{2.3b}$$

$$g(R(E_2,E_3).JX,Y) = -\nabla_{[E_2,E_3]}\Omega + E_3\alpha\Psi - \alpha p(E_3)\Phi, \tag{2.3c}$$

$$g(R(E_4,E_2).JX,Y) = -\nabla_{[E_4,E_2]}\Omega - E_4\alpha\Psi + \alpha p(E_4)\Phi, \tag{2.3d}$$

$$g(R(E_1,E_2).JX,Y) = -\nabla_{[E_1,E_2]}\Omega + (E_1\alpha - \alpha p(E_2))\Psi - (\alpha p(E_1) + E_2\alpha)\Phi, \tag{2.3e}$$

where as usual $R(X,Y).J = \nabla_X(\nabla_Y J) - \nabla_Y(\nabla_X J) - \nabla_{[X,Y]}J$. Recall that

$$R(X,Y).J = R(X,Y) \circ J - J \circ R(X,Y),$$

i.e. $R(X,Y)$ acts on the tensor J as a derivation. Since $\mathcal{R}(LM) \subset \bigwedge^+ M$ it is clear that

$$g(R(E_1,E_3).JX,Y) = g(R(E_4,E_2).JX,Y), \tag{2.4a}$$
$$g(R(E_3,E_2).JX,Y) = g(R(E_4,E_1).JX,Y). \tag{2.4b}$$

Consequently, from (2.3) and (2.4), using the condition $\mathcal{R}(LM) \subset \bigwedge^+ M$, we get

$$\tfrac{1}{2}\mathcal{R}(\Phi,\Psi) = -g(R(E_1,E_3).JE_1,E_3) = E_3\alpha + \alpha\theta_1([E_1,E_3]) = E_3\alpha - \alpha\Gamma^3_{11}, \tag{2.5a}$$

$$\tfrac{1}{2}\mathcal{R}(\Phi,\Psi) = g(R(E_2,E_3).JE_3,E_2) = -(E_3\alpha + \alpha\theta_2([E_2,E_3])) = -E_3\alpha + \alpha\Gamma^3_{22}, \tag{2.5b}$$

$$\tfrac{1}{2}\mathcal{R}(\Phi,\Phi) = -g(R(E_4,E_2).JE_3,E_2) = E_4\alpha - \alpha\theta_2([E_4,E_2]) = E_4\alpha - \alpha\Gamma^4_{22}, \tag{2.5c}$$

$$\tfrac{1}{2}\mathcal{R}(\Psi,\Psi) = -g(R(E_1,E_4).JE_1,E_3) = -(-E_4\alpha - \alpha\theta_1([E_1,E_4])) = E_4\alpha - \alpha\Gamma^4_{11}. \tag{2.5d}$$

$$\begin{aligned}
&(2.5e) \quad \tfrac{1}{2}\mathcal{R}(\Phi,\Psi) = -g(R(E_1,E_4).JE_1,E_4) = -(\alpha p(E_4)+\alpha\theta_2([E_1,E_4]))\\
&\qquad = \alpha\Gamma^3_{44} - \alpha\Gamma^2_{14},\\
&(2.5f) \quad \tfrac{1}{2}\mathcal{R}(\Phi,\Psi) = -g(R(E_4,E_2).JE_1,E_3) = \alpha p(E_4)+\alpha\theta_1([E_4,E_2])\\
&\qquad = -\alpha\Gamma^1_{24} - \alpha\Gamma^3_{44},\\
&(2.5g) \quad \tfrac{1}{2}\mathcal{R}(\Phi,\Phi) = g(R(E_1,E_3).JE_1,E_4) = -\alpha\theta_2([E_1,E_3]) - \alpha p(E_3)\\
&\qquad = -\alpha\Gamma^2_{13} - \alpha\Gamma^4_{33},\\
&(2.5a) \quad \tfrac{1}{2}\mathcal{R}(\Psi,\Psi) = -g(R(E_2,E_3).JE_1,E_3) = \alpha\theta_1([E_1,E_3]) - \alpha p(E_3)\\
&\qquad = \alpha\Gamma^1_{23} - \alpha\Gamma^4_{33}.
\end{aligned}$$

Note that for a Lee form θ of (M,g,J) we have $\theta = -2\alpha\theta_4$. Write $X = 2\alpha E_3$. Then $d\Omega = -2\alpha\theta_4\wedge\Omega$ and

$$L_X\Omega = d(i_X\Omega) + i_X(d\Omega) = 2d(\alpha\theta_4) = -d\theta.$$

Hence

$$L_X\Omega^2 = 2L_X\Omega\wedge\Omega = -2d\theta\wedge\Omega.$$

Since $d\Omega = \theta\wedge\Omega$ we have $d\theta\wedge\Omega = 0$. Thus $L_X\Omega^2 = 0$ and $\mathrm{div}X = 0$. This means that

$$\Gamma^3_{11} + \Gamma^3_{22} + \Gamma^3_{44} = E_3 \ln\alpha. \tag{2.6}$$

From Lemma B, (2.5) and (2.6) we get (a)–(c) of Lemma C. Since $\theta = -\delta\Omega\circ J$ we have $\nabla J(E_1,E_1) = \nabla J(E_2,E_2) = \alpha E_3$ and $\nabla J(E_3,E_3) = \nabla J(E_4,E_4) = 0$. Consequently,

$$\alpha E_3 + J(\nabla_{E_1}E_1) = \nabla_{E_1}E_2, \qquad \alpha E_3 + J(\nabla_{E_2}E_2) = -\nabla_{E_2}E_1.$$

Thus $\Gamma^3_{12}+\Gamma^4_{11} = \alpha$ and $-\Gamma^3_{21}+\Gamma^4_{22} = \alpha$ and (d) follows. On the other hand $\mathcal{R}(\Phi,\Phi)+\mathcal{R}(\Psi,\Psi) = 4(E_4\alpha - \alpha\Gamma^4_{11}) = 4(\alpha\Gamma^1_{23} - \alpha\Gamma^4_{33})$ and (e) follows. ∎

Let us recall that Apostolov and Gauduchon [A-G-1] proved that a Hermitian surface with J-invariant Ricci tensor has symmetric $*$-Ricci tensor (equivalently Ω is an eigenfield of W^+). It follows that every Hermitian surface with J-invariant Ricci tensor satisfies condition (G_3) of Gray. Recall also that the Ricci tensor ϱ of (M,g) is J-invariant if and only if $\mathcal{R}(LM)\subset\bigwedge^+ M$. Note that Lemma D below can also be deduced from [De-1] and [A-G-1] if we additionally assume that (M,g,J) is l.c.K. (locally conformally Kähler—since the nullity foliation is then spanned by a holomorphic Killing vector field ξ and $J\xi$ and clearly $[\xi,J\xi]=0$). In the compact case (M,g,J) is l.c.K. but in general it may not be l.c.K.

Lemma D. *Let (M,g,J) be a Hermitian 4-dimensional manifold whose curvature tensor $\mathcal{R}$ satisfies $\mathcal{R}(LM)\subset\bigwedge^+ M$. Then the Kähler form Ω of (M,g,J) is an eigenform of the Weyl positive tensor W^+, i.e. $W^+\Omega = \lambda\Omega$*

for $\lambda \in C^\infty(M)$ (or equivalently (M,g,J) has symmetric $*$-Ricci tensor) and the nullity distribution $\mathcal{D}$ is involutive.

Proof. Note that it is enough to prove the lemma for (M_0,g,J). Thus we can assume that $\mathcal{D}$ is a 2-dimensional J-invariant distribution. Let $\{E_3,E_4\}$ be a local orthonormal basis in $\mathcal{D}$ such that $E_4 = JE_3$. Hence

$$\nabla_{E_3} J = 0, \tag{2.7a}$$

$$\nabla_{E_4} J = 0. \tag{2.7b}$$

Consequently, we obtain

$$\nabla^2_{E_4E_3} J + \nabla_{\nabla_{E_4}E_3} J = 0, \tag{2.8a}$$

$$\nabla^2_{E_3E_4} J + \nabla_{\nabla_{E_3}E_4} J = 0. \tag{2.8b}$$

Thus $\nabla^2_{E_3E_4}J - \nabla^2_{E_4E_3}J + \nabla_{[E_3,E_4]}J = 0$. Hence

$$R(E_3,E_4).J = -\nabla_{[E_3,E_4]}J. \tag{2.9}$$

Choose a local orthonormal basis $\{E_1,E_2\}$ of $\mathcal{D}^\perp$ such that $JE_1 = E_2$ and (2.1) holds. From (2.7) we obtain

$$R(E_3,E_4,JX,Y) + R(E_3,E_4,X,JY) = -\nabla_{[E_3,E_4]}\Omega(X,Y). \tag{2.10}$$

Consequently,

$$\begin{aligned}\mathcal{R}(E_3\wedge E_4, E_2\wedge E_3 + E_1\wedge E_4) &= \mathcal{R}(E_3\wedge E_4,\Psi)\\ &= \alpha\theta_1([E_3,E_4]),\end{aligned} \tag{2.11a}$$

$$\begin{aligned}\mathcal{R}(E_3\wedge E_4, E_1\wedge E_3 - E_2\wedge E_4) &= \mathcal{R}(E_3\wedge E_4,\Phi)\\ &= \alpha\theta_2([E_3,E_4]).\end{aligned} \tag{2.11b}$$

Set $a = \mathcal{R}(E_3\wedge E_4,\Psi)$, $b = \mathcal{R}(E_3\wedge E_4,\Phi)$, $c = \mathcal{R}(E_1\wedge E_2,\Psi)$, $d = \mathcal{R}(E_1\wedge E_2,\Phi)$. Note that the form $\overline{\Omega} = E_1\wedge E_2 - E_3\wedge E_4$ is anti-self-dual ($\overline{\Omega} \in \bigwedge^- M$). Thus $c - a = 0 = d - b$. We also have $\mathcal{R}(\Omega,\Phi) = b + d$, $\mathcal{R}(\Omega,\Psi) = a + c$. Consequently,

$$\mathcal{R}(\Omega,\Phi) = 2b = 2\alpha\theta_2([E_3,E_4]), \qquad \mathcal{R}(\Omega,\Psi) = 2a = 2\alpha\theta_1([E_3,E_4]). \tag{2.12}$$

It is clear that Ω is an eigenform of W^+ if and only if $\mathcal{R}(\Omega,\Phi) = 0, \mathcal{R}(\Omega,\Psi) = 0$. The last two equations are equivalent to the symmetry of the $*$-Ricci tensor (it also means that the component W^+_2 of the positive Weyl tensor vanishes). Note also that $\mathcal{D}$ is a minimal foliation. ∎

The first part of the next lemma is well known (see [A-G-1]).

LEMMA E. *Let (M,g,J) be a Hermitian surface with J-invariant Ricci tensor. Then $\Gamma^4_{13} = -E_2\ln\alpha$, $\Gamma^4_{23} = E_1\ln\alpha$ and $d\theta$ is anti-self-dual. In particular if M is compact then (M,g,J) is locally conformally Kähler. In addition (M,g,J) is l.c.K. if and only if $E_3\alpha = 0$, $\Gamma^1_{34} = E_2\ln\alpha$, $\Gamma^2_{34} = -E_1\ln\alpha$.*

Proof. From [A-G-1, Th. 2] it follows that Ω is an eigenform of W^+. Let $\{E_1, E_2, E_3, E_4\}$ be a local special frame. From (2.3e) we deduce that the equation $\mathcal{R}(\Omega, \Phi) = 0$ is equivalent to $-\alpha\theta_1([E_1, E_2]) - (\alpha p(E_1) + E_2\alpha) = 0$. Analogously the equation $\mathcal{R}(\Omega, \Psi) = 0$ is equivalent to $-\alpha\theta_2([E_1, E_2]) - \alpha p(E_2) + E_1\alpha = 0$. We get $\Gamma^4_{13} = -E_2 \ln\alpha$, $\Gamma^4_{23} = E_1 \ln\alpha$ after some easy computation. Note that

$$\text{(2.13a)} \quad d\theta_4(E_3, E_4) = -\theta_4([E_3, E_4]) = -E_3 \ln\alpha,$$

$$\text{(2.13b)} \quad d\theta_4(E_1, E_2) = -\theta_4([E_1, E_2]) = -\Gamma^4_{12} + \Gamma^4_{21} = -2E_3 \ln\alpha,$$

$$\text{(2.13c)} \quad d\theta_4(E_1, E_3) = -\theta_4([E_1, E_3]) = -\Gamma^4_{13} + \Gamma^4_{31},$$

$$\text{(2.13d)} \quad d\theta_4(E_2, E_3) = -\theta_4([E_2, E_3]) = -\Gamma^4_{23} + \Gamma^4_{32}.$$

We also have $d\theta = -2d\alpha \wedge \theta_4 - 2\alpha d\theta_4$. From (2.13) we get

$$\text{(2.14)} \qquad -\tfrac{1}{2}d\theta = 2E_3\alpha\overline{\Omega} + (-\alpha\Gamma^1_{34} + E_2\alpha)\overline{\Phi} + (-\alpha\Gamma^4_{32} + E_1\alpha)\overline{\Psi},$$

where $\overline{\Phi} = \theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_4$, $\overline{\Psi} = \theta_1 \wedge \theta_4 - \theta_2 \wedge \theta_3$. Consequently, $d\theta$ is anti-self-dual. ■

If (M, g, J) is a Hermitian surface with $|\nabla J| \neq 0$ on M then the distributions $\mathcal{D}, \mathcal{D}^\perp$ define a natural opposite almost Hermitian structure $\overline{J}$ on M. This structure is defined as follows: $\overline{J}|_{\mathcal{D}} = -J|_{\mathcal{D}}$, $\overline{J}|_{\mathcal{D}^\perp} = J|_{\mathcal{D}^\perp}$. In the special basis we just have $\overline{J}E_1 = E_2$, $\overline{J}E_3 = -E_4$.

LEMMA F. *Let (M, g, J) be a Hermitian l.c.K. 4-manifold with Hermitian Ricci tensor. Assume that $|\nabla J| \neq 0$ on M. Then the following conditions are equivalent*:

(a) $(M, g, \overline{J})$ *is a Hermitian surface.*
(b) $\nabla\alpha \parallel E_4 = -\frac{1}{2\alpha}\theta^\sharp$.
(c) $\mathcal{D}$ *is a totally geodesic foliation.*
(d) $\mathcal{D}$ *is contained in the nullity of* $\overline{J}$.
(e) $\nabla_{E_4}E_4 = 0$ (*equivalently* $\nabla_{\theta^\sharp}\theta^\sharp \wedge \theta^\sharp = 0$).
(f) $d|\theta|^2 \wedge \theta = 0$.

Proof. Choose a local orthonormal frame $\{E_1, \ldots, E_4\}$ such that (2.1) holds. Since (M, g, J) is l.c.K. we have $d\theta = 0$ and consequently

$$\text{(2.15)} \qquad E_3 \ln\alpha = 0, \qquad \Gamma^1_{34} = E_2 \ln\alpha, \qquad \Gamma^2_{34} = -E_1 \ln\alpha.$$

From the equalities $J(\nabla_{E_3}E_3) = \nabla_{E_3}E_4$ and $J(\nabla_{E_4}E_4) = -\nabla_{E_4}E_3$ we obtain

$$\text{(2.16)} \quad -\Gamma^1_{33} = \Gamma^1_{44} = -\Gamma^2_{34} = E_1 \ln\alpha, \qquad \Gamma^2_{44} = \Gamma^1_{34} = -\Gamma^2_{33} = E_2 \ln\alpha.$$

Note that (we write $\nabla_X\theta_i = \omega^j_i(X)\theta_j$, $\overline{\Phi} = \theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_4$, $\overline{\Psi} = \theta_1 \wedge \theta_4 - \theta_2 \wedge \theta_3$)

$$\nabla(\theta_1 \wedge \theta_2) = \tfrac{1}{2}\{\Phi(\omega^4_1 + \omega^3_2) + \Psi(\omega^1_3 + \omega^4_2) + \overline{\Phi}(-\omega^4_1 + \omega^3_2) + \overline{\Psi}(-\omega^1_3 + \omega^4_2)\}.$$

Analogously

$$\nabla(\theta_3 \wedge \theta_4) = \tfrac{1}{2}\{\varPhi(\omega_1^4 + \omega_2^3) + \varPsi(\omega_3^1 + \omega_2^4) - \overline{\varPhi}(-\omega_1^4 + \omega_2^3) - \overline{\varPsi}(-\omega_3^1 + \omega_2^4)\}.$$

Note that $\nabla\varOmega = a \otimes \varPhi + b \otimes \varPsi$ and $\nabla\overline{\varOmega} = a' \otimes \overline{\varPhi} + b' \otimes \overline{\varPsi}$ where under our assumptions $a = \alpha\theta_1$ and $b = -\alpha\theta_2$. It is clear that $(M, g, \overline{J})$ is Hermitian if and only if

$$a' = b' \circ J.$$

On the other hand $a = \omega_1^4 + \omega_2^3, b = \omega_3^1 + \omega_2^4$ and

$$\text{(2.17a)} \qquad \alpha\theta_1 = \omega_1^4 + \omega_2^3, \qquad -\alpha\theta_2 = \omega_3^1 + \omega_2^4,$$

$$\text{(2.17b)} \qquad a' = -\omega_1^4 + \omega_2^3, \qquad b' = \omega_3^1 - \omega_2^4.$$

It is clear from (2.15), (2.16) that $\mathcal{D}$ is in the nullity of $\overline{J}$ if and only if $E_1\alpha = E_2\alpha = 0$. The last condition is also equivalent to $\mathcal{D}$ being totally geodesic. Recall that $\varGamma_{kj}^i = \omega_j^i(E_k)$. It is also clear from (2.17) that $(M, g, \overline{J})$ is Hermitian if and only if (b) holds. Since $\varGamma_{44}^1 = E_1 \ln\alpha$, $\varGamma_{44}^2 = E_2 \ln\alpha$, $\varGamma_{44}^3 = -E_3 \ln\alpha$, (e) is equivalent to (b). Note that $|\theta|^2 = 4\alpha^2$, thus (f) is equivalent to (b). ∎

LEMMA G. *Let (M, g, J) be a Hermitian 4-manifold. Assume that $|\nabla J| \neq 0$ on M. If (M, g, J) has Hermitian Ricci tensor then $d\overline{\varOmega} = 2(\varGamma_{12}^3 - \varGamma_{22}^4)\theta_4 \wedge \theta_1 \wedge \theta_2$. If $\delta W^+ = 0$ and $|W^+|$ is non-vanishing on M then $\alpha = \frac{1}{3}E_4 \ln|\kappa|$ (equivalently $2\alpha^2 = -\frac{1}{3}\theta^\sharp \ln|\kappa|$) and if (M, g) is Einstein then $\nabla\tau^* \parallel E_4$.*

Proof. Note that from the Cartan structure equations, $d\theta_i = -\sum_{p=1}^4 \omega_p^i \wedge \theta_p$. Hence using the above results we obtain

$$\text{(2.18a)} \qquad d(\theta_1 \wedge \theta_2) = -2\varGamma_{11}^4\theta_4 \wedge \theta_1 \wedge \theta_2,$$

$$\text{(2.18a)} \qquad d(\theta_3 \wedge \theta_4) = -2\varGamma_{12}^3\theta_4 \wedge \theta_1 \wedge \theta_2.$$

Thus

$$\text{(2.19)} \qquad d\overline{\varOmega} = 2(\varGamma_{12}^3 - \varGamma_{22}^4)\theta_4 \wedge \theta_1 \wedge \theta_2.$$

Now assume that $\delta W^+ = 0$ and $|W^+| \neq 0$ on M. Then (see [A-G-1, p. 431]) we have

$$\theta = -\tfrac{2}{3} d\ln|\kappa| = -df,$$

where $f = \frac{2}{3}\ln|3\tau^* - \tau|$. Note that since $\theta = -2\alpha\theta_4$ we get $E_1 f = E_2 f = E_3 f = 0$. Consequently, $\theta = -2\alpha\theta_4 = -E_4 f\theta_4$ and $2\alpha = E_4 f$. If (M, g) is Einstein then $\delta W = 0$ and either $\kappa = 0$ on M or $|W^+| \neq 0$ on M. Since τ is constant it follows that in the first case $\tau^* = \frac{1}{3}\tau$ is constant, while $E_1\tau^* = E_2\tau^* = E_3\tau^* = 0$ in the second case. ∎

Recall that an Einstein-Hermitian 4-manifold is l.c.K. unless it is ASD, i.e. $W^+ = 0$ (see [De-1] and [A-G-1]).

Corollary. *Let* (M, g, J) *be an Einstein-Hermitian* 4*-manifold which is not ASD. Assume that* $|\nabla J| \neq 0$ *on* M*. Then the following conditions are equivalent*:

(a) $(M, g, \overline{J})$ *is a Hermitian surface*,
(b) $\Delta\theta = \lambda\theta$ *for some* $\lambda \in C^\infty(M)$.

Proof. From (1.6) it follows that $d\kappa = -\frac{3}{2}(d|\theta|^2 + 2\Delta\theta)$. Since $d\kappa = -\frac{2}{3}\kappa\theta$ we get $d|\theta|^2 \wedge \theta = -2\Delta\theta \wedge \theta$ and the result is a consequence of Lemma F(f). ∎

Lemma H. *Let* (M, g, J) *be a Hermitian* 4*-manifold with Hermitian Ricci tensor. If* $\delta W^+ = 0$ *and* $|W^+|$ *is non-vanishing on* M *then* $\kappa \neq 0$ *on* M *and the field* $X = J(\nabla\kappa^{-1/3})$ *is a holomorphic Killing vector field for* (M, g, J)*. What is more,* $X = \frac{1}{2}\kappa^{-1/3}J(\theta^\sharp)$ *and* $|X| = \alpha\kappa^{-1/3}$*. In particular the set* $\alpha^{-1}(0)$ *is a totally geodesic submanifold of* (M, g)*.*

Proof. The first statement can be proved analogously to [De-1, Prop. 4 and Prop. 5] and [A-G-1, Prop. 1]). Note that

$$X = J(\nabla\kappa^{-1/3}) = -\tfrac{1}{3}J(\nabla\kappa)\kappa^{-4/3} = -\tfrac{1}{3}J(\nabla \ln|\kappa|)\kappa^{-1/3} = \tfrac{1}{2}\kappa^{-1/3}J(\theta^\sharp).$$

Since $\theta^\sharp = -2\alpha E_4$ we get $|X| = \alpha\kappa^{-1/3}$. ∎

Since X is a holomorphic vector field and $X_x = 0$ if and only if $\alpha(x) = 0$ we obtain

Corollary. *Let* (M, g, J) *be a Hermitian non-Kähler* 4*-manifold with Hermitian Ricci tensor. If* $\delta W^+ = 0$ *and* $|W^+|$ *is non-vanishing on* M *then the set* $F = \{x \in M : |\nabla J_x| = 0\}$ *is nowhere dense* (*i.e.* $U = M - F$ *is an open dense subset of* M).

Remark. It is easy to see exactly as above that also the following statement holds:

Let (M, g, J) be a Hermitian non-Kähler 4-manifold with Hermitian Ricci tensor. Assume that (M, g, J) is conformally Kähler and let A be a smooth function such that $dA = \frac{1}{2}\theta$. Thus $(M, \overline{g}, J)$ is Kähler where $\overline{g} = \exp(-2A)g$. Then the field $X = J(\overline{\nabla}\exp(-A)) = e^A J(\nabla A) = \frac{1}{2}e^A(\delta\Omega)^\sharp$ is a holomorphic Killing vector field for (M, g, J) and $|X| = e^A\alpha$. The set $F = \{x \in M : |\nabla J_x| = 0\}$ is totally geodesic and nowhere dense.

Lemma I. *Let* (M, g, J) *be a compact Hermitian Einstein non-Kähler* 4*-manifold. Assume that the natural opposite almost Hermitian structure* $\overline{J}$ *defined on the set* $U = M - \alpha^{-1}(0)$ *is Hermitian. Then* $\overline{J}$ *extends smoothly to a global opposite Hermitian structure* $\overline{J}$ *on* M*.*

Proof. Since (M, g) is Einstein it follows that either $W^- = 0$ or $W^- \neq 0$ everywhere. From our assumptions it follows that the scalar curvature τ of (M, g) is positive (see [G-M, Th. 1.1] and [C-S-V, Th. 2.1] or [LeB]). Thus in view of [H] and [A-D-M] we get $W^- \neq 0$ on M. On the open dense subset U

the tensor W^- has exactly two eigenvalues. Since $\operatorname{tr} W^- = 0$ and $W^- \neq 0$ everywhere it is clear that W^- has two eigenvalues everywhere. The Kähler form $\overline{\Omega}$ of $(U, g, \overline{J})$ is a simple eigenform of $W^-|_U$. In view of the above results it extends to a global simple eigenform of W^-. ■

For the description of D. Page's Einstein-Hermitian metric on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ we refer to [K] (see also [B]). The Hermitian structure on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ is given by $JH = X/f$, $JY = Z$ where X, Y, Z are left invariant vector fields on $\mathrm{SU}(2) = S^3$ and the metric on the open dense subset $U = (0, l) \times S^3$ of $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ is given by $g = dt^2 + g_t$ where $g_t = f(t)^2\theta_1^2 + g(t)^2(\theta_2^2 + \theta_3^2)$ where $\{\theta_1, \theta_2, \theta_3\}$ is the left invariant co-frame on SU(2) dual to the frame $\{X, Y, Z\}$. It is easy to show using the O'Neill formulas that $\nabla J(H, H) = 0$, which means that $H \in \mathcal{D}$. We also have $\delta\Omega \parallel X$. Hence we obtain (in Koda's notation) $\mathcal{D} = \operatorname{span}\{H, X\}$ and $E_4 = H$. Since $\nabla_H H = 0$ it follows that the natural opposite almost Hermitian structure $\overline{J}$ given by $\overline{J}H = -X/f$, $\overline{J}Y = Z$ is Hermitian. Clearly this structure extends to a global opposite Hermitian structure on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. From [LeB] it follows that an Eistein-Hermitian non-Kähler surface (M, g) is a blow-up of $\mathbb{C}P^2$ at one, two or three points in general position. C. LeBrun also proves that the Einstein-Hermitian metric on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ is uniquely determined up to isometry. It follows that the only compact Einstein-Hermitian surface with opposite Hermitian structure is isometrically biholomorphic to $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ with D. Page's metric. Indeed, if (M, g, J) is a compact Einstein surface then it satisfies $\tau(M) \leq 0$ as a blow-up of $\mathbb{C}P^2$ where $\tau(M)$ denotes the signature of (M, g, J). Since $(M, g, \overline{J})$ is also a compact Hermitian surface we have $\tau(M, \overline{J}) = -\tau(M, J) \leq 0$. Consequently, $\tau(M) = 0$ and (M, g, J) must be isometrically biholomorphic to $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ with D. Page's metric. Hence we have from [LeB] and Lemma I:

PROPOSITION 2. *Let (M, g, J) be a compact Einstein-Hermitian non-Kähler 4-manifold. Assume that the natural opposite almost Hermitian structure $\overline{J}$ defined on the set $U = \{x \in M : |\nabla J_x| \neq 0\}$ is Hermitian. Then (M, g, J) is isometrically biholomorphic to $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ with D. Page's metric.*

3. Einstein-Hermitian metrics of cohomogeneity 1. Our present aim is to show how to construct all non-compact examples of Hermitian surfaces with Hermitian natural opposite structure. Since an Einstein-Hermitian non-Kähler manifold M satisfies the condition $\nabla J \neq 0$ on an open dense subset of M we shall assume that a non-Kähler Einstein-Hermitian manifold M satisfies $\nabla J \neq 0$ on the whole of M. Recall also that a homogeneous Einstein 4-dimensional manifold is locally symmetric and that an Einstein-Hermitian surface (M, g, J) satisfies one of the two conditions: (1) (M, g) is ASD, i.e. $W^+ = 0$, or (2) $W^+ \neq 0$ on the whole of M (see [De-1], [A-G-1]). We shall prove

THEOREM 1. *Let (M, g, J) be an Einstein-Hermitian non-Kähler 4-manifold. Assume that W^+ is non-vanishing on M. Then the following conditions are equivalent*:

(a) *(M, g) is (locally) of cohomogeneity 1*,

(b) *$(M, g, \overline{J})$ is a Hermitian surface*,

(c) *M is locally isometric to the manifold $\widetilde{M} = \mathbb{R} \times P_0$, where (P_0, g_0) is a 3-dimensional naturally reductive manifold (the total space of a Riemannian submersion $p : P_0 \to M_0$ over a Riemannian surface M_0 of constant sectional curvature $K \in \{-1, 0, 1\}$) with a metric*

$$g = dt^2 + a(t)^2\theta^2 + b(t)^2 p^* g_{\mathrm{can}}, \qquad (*)$$

where $g_0 = \theta^2 + p^* g_{\mathrm{can}}$, θ is the connection form of P_0 such that $p^* d\theta = c\,\mathrm{vol}_{\mathrm{can}}$, $c \in \mathbb{R}$ and $\mathrm{vol}_{\mathrm{can}}$ is the volume form of the canonical metric on M_0.

Proof. We shall prove the implications (b)⇒(c)⇒(a)⇒(b). We start with the proof of (b)⇒(c). Note that $\overline{J}$ is a global opposite Hermitian structure on M. Let $\{E_1, E_2, E_3, E_4\}$ be a local standard frame in U. Denote by $\overline{\tau}^*$ the $*$-scalar curvature of $(M, g, \overline{J})$. Also write $\beta = \frac{1}{2\sqrt{2}}|\nabla \overline{J}|$. Then $d\overline{\Omega} = -2\varepsilon\beta\theta_4 \wedge \theta_1 \wedge \theta_2$ and $\delta\overline{\Omega} = 2\varepsilon\beta E_3, \overline{\theta} = -2\varepsilon\beta\theta_4$ where $\varepsilon \in \{-1, 1\}$, $\mathcal{D} = \{X : i_X(\Omega - \overline{\Omega}) = 0\}$. Consequently, the distribution $\mathcal{D}^\perp$ on U also extends to a globally defined distribution. From Lemma F it follows that $\nabla_{E_4} E_4 = 0$, $\nabla\alpha \parallel E_4$ and the foliation $\mathcal{D}$ is totally geodesic. Since $\mathcal{R}(\Omega, \Omega) = \frac{1}{2}\tau^*$ and $\mathcal{R}(\overline{\Omega}, \overline{\Omega}) = \frac{1}{2}\overline{\tau}^*$ we get $R_{1212} + R_{3434} = -\frac{1}{4}(\tau^* + \overline{\tau}^*)$. Since (M, g) is an Einstein space we have $R_{1212} = R_{3434}$. Thus we get $K_{\mathcal{D}} = -R_{3434} = \frac{1}{8}(\tau^* + \overline{\tau}^*)$ where $K_{\mathcal{D}}$ denotes the sectional curvature of leaves of the foliation $\mathcal{D}$. Note that τ^* is non-constant, the distribution $S = \mathrm{span}\{E_1, E_2, E_3\}$ is involutive and its leaves are $M(c) = \tau^{*-1}(c)$. Choose local coordinates (t, x_1, x_2, x_3) such that $E_4 = \partial/\partial t$. Then $c = c(t)$ and we can parameterize $M(c)$ as $M(c(t)) = M(t)$.

We shall show that every leaf of S is a 3-dimensional naturally reductive space P_0 which is the total space of a Riemannian submersion over a Riemannian surface of constant sectional curvature. Note that P_0 is one of the Lie groups $\mathrm{SU}(2), H, \mathrm{SL}(2, \mathbb{R})$ with a left invariant metric or is a Riemannian product $\mathbb{R} \times M_0$ where H denotes the Heisenberg group and M_0 is a (real) surface of constant curvature $K \in \{-1, 0, 1\}$. In view of the results of Pedersen and Tod [P-T] it is enough to show that every leaf S of S is an $\mathcal{A}$-manifold.

Using the formula

$$R_{ijks} = g(R(E_i,E_j)E_k,E_s) \\ = E_i\Gamma^s_{jk} - E_j\Gamma^s_{ik} + \Gamma^p_{jk}\Gamma^s_{ip} - \Gamma^p_{ik}\Gamma^s_{jp} - (\Gamma^p_{ij} - \Gamma^p_{ji})\Gamma^s_{pk},$$

in view of $E_1\alpha = E_2\alpha = E_3\alpha = 0$ we obtain

$$R_{1313} = R_{2424} = \Gamma^4_{11}\Gamma^4_{33} - (\Gamma^3_{12})^2, \tag{3.1.a}$$

$$R_{2323} = R_{1414} = \Gamma^4_{11}\Gamma^4_{33} - (\Gamma^3_{12})^2, \tag{3.1.b}$$

$$R_{3434} = R_{1212} = E_4^2 \ln\alpha - E_4\alpha + (\Gamma^4_{33})^2, \tag{3.1.c}$$

$$R_{1234} = 2\Gamma^4_{11}\Gamma^3_{12} - 2\Gamma^3_{12}\Gamma^4_{33}. \tag{3.1.d}$$

Denote by R^{S} the curvature tensor of the leaf S of foliation S. Then

$$g(R(X,Y)Z,W) = g(R^{\mathrm{S}}(X,Y)Z,W) + g(A(X,Z),A(Y,W)) \\ - g(A(Y,Z),A(X,W))$$

where A denotes the second fundamental form of the hypersurface S. Since also $R(E_3,E_4)\overline{J} = 0$ we obtain

$$R_{3413} = R_{3424} = R_{3423} = R_{3424} = R_{1213} \\ = R_{1224} = R_{1223} = R_{1224} = 0. \tag{3.2}$$

It is easy to see that $\varrho^{\mathrm{S}}(E_1,E_3) = \varrho^{\mathrm{S}}(E_2,E_3) = \varrho^{\mathrm{S}}(E_1,E_2) = 0$ where ϱ^{S} is the Ricci tensor of the hypersurface S. We also have

$$R^{\mathrm{S}}_{1212} = R_{1212} - \Gamma^4_{11}\Gamma^4_{22} = R_{1212} - (\Gamma^4_{11})^2, \tag{3.3a}$$

$$R^{\mathrm{S}}_{1313} = R_{1313} - \Gamma^4_{11}\Gamma^4_{33}, \tag{3.3b}$$

$$R^{\mathrm{S}}_{2323} = R_{2323} - \Gamma^4_{22}\Gamma^4_{33}. \tag{3.3c}$$

Note that from Lemma C(d) and (2.19) we have

$$\Gamma^3_{12} = \frac{\alpha - \varepsilon\beta}{2}, \qquad \Gamma^4_{11} = \frac{\alpha + \varepsilon\beta}{2}. \tag{3.4}$$

Consequently, the Ricci tensor of $(\mathrm{S}, g|_{\mathrm{S}})$ has two eigenvalues λ,μ such that

$$\lambda = \varrho^{\mathrm{S}}(E_3,E_3) = 2(\Gamma^3_{12})^2 = 2\left(\frac{\alpha - \varepsilon\beta}{2}\right)^2, \tag{3.5a}$$

$$\mu = \varrho^{\mathrm{S}}(E_1,E_1) = \varrho^{\mathrm{S}}(E_2,E_2) \\ = (\Gamma^4_{11})^2 - (\Gamma^4_{33})^2 + E_4\alpha - E_4^2\ln\alpha + (\Gamma^3_{12})^2 \\ = (\Gamma^4_{11})^2 + \frac{\tau^* + \overline{\tau}^*}{8} + (\Gamma^3_{12})^2. \tag{3.5b}$$

It is clear that λ,μ are constant on every leaf S of the foliation S. We show that $(\mathrm{S}, g|_{\mathrm{S}})$ is an $\mathcal{A}$-manifold. It is enough to show (see [J]) that $\nabla^{\mathrm{S}}_{E_1}E_1, \nabla^{\mathrm{S}}_{E_2}E_2$, $\nabla^{\mathrm{S}}_{E_1}E_2 + \nabla^{\mathrm{S}}_{E_2}E_1 \in \mathcal{D}^\perp$ and $\nabla^{\mathrm{S}}_{E_3}E_3 = 0$ where ∇^{S} is the induced Levi-Civita connection of $(\mathrm{S}, g|_{\mathrm{S}})$. The above conditions are consequences of the equations $\Gamma^3_{11} = \Gamma^3_{22} = \Gamma^3_{12} + \Gamma^3_{21} = \Gamma^1_{33} = \Gamma^2_{33} = 0$, which hold true in view of Lemma C and (2.16). It follows that every leaf $(\mathrm{S}, g|_{\mathrm{S}})$ of S is a 3-dimensional $\mathcal{A}$-manifold. Note that it also means that the metric fg of the foliated manifold $(M, \mathcal{D}, fg)$ is bundle-like for a positive function f satisfying the equation $E_4 \ln f = 2\Gamma_{11}^4 = \alpha + \varepsilon\beta$. Note that the differential form $\omega = (\alpha + \varepsilon\beta)\theta_4$ is well defined and closed (note that $d\theta_4 = 0$), thus (we can assume $H^1(M, \mathbb{R}) = 0$) there exists a function $F \in C^\infty(M)$ such that $dF = \omega$. We take $f = \exp F$. It follows that the distribution $\mathcal{D}^\perp$ is geodesic in (M, fg) (i.e. $\nabla^f_X X \in \Gamma(\mathcal{D}^\perp)$ if $X \in \Gamma(\mathcal{D}^\perp)$ where ∇^f is the Levi-Civita connection of (M, fg)), which means that $(M, fg, \mathcal{D})$ has a bundle-like metric. Thus (locally) M is a locally trivial bundle over the space of leaves $M/\mathcal{D}$ and the natural projection $p : M \to M/\mathcal{D}$ is a Riemannian submersion (this is a consequence of [M]), precisely for every $x_0 \in M$ there exists a neighborhood U of x_0 such that $p : (U, fg) \to (U/\mathcal{D}, g_*)$ is a Riemannian submersion onto the Riemannian manifold $(U/\mathcal{D}, g_*)$ with the induced Riemannian metric g_*.

It follows that the manifold $M(t)$ is isometric to a locally trivial bundle over $(M_0, b(t)^2\mathrm{can})$ (where can denotes the standard metric on M_0), i.e. it is the total space of a Riemannian submersion $p : M(c) \to M_0$ with a metric

$$g_c = a(t)^2\eta \otimes \eta + b(t)^2 p^*\mathrm{can},$$

where $\eta = \frac{1}{2\alpha}\delta\Omega$ and $a(t), b(t)$ depend only on t. Note that the horizontal space of any such fibration coincides with $\mathcal{D}^\perp$. Thus M has a metric

$$g = dt^2 + a(t)^2\eta \otimes \eta + b(t)^2 p^*\mathrm{can}, \tag{3.6}$$

where a, b are smooth functions depending on t. Note that $a = l_0|X| = l_0\kappa^{-1/3}\alpha$ for some constant $l_0 \in \mathbb{R}_+$. Every $\mathcal{A}$-manifold $M(t)$ with the metric $g_t = a(t)^2\eta \otimes \eta + b(t)^2 p^*\mathrm{can}$ admits a 3-dimensional group G of isometries whose Lie algebra $\mathfrak{g}$ consists of lifts of Killing fields on (M_0, can) (they correspond to the Killing fields on G with any left invariant metric which are right invariant vector fields on G). Thus from (3.6) it is clear that M admits a 3-dimensional group of isometries G such that G preserves every $M(t)$ and $[X, Y] = 0$ (where $X = J(\nabla\kappa^{-1/3})$) for any Killing vector field $Y \in \mathfrak{g}$. It is easy to see that the action of G extends to M and the dimension of the isometry group $\mathrm{Iso}(M, g)$ of (M, g) is at least 4.

(c)⇒(a). This is trivial, from $(*)$ it follows that the group G of isometries of P_0 acts as isometries on M with orbits P_0.

(a)⇒(b). Since $W^+ \neq 0$ it follows that every isometry of M is holomorphic. Consequently, $X\alpha = 0$ if $X \in \mathfrak{iso}(M)$. Analogously $X|W^+|^2 = 0$, thus $X\kappa = 0$ if $X \in \mathfrak{iso}(M)$. Since $\theta = -\frac{2}{3}d\ln|\kappa|$ it follows that $\nabla\alpha \parallel \theta^\sharp$. Thus from Lemma F we get (b). ■

As a corollary from the above theorem we get

THEOREM 2. *Let (M,g) be an oriented Einstein 4-manifold. Assume that (M,g) is not locally symmetric. Then the following conditions are equivalent*:

(a) *(M,g) is (locally) of cohomogeneity* 1 (*at least on an open dense subset of M*), *with the group* $\mathrm{Iso}(M)$ *of* (*local*) *isometries of dimension at least* 4.

(b) *(M,g) admits* (*up to change of orientation and up to two-fold covering*) *a compatible non-Kähler Hermitian structure J such that $|W^+| \neq 0$ and the natural opposite almost Hermitian structure $\overline{J}$ is Hermitian.*

(c) *(M,g) admits* (*up to change of orientation and up to two-fold covering*) *a compatible non-Kähler Hermitian structure J such that the natural opposite almost Hermitian structure $\overline{J}$ is Hermitian.*

(d) *An open dense subset $(U, g_{|U}) \subset (M,g)$ is locally isometric to the manifold $\widetilde{M} = \mathbb{R} \times P_0$, where (P_0, g_0) is a 3-dimensional $\mathcal{A}$-manifold* (*the total space of a Riemannian submersion $p : P_0 \to M_0$ over a Riemannian surface M_0 of constant sectional curvature $K \in \{-1,0,1\}$*) *with a metric*

$$g = dt^2 + a(t)^2\theta^2 + b(t)^2 p^* g_{\mathrm{can}},$$

where $g_0 = \theta^2 + p^* g_{\mathrm{can}}$, θ is the connection form of P_0 such that $p^* d\theta = c\,\mathrm{vol}_{\mathrm{can}}$, $c \in \mathbb{R}$ and $\mathrm{vol}_{\mathrm{can}}$ is the volume form of the canonical metric on M_0.

Proof. (a)$\Rightarrow$(b). From [De-1, Lemma 9] it follows that (M,g) has both tensors W^+, W^- degenerate. Since (M,g) is not locally symmetric it follows from Proposition 1 that at least one of the functions $|W^+|$, $|W^-|$ is not constant on M. Choose an orientation such that $|W^+|$ is not constant, in particular does not vanish on M. Then (M,g) admits a positive Hermitian non-Kähler structure J (which corresponds to a simple eigenvalue of W^+) and from Theorem 1 it follows that it also admits an opposite natural Hermitian structure $\overline{J}$. Note that $\overline{J}$ is defined on an open and dense subset $U \subset M$. If $|W^-| \neq 0$ then $\overline{J}$ extends to the whole of M as a simple eigenvalue of W^- and if $|W^-|$ is a non-zero constant then $\overline{J}$ is Kähler.

(b)$\Rightarrow$(c), (c)$\Rightarrow$(d) and (d)$\Rightarrow$(a) are now trivial, where we take $U = \{x \in M : |\nabla J_x| \neq 0\}$. Note that the metric $(*)$ always has at least 4-dimensional group of isometries and if it is not locally symmetric then it is of cohomogeneity 1. Note also that if (M,g) admits a compatible non-Kähler Hermitian structure J such that the natural opposite almost Hermitian structure $\overline{J}$ is Hermitian and $W^+ = 0$ then $W^- \neq 0$ and $\overline{J}$ extends to a global non-Kähler Hermitian structure such that the natural opposite structure for $\overline{J}$ is the Hermitian structure J (see Lemma G), so (b) is equivalent to (c). ∎

REMARK. Note that there are many examples of non-compact Einstein-Hermitian manifolds with Hermitian natural opposite structure $\overline{J}$. For example, all the examples of A. Derdziński (see [De-2]) of self-dual Einstein-

Hermitian structures on $\mathbb{C}^2$ have globally defined and Hermitian natural opposite almost Hermitian structure $\overline{J}$ ($\overline{J}$ is given by $\overline{J}e_1 = -e_3$, $\overline{J}e_2 = e_4$ in Derdziński's notation, clearly $\mathcal{D} = \text{span}\ \{e_1, e_3\}$). Note that for arbitrary functions a, b the metric $(*)$ has two Hermitian opposite structures given by the foliation $\mathcal{D} = \text{span}\{\partial/\partial t, \theta^\sharp\}$ and distribution $\mathcal{D}^\perp$. The foliation $\mathcal{D}$ is totally geodesic and is contained in the nullity of J and $\overline{J}$. From our theorems it follows that all such examples are generally of the form $\mathbb{R}\times P$ where P is a 3-dimensional naturally reductive manifold (the total space of a Riemannian submersion $p : P \to M_0$ over a Riemannian surface (M_0, can) of constant curvature $K \in \{-1, 0, 1\}$) with the metric $g = dt^2 + a(t)^2\eta\otimes\eta + b(t)^2 p^*\text{can}$ where the functions a, b satisfy a system of ODE's obtained by means of the O'Neill formulas so that the Einstein condition is satisfied.

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Institute of Mathematics
Technical University of Cracow
Warszawska 24
31-155 Kraków, Poland
E-mail: wjelon@usk.pk.edu.pl

Institute of Mathematics
Polish Academy of Sciences
Cracow Branch
Św. Tomasza 30
31-027 Kraków, Poland

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