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Jung constants of Orlicz sequence spaces

by Tao Zhang (Shanghai)

Abstract. Estimation of the Jung constants of Orlicz sequence spaces equipped with either the Luxemburg norm or the Orlicz norm is given. The exact values of the Jung constants of a class of reflexive Orlicz sequence spaces are found by using new quantitative indices for $\mathcal{N}$-functions.

1. Preliminaries. Let X be a normed linear space and $A \subset X$ be a bounded set. The diameter of A is $d(A) = \sup\{\|x-y\| : x, y \in A\}$. If $z \in X$, we set $r(A, z) = \sup\{\|x-z\| : x \in A\}$. For $A, B \subset X$, $r(A, B) = \inf\{r(A, z) : z \in B\}$ is the *relative Chebyshev radius* of A (with respect to B), and the *Chebyshev center* of A with respect to B is defined by $z(A, B) = \{y \in B : \sup\{\|x-y\| : x \in A\} = r(A, B)\}$. Clearly, $r(A, z) = r(\overline{\mathrm{co}}(A), z)$, $r(A, B) = r(\overline{\mathrm{co}}(A), B)$ and $r(A, X) = r(\overline{\mathrm{co}}(A), X)$.

Definition 1.1 (Jung [7]). The *Jung constant* $\mathrm{JC}(X)$ of a normed linear space X is defined to be

$$\mathrm{JC}(X) = \sup\left\{\frac{r(A, X)}{d(A)} : A \subset X \text{ bounded}, \ d(A) > 0\right\}. \tag{1}$$

Clearly, always $1/2 \le \mathrm{JC}(X) \le 1$. Pichugov [12] computed $\mathrm{JC}(l^p)$ (see also Corollary 3.4 in Section 3). Amir [1] proved that if X is a dual space, then

$$\mathrm{JC}(X) = \sup\left\{\frac{r(A, X)}{d(A)} : A \subset X \text{ finite}, \ d(A) > 0\right\}. \tag{2}$$

By using (2), Amir obtained the following.

Lemma 1.2 (see [1, Proposition 2.5(b)]). *Let $(X_\alpha)_{\alpha \in D}$ be a net of linear subspaces of the Banach space X, directed by inclusion, such that $\overline{\bigcup_{\alpha \in D} X_\alpha} = X$. If X is a dual space and each X_α admits a norm-1 linear projection P_α, then $\mathrm{JC}(X) = \sup_{\alpha \in D} \mathrm{JC}(X_\alpha) = \lim_{\alpha \in D} \mathrm{JC}(X_\alpha)$.*

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LEMMA 1.3 (Pichugov [12]). *Let X_n be a real n-dimensional normed space and let A be a bounded closed convex set in X_n with $r(A, X_n)$ being its Chebyshev radius. Then the point x is the Chebyshev center of A if and only if there exists an integer $N \le n+1$ for which*

(a) *there are $x_i \in A$, $i \le N$, such that $\|x_i - x\| = r(A, X_n)$ for all $i \le N$;*

(b) *there are $f_i \in X_n^*$, the dual space of X_n, $i \le N$, such that $\|f_i\| = 1$ and $\langle x_i - x, f_i\rangle = \|x_i - x\|$ for all $i \le N$;*

(c) *there are $c_i \ge 0$, $i \le N$, such that $\sum_{i=1}^N c_i = 1$ and $\sum_{i=1}^N c_i f_i = 0$.*

In this case, $\sum_{i=1}^N \sum_{j=1}^N c_i c_j \langle x_i - x_j, f_i - f_j\rangle = 2r(A, X_n)$. If $1 \le \lambda \le 2$ and

$$\Lambda = \sum_{i=1}^N \sum_{j=1}^N c_i c_j \{\langle x_i - x_j, f_i - f_j\rangle\}^\lambda,$$

then

$$\frac{2^\lambda [r(A, X_n)]^\lambda}{\left(\frac{n}{n+1}\right)^{\lambda-1}} \le \Lambda \le [d(A)]^\lambda \sum_{i=1}^N \sum_{j=1}^N c_i c_j \|f_i - f_j\|^\lambda. \tag{3}$$

LEMMA 1.4 (Pichugov [12]). *Let X be a separable and dual space. If $\{x_1, x_2, \ldots\}$ is a dense set in X and $X_n = \operatorname{span}\{x_i : 1 \le i \le n\}$, then*

$$\mathrm{JC}(X) \le \liminf_{n\to\infty} \mathrm{JC}(X_n). \tag{4}$$

Recall that Bynum [2] defined the *normal structure coefficient* $N(X)$ of a Banach space X by

$$N(X) = \inf\left\{\frac{d(A)}{r(A, A)} : A \subset X \text{ closed bounded convex, } d(A) > 0\right\}.$$

Maluta [11] denoted $[N(X)]^{-1}$ by $\widetilde{N}(X)$ and proved that $2^{-1/2} \le \widetilde{N}(X)$ for every infinite-dimensional Banach space X. Amir [1] pointed out that for every Banach space X,

$$1/2 \le \mathrm{JC}(X) \le \widetilde{N}(X) \le 1. \tag{5}$$

Next we introduce some basic facts on Orlicz spaces. Let

$$\Phi(u) = \int_0^{|u|} \phi(t)\,dt \quad \text{and} \quad \Psi(v) = \int_0^{|v|} \psi(s)\,ds$$

be a pair of complementary $\mathcal{N}$-functions. The *Orlicz sequence space* l^Φ is defined to be the set

$$l^\Phi = \Big\{x = (x(1), x(2), \ldots) : \varrho_\Phi(\lambda x) = \sum_{i=1}^\infty \Phi(\lambda|x(i)|) < \infty \text{ for some } \lambda > 0\Big\}.$$

The *Luxemburg norm* and the *Orlicz norm* are defined respectively by

$$\|x\|_{(\Phi)} = \inf\{c > 0 : \varrho_\Phi(x/c) \le 1\}$$

and

$$\begin{aligned}\|x\|_\Phi &= \sup\Big\{\sum_{i=1}^{\infty}|x(i)y(i)| : \varrho_\Psi(y) = \sum_{i=1}^{\infty}\Psi(|y(i)|) \le 1\Big\}\\ &= \inf_{k>0}\frac{1}{k}\,[1 + \varrho_\Phi(kx)].\end{aligned}$$

The norms are equivalent: $\|x\|_{(\Phi)} \le \|x\|_\Phi \le 2\|x\|_{(\Phi)}$. The closed separable subspace h^Φ of l^Φ is defined to be the set

$$h^\Phi = \Big\{x \in l^\Phi : \varrho_\Phi(\lambda x) = \sum_{i=1}^{\infty}\Phi(\lambda|x(i)|) < \infty \text{ for any } \lambda > 0\Big\}.$$

An important parameter for analysis in an Orlicz space is the rate of growth of the underlying $\mathcal{N}$-function. An $\mathcal{N}$-function $\Phi(u)$ is said to satisfy the Δ_2-*condition* for large u (resp. for small u, for all $u \ge 0$), in symbols $\Phi \in \Delta_2(\infty)$ (resp. $\Phi \in \Delta_2(0)$, $\Phi \in \Delta_2$), if there exist $u_0 > 0$ and $K > 2$ such that $\Phi(2u) \le K\Phi(u)$ for $u \ge u_0$ (resp. for $0 < u \le u_0$, for $u \ge 0$). An $\mathcal{N}$-function $\Phi(u)$ is said to satisfy the ∇_2-*condition* for large u, in symbols $\Phi \in \nabla_2(\infty)$, if there exist $u_0 > 0$ and $a > 1$ such that $\Phi(u) \le \frac{1}{2a}\Phi(au)$ for $u \ge u_0$. Similarly we define $\Phi \in \nabla_2(0)$ and $\Phi \in \nabla_2$. The basic facts on Orlicz spaces can be found in [9], [10] and [13]. For instance, l^Φ is separable if and only if $\Phi \in \Delta_2(0)$; l^Φ is reflexive if and only if $\Phi \in \Delta_2(0) \cap \nabla_2(0)$.

New quantitative indices of $\Phi(u)$ are provided by the following six constants:

$$\alpha_\Phi = \liminf_{u\to\infty}\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \beta_\Phi = \limsup_{u\to\infty}\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \tag{6}$$

$$\alpha^0_\Phi = \liminf_{u\to 0}\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \beta^0_\Phi = \limsup_{u\to 0}\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \tag{7}$$

and

$$\begin{aligned}\overline{\alpha}_\Phi &= \inf\Big\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u < \infty\Big\},\\ \overline{\beta}_\Phi &= \sup\Big\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u < \infty\Big\}.\end{aligned} \tag{8}$$

The following result will play a leading role in this paper.

Theorem 1.5. (i) $\Phi \notin \Delta_2(\infty) \Leftrightarrow \beta_\Phi = 1$; $\Phi \notin \nabla_2(\infty) \Leftrightarrow \alpha_\Phi = 1/2$;
(ii) $\Phi \notin \Delta_2(0) \Leftrightarrow \beta^0_\Phi = 1$; $\Phi \notin \nabla_2(0) \Leftrightarrow \alpha^0_\Phi = 1/2$;
(iii) $\Phi \notin \Delta_2 \Leftrightarrow \overline{\beta}_\Phi = 1$; $\Phi \notin \nabla_2(\infty) \Leftrightarrow \overline{\alpha}_\Phi = 1/2$.

The proof of Theorem 1.5 can be found in [13, p. 23].

Other well known quantitative indices of Φ are provided by the following six constants:

$$A_\Phi = \liminf_{t\to\infty} \frac{t\phi(t)}{\Phi(t)}, \quad B_\Phi = \limsup_{t\to 0} \frac{t\phi(t)}{\Phi(t)}, \tag{9}$$

$$A_\Phi^0 = \liminf_{t\to 0} \frac{t\phi(t)}{\Phi(t)}, \quad B_\Phi^0 = \limsup_{t\to 0} \frac{t\phi(t)}{\Phi(t)}, \tag{10}$$

$$\overline{A}_\Phi = \inf\left\{\frac{t\phi(t)}{\Phi(t)} : 0 < t < \infty\right\}, \quad \overline{B}_\Phi = \sup\left\{\frac{t\phi(t)}{\Phi(t)} : 0 < t < \infty\right\}. \tag{11}$$

It is also known that $\Phi \notin \Delta_2(\infty) \Leftrightarrow B_\Phi = \infty$; $\Phi \notin \nabla_2(\infty) \Leftrightarrow A_\Phi = 1$; $\Phi \notin \Delta_2(0) \Leftrightarrow A_\Phi^0 = 1$; $\Phi \notin \Delta_2 \Leftarrow \overline{B}_\Phi = \infty$; and $\Phi \notin \nabla_2 \Leftarrow \overline{A}_\Phi = 1$. Furthermore, we have the following.

PROPOSITION 1.6. *Let Φ and Ψ be a pair of complementary $\mathcal{N}$-functions. Then*

$$\frac{1}{A_\Phi} + \frac{1}{B_\Psi} = \frac{1}{A_\Phi^0} + \frac{1}{B_\Psi^0} = \frac{1}{\overline{A}_\Phi} + \frac{1}{\overline{B}_\Psi} = 1. \tag{12}$$

PROPOSITION 1.7. *Let $\Phi(u)$ be an $\mathcal{N}$-function. Then*

$$2^{-1/A_\Phi} \le \alpha_\Phi \le \beta_\Phi \le 2^{-1/B_\Phi}, \tag{13}$$

$$2^{-1/A_\Phi^0} \le \alpha_\Phi^0 \le \beta_\Phi^0 \le 2^{-1/B_\Phi^0}, \tag{14}$$

$$2^{-1/\overline{A}_\Phi} \le \overline{\alpha}_\Phi \le \overline{\beta}_\Phi \le 2^{-1/\overline{B}_\Phi}. \tag{15}$$

The proofs of Propositions 1.6 and 1.7 can be found in [13, p. 27].

Finally, we need some properties of Hadamard matrices, which can found in [12], [6] and [5]. The *Hadamard matrix* $H_{(n+1)\times(n+1)}$ of order $n+1$ is defined to be a square matrix with entries ± 1 and with pairwise orthogonal rows. $H_{(n+1)\times(n+1)}$ is said to be in *normalized form* if its first column and row consist only of ones. Removing the first column of $H_{(n+1)\times(n+1)}$, we obtain the matrix $H_{n\times(n+1)}$, which is used in [12] and [6, Lemma 2].

THEOREM 1.8. *Let Φ and Ψ be a pair of complementary $\mathcal{N}$-functions. Then*

$$2\alpha_\Phi^0 \beta_\Psi^0 = 1 = 2\alpha_\Psi^0 \beta_\Phi^0.$$

EXAMPLE 1.9. If $n+1 = 4$, then

$$H_{4\times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad H_{3\times 4} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Let Φ, Ψ be a pair of complementary $\mathcal{N}$-functions. Then $l^{(\Phi)}(3) = \operatorname{span}\{e_i : 1 \le i \le 3\}$ is a subspace of $l^{(\Phi)}$, where $e_i = (\overbrace{0, 0, \ldots, 1}^{i}, 0, \ldots)$.

We denote by

$$[x_1, x_2, x_3, x_4] = \Phi^{-1}(1/3)(e_1, e_2, e_3)H_{3\times 4}$$

the fact that

$$\begin{aligned}
x_1 &= \Phi^{-1}(1/3)(e_1 + e_2 + e_3) = \Phi^{-1}(1/3)(1, 1, 1, 0, 0, \ldots),\\
x_2 &= \Phi^{-1}(1/3)(e_1 - e_2 - e_3) = \Phi^{-1}(1/3)(1, -1, -1, 0, 0, \ldots),\\
x_3 &= \Phi^{-1}(1/3)(-e_1 + e_2 - e_3) = \Phi^{-1}(1/3)(-1, -1, 1, 0, 0, \ldots),\\
x_4 &= \Phi^{-1}(1/3)(-e_1 - e_2 + e_3) = \Phi^{-1}(1/3)(-1, -1, 1, 0, 0, \ldots).
\end{aligned}$$

Then

$$\|x_i\|_{(\Phi)} = \Phi^{-1}(1/3)\| \pm e_1 \pm e_2 \pm e_3\|_{(\Phi)} = 1,$$

where $1 \le i \le 4$ and for $i \ne j$,

$$\|x_i - x_j\|_{(\Phi)} = \frac{2\Phi^{-1}(1/3)}{\Phi^{-1}(1/2)}.$$

Hence

$$d(A_4) = \frac{2\Phi^{-1}(1/3)}{\Phi^{-1}(1/2)}.$$

Put

$$y_i = \frac{x_i}{3[\Phi^{-1}(1/3)]^2}, \qquad 1 \le i \le 4.$$

Then $\|y_i\|_\Psi = 1$, $y_i \in l^\Psi(3) = (l^{(\Phi)})^*$. Since $\|\pm e_1 \pm e_2 \pm e_3\|_\Psi = 3\Psi^{-1}(1/3)$, if we put $c_i = 1/4$ then $\sum_{i=1}^4 c_i y_i = 0$ and $\sum_{i=1}^4 c_i = 1$. Therefore, by Lemma 1.3, the set A_4 has zero as its Chebyshev center in $l^{(\Phi)}(3) = \mathrm{span}\{e_i : 1 \le i \le 3\} \subset l^{(\Phi)}$ (see also [4, Lemma 2]). It follows from (1) that

$$\mathrm{JC}(l^{(\Phi)}(3)) \ge \frac{r(A_4, l^{(\Phi)}(3))}{d(A_4)} = \frac{\Phi^{-1}(1/2)}{2\Phi^{-1}(1/3)}.$$

In general, if for some n_0,

$$n_0 + 1 \in D = \{n + 1 \in \mathbb{N} : \text{the Hadamard matrix } H_{(n+1)\times(n+1)} \text{ exists}\},$$

then put

$$A_{n_0+1} = \{x_i : 1 \le i \le n_0 + 1\}, \qquad l^{(\Phi)}(n_0) = \mathrm{span}\{e_i : 1 \le i \le n_0\},$$

where

$$[x_1, \ldots, x_{n_0}, x_{n_0+1}] = \Phi^{-1}(1/n_0)[e_1, \ldots, e_{n_0}]H_{n_0\times(n_0+1)}.$$

Then $\|x_i\|_{(\Phi)} = 1$, $i \le n_0 + 1$. For $i \not\in j$,

$$d(A_{n_0+1}) = \|x_i - x_j\|_{(\Phi)} = \frac{2\Phi^{-1}\left(\frac{1}{n_0}\right)}{\Phi^{-1}\left(\frac{2}{n_0+1}\right)}.$$

Finally, it follows from Lemma 1.3 that

$$\mathrm{JC}(l^{(\Phi)}(n_0)) \geq \frac{r(A_{n_0+1}, l^{(\Phi)}(n_0))}{d(A_{n_0+1})} = \frac{\Phi^{-1}\left(\frac{2}{n_0+1}\right)}{2\Phi^{-1}\left(\frac{1}{n_0}\right)}.$$

LEMMA 1.10. *For an $\mathcal{N}$-function $\Phi \in \Delta_2(0) \cap \nabla_2(0)$, we have*

$$\widetilde{N}(l^{(\Phi)}) < 1 \quad \text{and} \quad \widetilde{N}(l^{\Phi}) < 1.$$

COROLLARY 1.11. *If an $\mathcal{N}$-function Φ is in $\Delta_2(0) \cap \nabla_2(0)$ then*

$$\max\{\mathrm{JC}(l^{(\Phi)}), \mathrm{JC}(l^{\Phi})\} < 1.$$

2. Lower bounds of $\mathrm{JC}(l^{(\Phi)})$

THEOREM 2.1. *For any $\mathcal{N}$-function Φ, we have*

$$\beta_\Phi^0 \leq \mathrm{JC}(h^{(\Phi)}), \tag{16}$$

$$\beta_\Phi^0 \leq \mathrm{JC}(l^{(\Phi)}). \tag{17}$$

Proof. Since $\beta_\Phi^0 = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$, there exist $1/2 > u_k \searrow 0$ such that

$$\lim_{k\to\infty} \frac{\Phi^{-1}(u_k)}{\Phi^{-1}(2u_k)} = \beta_\Phi^0.$$

For any given $0 < \varepsilon < 1/2$, there is a $u_0 \in \{u_k : k \geq 1\}$ such that

$$u_0 < \varepsilon/4 \tag{18}$$

and

$$\frac{\Phi^{-1}(u_0)}{\Phi^{-1}(2u_0)} > \beta_\Phi^0 - \varepsilon. \tag{19}$$

Put $k_0 = [1/(2u_0)]$. Then $k_0 \leq 1/(2u_0) < k_0 + 1$, and thus

$$u_0 \leq \frac{1}{2k_0} \quad \text{and} \quad \frac{1}{k_0} < \frac{2u_0}{1 - 2u_0}. \tag{20}$$

By (18) we have $2u_0 < \varepsilon/2 < \varepsilon/(1+\varepsilon)$, and so

$$1 - 2u_0 > 1 - \frac{\varepsilon}{1+\varepsilon} = \frac{1}{1+\varepsilon}. \tag{21}$$

We first show (16). Put

$$C_0 = \Phi^{-1}(1/k_0). \tag{22}$$

Put $X_0 = (\overbrace{C_0, \ldots, C_0}^{k_0})$, $Y_0 = (\overbrace{0, \ldots, 0}^{k_0})$, where $\dim X_0 = \dim Y_0 = k_0$. Set

$$x_1 = (X_0, Y_0, Y_0, \ldots),\ x_2 = (\overbrace{Y_0, X_0}^{2}, Y_0, \ldots), \ldots,\ x_i = (\overbrace{Y_0, \ldots, Y_0, X_0}^{i}, 0, 0, \ldots),$$

i.e.,

$$x_i = C_0\chi[1+(i-1)k_0, ik_0].$$

For $n<m$, we define

$$\chi[n,m] = (\overbrace{\underbrace{0,\dots,0,1}_{n},1,\dots,1}^{m},0,0,\dots).$$

For any $z=(z(1),z(2),\dots)\in h^{(\Phi)}$, we define

$$z\chi[n,m] = \{0,\dots,0,z(n),z(n+1),\dots,z(m),0,\dots\}.$$

We have $A=\{x_i : i\ge 1\}\subset S(h^{(\Phi)})$, since

$$\|x_i\|_{(\Phi)} = \frac{C_0}{\Phi^{-1}\left(\frac{1}{k_0}\right)} = 1, \qquad i\ge 1.$$

For $i\ne j$, by (19)–(21), we have

$$\begin{aligned}\|x_i-x_j\|_{(\Phi)} &= \frac{\Phi^{-1}\left(\frac{1}{k_0}\right)}{\Phi^{-1}\left(\frac{1}{2k_0}\right)} < \frac{\Phi^{-1}\left(\frac{2u_0}{1-2u_0}\right)}{\Phi^{-1}(u_0)} < \frac{\Phi^{-1}((1+\varepsilon)2u_0)}{\Phi^{-1}(u_0)}\\ &< \frac{(1+\varepsilon)\Phi^{-1}(2u_0)}{\Phi^{-1}(u_0)} < \frac{1+\varepsilon}{\beta_\Phi^0-\varepsilon}.\end{aligned}$$

Hence

$$d(A) < \frac{1+\varepsilon}{\beta_\Phi^0-\varepsilon}. \tag{23}$$

For any $z=(z(1),z(2),\dots)\in h^{(\Phi)}$, put

$$P_n z = (z(1),\dots,z(n),0,0,\dots).$$

Then

$$\|z-P_n z\|_{(\Phi)}\to 0 \quad \text{as } n\to\infty.$$

Since

$$\|P_m z-P_n z\|_{(\Phi)} \le \|P_m z - z\|_{(\Phi)} + \|z-P_n z\|_{(\Phi)}$$

we get

$$\lim_{n,m\to\infty}\|P_m z-P_n z\|_{(\Phi)} = 0.$$

Hence

$$\|z\chi[1+(i-1)k_0, ik_0]\|_{(\Phi)} = \|P_{ik_0}z - P_{(i-1)k_0}z\|_{(\Phi)} \to 0 \qquad (i\to\infty).$$

Then

$$\begin{aligned} r(A,z) &= \sup_{i\ge 1}\|x_i - z\|_{(\Phi)} \ge \limsup_{i\to\infty}\|x_i - z\|_{(\Phi)} \\ &\ge \limsup_{i\to\infty}\|(x_i - z)\chi[1+(i-1)k_0, ik_0]\|_{(\Phi)} \\ &= \limsup_{i\to\infty}\|x_i - z\chi[1+(i-1)k_0, ik_0]\|_{(\Phi)} \\ &\ge \limsup_{i\to\infty}\{\|x_i\|_{(\Phi)} - \|z\chi[1+(i-1)k_0, ik_0]\|_{(\Phi)}\} = 1. \end{aligned}$$

We see that $r(A,h^{(\Phi)}) = \inf\{r(A,z) : z \in h^{(\Phi)}\} \ge 1$. Since $z \in h^{(\Phi)}$ is arbitrary, by (23) and the definition of $\mathrm{JC}(X)$ we have

$$\mathrm{JC}(h^{(\Phi)}) \ge \frac{r(A,h^{(\Phi)})}{d(A)} > \frac{\beta_\Phi^0 - \varepsilon}{1+\varepsilon}.$$

We have thus proved (16) since ε is arbitrary.

Next we show (17). For any $z = \{z(j)\} \in l^{(\Phi)}$, we have $\lim_{i\to\infty}\sup_{j>i} z(j) = 0$. Since $l^{(\Phi)} \subset c_0$ (Chen [3, p. 169]), it follows that

$$\begin{aligned} r(A,z) &= \sup_{i\ge 1}\|x_i - z\|_{(\Phi)} \ge \limsup_{i\to\infty}\|x_i - z\|_{(\Phi)} \\ &\ge \limsup_{i\to\infty}\|(x_i - z)\chi[1+(i-1)k_0, ik_0]\|_{(\Phi)} \\ &\ge \limsup_{i\to\infty}\{\|x_i\|_{(\Phi)} - \sup_{1+(i-1)k_0\le j\le ik_0}|z(j)|\,\|\chi[1+(i-1)k_0, ik_0]\|_{(\Phi)}\} \\ &= 1 - \frac{1}{\Phi^{-1}(1/k_0)}\lim_{i\to\infty}\sup_{1+(i-1)k_0\le j}|z(j)| = 1, \end{aligned}$$

proving (17).

Corollary 2.2. (i) *For any $\mathcal{N}$-function $\Phi \notin \Delta_2(0)$, we have*

$$\mathrm{JC}(l^{(\Phi)}) = \mathrm{JC}(h^{(\Phi)}) = 1.$$

(ii) *For any $\mathcal{N}$-function Φ, we have* $\mathrm{JC}(l^{(\Phi)}) = \mathrm{JC}(h^{(\Phi)})$.

Proof. (i) If $\Phi \notin \Delta_2(0)$, then $\beta_\Phi^0 = 1$. By Theorem 2.1 we have $1 \le \mathrm{JC}(l^{(\Phi)})$, $1 \le \mathrm{JC}(h^{(\Phi)})$. For any Banach space X, we have $1/2 \le \mathrm{JC}(X) \le 1$. Hence (i) always holds.

(ii) For any $\mathcal{N}$-function Φ, either $\Phi \notin \Delta_2(0)$, or $\Phi \in \Delta_2(0)$. If $\Phi \notin \Delta_2(0)$ then by (i) we have (ii); if $\Phi \in \Delta_2(0)$ we get $h^{(\Phi)} = l^{(\Phi)}$, hence $\mathrm{JC}(l^{(\Phi)}) = \mathrm{JC}(h^{(\Phi)})$.

Theorem 2.3. *If $\Phi \in \Delta_2(0)$, then*

$$\frac{1}{2\alpha_\Phi^0} \le \mathrm{JC}(l^{(\Phi)}). \tag{24}$$

Proof. By definition of α_Φ^0, for any $n \in \mathbb{N}$ and $n \geq 2$, there are $0 < u_n < 1/(n2^n)$ such that

$$\frac{\Phi^{-1}(u_n)}{\Phi^{-1}(2u_n)} < \alpha_\Phi^0 + \frac{1}{n}. \tag{25}$$

Let $k_n = [1/(2^n u_n)]$. Then $k_n \leq 1/(2^n u_n) < k_n + 1$. Since $2^n u_n < 1/n$, we have

$$1 - 1/n < 1 - 2^n u_n < k_n 2^n u_n \leq 1. \tag{26}$$

Define $A = \{x_i : 1 \leq i \leq 2^n\}$, where

$$[x_1, \ldots, x_{2^n}] = a_n[e_1, \ldots, e_{k_n(2^n-1)}] \left.\begin{bmatrix} H_{(2^n-1)\times 2^n} \\ \vdots \\ H_{(2^n-1)\times 2^n} \end{bmatrix}\right\} k_n$$

with

$$a_n = \Phi^{-1}\left(\frac{1}{k_n(2^n-1)}\right).$$

For $n = 2$, $k_n = 3$, $k_n(2^n - 1) = 9$, we have

$$[x_1, x_2, x_3, x_4] = a_2[e_1, \ldots, e_9] \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

i.e.,

$$\begin{aligned} x_1 &= a_2(\ \ 1,\ \ 1,\ \ 1;\ \ 1,\ \ 1,\ \ 1;\ \ 1,\ \ 1,\ \ 1;\ 0, \ldots), \\ x_2 &= a_2(\ \ 1, -1, -1;\ \ 1, -1, -1;\ \ 1, -1, -1;\ 0, \ldots), \\ x_3 &= a_2(-1,\ \ 1, -1; -1,\ \ 1, -1; -1,\ \ 1, -1;\ 0, \ldots), \\ x_4 &= a_2(-1, -1,\ \ 1; -1, -1,\ \ 1; -1, -1,\ \ 1;\ 0, \ldots). \end{aligned}$$

Hence $\|x_i\|_{(\Phi)} = 1$, $1 \leq i \leq 2^n$. Let Ψ be the complementary $\mathcal{N}$-function to Φ and

$$y_i = \frac{x_i}{a_n^2 k_n(2^n-1)}, \qquad c_i = \frac{1}{2^n}, \qquad 1 \leq i \leq 2^n.$$

Then $\|y_i\|_\Psi = 1$, $\sum_{i=1}^{2^n} c_i y_i = 0$, $\langle x_i, y_j\rangle = 0$ $(i \neq j)$ and $\langle x_i, y_i\rangle = 1 = \|x_i\|_{(\Phi)}$. Let

$$l^{(\Phi)}(k_n(2^n-1)) = \operatorname{span}\{e_i : 1 \leq i \leq k_n(2^n-1)\}.$$

By Lemma 1.3, 0 is the Chebyshev center of A and

$$r(A, l^{(\Phi)}(k_n(2^n-1))) = 1. \tag{27}$$

We will prove that

$$\|x_i - x_j\|_{(\Phi)} = \frac{2a_n}{\Phi^{-1}\big(\frac{2}{k_n 2^n}\big)}, \qquad i \neq j. \tag{28}$$

By (26) we have

$$k_n(2^n-1)u_n > 1 - \frac{1}{n} - k_n u_n \geq 1 - \frac{1}{n} - \frac{1}{2^n} \tag{29}$$

and

$$2a_n = 2\Phi^{-1}\left(\frac{u_n}{k_n(2^n-1)u_n}\right) < 2\Phi^{-1}\left(\frac{u_n}{1-\frac{1}{n}-\frac{1}{2^n}}\right) < \frac{2\Phi^{-1}(u_n)}{1-\frac{1}{n}-\frac{1}{2^n}}. \tag{30}$$

On the other hand, by (26),

$$\Phi^{-1}\left(\frac{2}{k_n 2^n}\right) = \Phi^{-1}\left(\frac{2u_n}{k_n 2^n u_n}\right) \geq \Phi^{-1}(2u_n). \tag{31}$$

By (28), (30), (31) and (25) we have

$$d(A) < \frac{2\big(\alpha_\Phi^0 + \frac{1}{n}\big)}{1-\frac{1}{n}-\frac{1}{2^n}}. \tag{32}$$

By (27), (32) and the definition of JC(X) we get

$$\mathrm{JC}(l^{(\Phi)}(k_n(2^n-1))) \geq \frac{r(A, l^{(\Phi)}(k_n(2^n-1)))}{d(A)} > \frac{1-\frac{1}{n}-\frac{1}{2^n}}{2\big(\alpha_\Phi^0 + \frac{1}{n}\big)}. \tag{33}$$

Since $k_m(2^m-1) \geq k_n(2^n-1)$ we have $l^{(\Phi)}(k_n(2^n-1)) \subset l^{(\Phi)}(k_m(2^m-1))$. As $\Phi \in \Delta_2(0)$, it follows that

$$\overline{\bigcup_{n\geq 2} l^{(\Phi)}(k_n(2^n-1))} = l^{(\Phi)}.$$

Define $P_{k_n(2^n-1)} : l^{(\Phi)} \to l^{(\Phi)}(k_n(2^n-1))$ for $x = (x(1), x(2), \ldots) \in l^{(\Phi)}$ by

$$P_{k_n(2^n-1)}x = (x(1), \ldots, x(k_n(2^n-1)), 0, 0, \ldots).$$

Then $\|P_{k_n(2^n-1)}\| = 1$. Moreover, $l^{(\Phi)} = (h^\Psi)^*$. Hence, by Theorem 1.3 and (33) we get (24):

$$\begin{aligned}\mathrm{JC}(l^{(\Phi)}) &= \sup_{n\geq 2} \mathrm{JC}(l^{(\Phi)}(k_n(2^n-1))) \\ &\geq \sup_{n\geq 2} \frac{1-\frac{1}{n}-\frac{1}{2^n}}{2\big(\alpha_\Phi^0+\frac{1}{n}\big)} = \frac{1}{2\alpha_\Phi^0}.\end{aligned}$$

COROLLARY 2.4. (i) $l^{(\Phi)}$ *is not reflexive if and only if* $\mathrm{JC}(l^{(\Phi)}) = 1$.

(ii) *If* $\Phi \in \Delta_2(0) \cap \nabla_2(0)$, *then*

$$\frac{1}{\sqrt{2}} \le \max\left(\beta_\Phi^0, \frac{1}{2\alpha_\Phi^0}\right) \le \mathrm{JC}(l^{(\Phi)}). \tag{34}$$

Proof. (i) If $l^{(\Phi)}$ is not reflexive, then either $\Phi \notin \Delta_2(0)$ or $\Phi \in \Delta_2(0)\backslash\nabla_2(0)$. For $\Phi \notin \Delta_2(0)$, the conclusion follows from Corollary 2.2. For $\Phi \in \Delta_2(0)\backslash\nabla_2(0)$, by Theorem 1.5(ii) we get $\alpha_\Phi^0 = 1/2$. By (24), $1 = 1/(2\alpha_\Phi^0) \le \mathrm{JC}(l^{(\Phi)}) \le 1$. i.e., $\mathrm{JC}(l^{(\Phi)}) = 1$. If $l^{(\Phi)}$ is reflexive, by Corollary 1.10 we have $\mathrm{JC}(l^{(\Phi)}) < 1$.

(ii) If $\Phi \in \Delta_2(0) \cap \nabla_2(0)$, it is sufficient to show that $1/\sqrt{2} \le \max(\beta_\Phi^0, 1/(2\alpha_\Phi^0))$.

Indeed, otherwise $1/\sqrt{2} > \beta_\Phi^0$ and $1/\sqrt{2} > 1/(2\alpha_\Phi^0)$. Hence $\alpha_\Phi^0 > 1/\sqrt{2} > \beta_\Phi^0$, a contradiction, since $\alpha_\Phi^0 \le \beta_\Phi^0$ always holds.

Now let us turn to Orlicz sequence spaces equipped with the Orlicz norm. If Ψ is the complementary $\mathcal{N}$-function to Φ, then

$$\alpha_\Psi^0 = \liminf_{v\to 0} \frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)}. \tag{35}$$

THEOREM 2.5. *For any* $\mathcal{N}$*-function* Φ, *we have*

$$\frac{1}{2\alpha_\Psi^0} \le \mathrm{JC}(h^\Phi), \tag{36}$$

$$\frac{1}{2\alpha_\Psi^0} \le \mathrm{JC}(l^\Phi). \tag{37}$$

Proof. By (35) there exist $1/4 > v_k \searrow 0$ such that

$$\lim_{k\to\infty} \frac{\Psi^{-1}(v_k)}{\Psi^{-1}(2v_k)} = \alpha_\Psi^0.$$

For any $0 < \varepsilon < 1$, exist some $v_0 \in \{v_k : k \ge 1\}$ such that $v_0 < \varepsilon/4$ and

$$\frac{\Psi^{-1}(v_0)}{\Psi^{-1}(2v_0)} < \alpha_\Psi^0 + \varepsilon. \tag{38}$$

Put $k_0 = [1/(2v_0)]$. Then $k_0 \le 1/(2v_0) < k_0 + 1$, hence

$$\frac{1}{k_0+1} < 2v_0 \le \frac{1}{k_0} \tag{39}$$

and

$$\frac{1}{k_0} < \frac{2v_0}{1-2v_0} < 4v_0 < \varepsilon. \tag{40}$$

Put

$$C_0 = \frac{1}{k_0 \Psi^{-1}(1/k_0)}. \tag{41}$$

We first show (36). Define $A = \{x_i : i \geq 1\}$, where

$$x_1 = C_0(\overbrace{1, \ldots, 1}^{k_0}, 0, 0, \ldots),$$
$$x_2 = C_0(\overbrace{0, \ldots, 0}^{k_0}, \overbrace{1, \ldots, 1}^{k_0}, 0, 0, \ldots),\ \ldots,$$

i.e., $x_i = C_0\chi[1 + (i-1)k_0, ik_0]$. By (41) we have $\|x_i\|_\Phi = 1$ $(i \geq 1)$. For $0 < u_1 \leq u_2$,

$$\frac{\Psi^{-1}(u_2)}{u_2} \leq \frac{\Psi^{-1}(u_1)}{u_1}.$$

By (38)–(40), for $i \neq j$ we have

$$\begin{aligned}
\|x_i - x_j\|_\Phi &= C_0 2k_0 \Psi^{-1}\left(\frac{1}{2k_0}\right) \leq C_0\, \frac{\Psi^{-1}(v_0)}{v_0} \\
&< 2(\alpha_\Psi^0 + \varepsilon) C_0\, \frac{\Psi^{-1}(2v_0)}{2v_0} \\
&< 2(\alpha_\Psi^0 + \varepsilon) C_0 (k_0 + 1) \Psi^{-1}\left(\frac{1}{k_0 + 1}\right) \\
&< 2(\alpha_\Psi^0 + \varepsilon) C_0 (k_0 + 1) \Psi^{-1}\left(\frac{1}{k_0}\right) \\
&= 2(\alpha_\Psi^0 + \varepsilon)\left(1 + \frac{1}{k_0}\right) < 2(1+\varepsilon)(\alpha_\Psi^0 + \varepsilon),
\end{aligned}$$

i.e.,

$$d(A) < 2(1+\varepsilon)(\alpha_\Psi^0 + \varepsilon). \tag{42}$$

For any $z = (z(1), z(2), \ldots) \in h^\Phi$, put

$$P_n z = (z(1), \ldots, z(n), 0, 0, \ldots).$$

Then

$$\|z\chi[1 + (i-1)k_0, ik_0]\|_\Phi = \|P_{ik_0} z - P_{(i-1)k_0} z\|_\Phi \to 0, \qquad i \to \infty.$$

Hence

$$\begin{aligned}
r(A, z) &= \sup_{i \geq 1} \|x_i - z\|_\Phi \geq \limsup_{i\to\infty} \|x_i - z\|_\Phi \\
&\geq \limsup_{i\to\infty} \|(x_i - z)\chi[1 + (i-1)k_0, ik_0]\|_\Phi \\
&= \limsup_{i\to\infty} \|x_i - z\chi[1 + (i-1)k_0, ik_0]\|_\Phi \\
&\geq \limsup_{i\to\infty} \{\|x_i\|_\Phi - \|z\chi[1 + (i-1)k_0, ik_0]\|_\Phi\} = 1.
\end{aligned}$$

Since $z \in h^{\Phi}$ is arbitrary, we have $r(A, h^{\Phi}) = \inf\{r(A, z) : z \in h^{\Phi}\} \geq 1$. By (42),

$$\mathrm{JC}(h^{\Phi}) \geq \frac{r(A, h^{\Phi})}{d(A)} > \frac{1}{2(\alpha_{\Psi}^{0} + \varepsilon)(1+\varepsilon)}.$$

As ε is arbitrary we get (36).

Next we show (37). If $z = (z(1), z(2), \ldots) \in l^{\Phi}$, then

$$\begin{aligned}
&\|z\chi[1 + (i-1)k_0, ik_0]\|_{\Phi} \\
&\quad \leq \sup\{|z(j)| : 1 + (i-1)k_0 \leq j \leq ik_0\} \|\chi[1 + (i-1)k_0, ik_0]\|_{\Phi} \\
&\quad \leq k_0 \Psi^{-1}\left(\frac{1}{k_0}\right) \sup\{|z(j)| : j \geq 1 + (i-1)k_0\} \to 0 \qquad (i \to \infty).
\end{aligned}$$

Hence $r(A, l^{\Phi}) \geq 1$ and so

$$\mathrm{JC}(l^{\Phi}) \geq \frac{1}{2(1+\varepsilon)(\alpha_{\Psi}^{0} + \varepsilon)}.$$

Since ε is arbitrary we get (37).

COROLLARY 2.6. (i) *If* $\Phi \notin \Delta_2(0)$, *then* $\mathrm{JC}(l^{\Phi}) = \mathrm{JC}(h^{\Phi}) = 1$.

(ii) *For any* $\mathcal{N}$*-function* Φ *we have* $\mathrm{JC}(l^{\Phi}) = \mathrm{JC}(h^{\Phi})$.

Proof. (i) If $\Phi \notin \Delta_2(0)$, then $\Psi \notin \nabla_2(0)$. Hence $\alpha_{\Psi}^{0} = 1/2$, which implies (i) by Theorem 2.5.

(ii) For $\Phi \notin \Delta_2(0)$ see (i). For $\Phi \in \Delta_2(0)$ we have $h^{\Phi} = l^{\Phi}$. The result follows from the proof of Theorem 2.5.

THEOREM 2.7. *If* $\Phi \in \Delta_2(0)$, *then*

$$\beta_{\Psi}^{0} \leq \mathrm{JC}(l^{\Phi}), \tag{43}$$

where Ψ *is the complementary* $\mathcal{N}$*-function to* Φ *and*

$$\beta_{\Psi}^{0} = \limsup_{v \to 0} \frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)}.$$

Proof. By definition of β_{Ψ}^{0}, for any $n \in \mathbb{N}$ and $n \geq 2$, there exist $0 < v_n < 1/(n2^n)$ such that

$$\frac{\Psi^{-1}(v_n)}{\Psi^{-1}(2v_n)} > \beta_{\Psi}^{0} - \frac{1}{n}. \tag{44}$$

Let $k_n = [1/(2^n v_n)]$. Then $k_n \leq 1/(2^n v_n) < k_n + 1$, hence

$$1 - 1/n < 1 - 2^n v_n < k_n 2^n v_n \leq 1. \tag{45}$$

Define $B = \{x_i : 1 \le i \le 2^n\}$, where

$$[x_1, \dots, x_{2^n}] = b_n[e_1, \dots, e_{k_n(2^n-1)}] \left.\begin{bmatrix} H_{(2^n-1)\times 2^n} \\ \vdots \\ H_{(2^n-1)\times 2^n} \end{bmatrix}\right\} k_n$$

and

$$b_n = \frac{1}{k_n(2^n-1)\Psi^{-1}\left(\frac{1}{k_n(2^n-1)}\right)}.$$

Hence $\|x_i\|_\Phi = 1$, $1 \le i \le 2^n$. We have

$$r(B, l^\Phi(k_n(2^n-1))) = 1. \tag{46}$$

We will prove that

$$\|x_i - x_j\|_\Phi = 2b_n\left[\frac{k_n 2^n}{2}\,\Psi^{-1}\left(\frac{2}{k_n 2^n}\right)\right]. \tag{47}$$

By (45) we have

$$\Psi^{-1}\left(\frac{2}{k_n 2^n}\right) = \Psi^{-1}\left(\frac{2v_n}{k_n 2^n v_n}\right) < \Psi^{-1}\left(\frac{2v_n}{1-1/n}\right) < \frac{1}{1-1/n}\,\Psi^{-1}(2v_n).$$

By (45) we also have

$$k_n(2^n-1)v_n = k_n 2^n v_n - k_n v_n \le 1 - k_n v_n < 1 - \frac{1}{2^n}\left(1 - \frac{1}{n}\right) < 1.$$

Hence

$$\Psi^{-1}\left(\frac{1}{k_n(2^n-1)}\right) = \Psi^{-1}\left(\frac{v_n}{k_n(2^n-1)v_n}\right) > \Psi^{-1}(v_n)$$

and

$$\begin{aligned} d(B) &< \frac{k_n 2^n \Psi^{-1}(2v_n)}{k_n(2^n-1)(1-1/n)\Psi^{-1}(v_n)} \\ &< \frac{1}{(1-1/2^n)(1-1/n)(\beta_\Psi^0 - 1/n)}. \end{aligned}$$

Hence

$$\begin{aligned} J(l^\Phi(k_n(2^n-1))) &\ge \frac{r(B, l^\Phi(k_n(2^n-1)))}{d(B)} \\ &\ge \left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{n}\right)\left(\beta_\Psi^0 - \frac{1}{n}\right). \end{aligned} \tag{48}$$

Since $\Phi \in \Delta_2(0)$ we get

$$\overline{\bigcup_{n\ge 2} l^\Phi(k_n(2^n-1))} = l^\Phi.$$

By Lemma 1.2 and (48) we get (43):

$$\begin{aligned}\mathrm{JC}(l^{\Phi}) &= \sup_{n\geq 2}\mathrm{JC}(l^{\Phi}(k_n(2^n-1)))\\ &\geq \sup_{n\geq 2}\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{n}\right)\left(\beta^0_{\Psi}-\frac{1}{n}\right)=\beta^0_{\Psi}.\end{aligned}$$

COROLLARY 2.8. (i) l^{Φ} *is not reflexive if and only if* $\mathrm{JC}(l^{\Phi})=1$.

(ii) *If* $\Phi\in\Delta_2(0)\cap\nabla_2(0)$, *then*

$$\frac{1}{\sqrt{2}}\leq\max\left(\frac{1}{2\alpha^0_{\Psi}},\beta^0_{\Psi}\right)\leq\mathrm{JC}(l^{\Phi}). \tag{49}$$

Proof. (i) If l^{Φ} is not reflexive, then either $\Phi\notin\Delta_2(0)$ or $\Phi\in\Delta_2(0)\backslash\nabla_2(0)$. If $\Phi\notin\Delta_2(0)$, the argument is similar to that of Corollary 2.6(i). If $\Phi\in\Delta_2(0)\backslash\nabla_2(0)$, then $\Psi\notin\Delta_2(0)$, hence $\beta^0_{\Psi}=1$. By (43) we get $\mathrm{JC}(l^{\Phi})=1$.

(ii) If l^{Φ} is reflexive, (49) follows from the proof of Theorems 2.5 and 2.7 and Corollary 2.4.

THEOREM 2.9. *Let* Φ,Ψ *be a pair of complementary* $\mathcal{N}$*-functions. If* $\Phi\in\Delta_2(0)\cap\nabla_2(0)$, *then*

$$\max\left(\beta^0_{\Phi},\frac{1}{2\alpha^0_{\Phi}}\right)\leq\min\{\mathrm{JC}(l^{(\Phi)}),\mathrm{JC}(l^{\Psi}),\mathrm{JC}(l^{\Phi}),\mathrm{JC}(l^{(\Psi)})\}. \tag{50}$$

Proof. By Corollaries 2.4(ii) and 2.8(ii), if $\Phi\in\Delta_2(0)\cap\nabla_2(0)$, then

$$\max\left(\beta^0_{\Phi},\frac{1}{2\alpha^0_{\Phi}}\right)\leq\min\{\mathrm{JC}(l^{(\Phi)}),\mathrm{JC}(l^{\Psi})\}, \tag{51}$$

$$\max\left(\frac{1}{2\alpha^0_{\Psi}},\beta^0_{\Psi}\right)\leq\min\{\mathrm{JC}(l^{\Phi}),\mathrm{JC}(l^{(\Psi)})\}. \tag{52}$$

By Theorem 1.8, $\beta^0_{\Phi}=1/(2\alpha^0_{\Psi})$, $1/(2\alpha^0_{\Phi})=\beta^0_{\Psi}$. Hence we get (50).

EXAMPLE 2.10. Let $1<p<\infty$ and $1/p+1/q=1$. Then

$$\max(2^{-1/p},2^{1/p-1})\leq\{\mathrm{JC}(l^p),\mathrm{JC}(l^q)\}. \tag{53}$$

In fact, let $\Phi_p(u)=|u|^p$. Then $l^{(\Phi_p)}=l^p$, $\Phi_p\in\Delta_2(0)\cap\nabla_2(0)$. For $u>0$,

$$\frac{\Phi_p^{-1}(u)}{\Phi_p^{-1}(2u)}=\frac{u^{1/p}}{(2u)^{1/p}}=2^{-1/p}.$$

Hence $\alpha^0_{\Phi_p}=\beta^0_{\Phi_p}=2^{-1/p}$ and we get (53) by (51).

EXAMPLE 2.11. Let $\Phi_r(u)=e^{|u|^r}-1$, $1<r<\infty$ and let Ψ_r be the complementary $\mathcal{N}$-function to Φ_r. Then

$$\max(2^{-1/r},2^{1/r-1})\leq\min\{\mathrm{JC}(l^{(\Phi_r)}),\mathrm{JC}(l^{\Phi_r}),\mathrm{JC}(l^{\Psi_r}),\mathrm{JC}(l^{(\Psi_r)})\}. \tag{54}$$

In fact, $\Phi_r^{-1}(u) = [\ln(1+u)]^{1/r}$. Hence

$$\alpha^0_{\Phi_r} = \beta^0_{\Phi_r} = \gamma^0_{\Phi_r} = \lim_{u\to 0} \frac{\Phi_r^{-1}(u)}{\Phi_r^{-1}(2u)} = 2^{-1/r}.$$

Hence $\Phi_r \in \Delta_2(0) \cap \nabla_2(0)$ and we get (54) by (50).

EXAMPLE 2.12. Let $\Phi(u) = e^{|u|} - |u| - 1$, $\Psi(v) = (1+|v|)\ln(1+|v|) - |v|$. Since

$$C^0_\Phi = \lim_{t\to 0} \frac{t\phi(t)}{\Phi(t)} = \lim_{t\to 0} \frac{t(e^t - 1)}{e^t - t - 1} = 2$$

we get $\Phi, \Psi \in \Delta_2(0) \cap \nabla_2(0)$ and $\alpha^0_\Phi = \beta^0_\Phi = 2^{-1/2}$. By (50) we have

$$\frac{1}{\sqrt{2}} \le \min\{\mathrm{JC}(l^{(\Phi)}), \mathrm{JC}(l^{\Phi}), \mathrm{JC}(l^{(\Psi)}), \mathrm{JC}(l^{\Psi})\}.$$

EXAMPLE 2.13. Let

$$\Phi(u) = (1+|u|)^{\sqrt{\ln(1+|u|)}} - 1.$$

Then

$$C^0_\Phi = \lim_{t\to 0} \frac{t\phi(t)}{\Phi(t)} = \frac{3}{2}.$$

Hence $\Phi \in \Delta_2(0) \cap \nabla_2(0)$, $\alpha^0_\Phi = \beta^0_\Phi = \gamma^0_\Phi = 2^{-1/C^0_\Phi} = 2^{-2/3}$. By (50),

$$\frac{1}{\sqrt[3]{2}} = \max\left(\gamma^0_\Phi, \frac{1}{2\gamma^0_\Phi}\right) \le \min\{\mathrm{JC}(l^{(\Phi)}), \mathrm{JC}(l^{\Phi}), \mathrm{JC}(l^{(\Psi)}), \mathrm{JC}(l^{\Psi})\}.$$

3. $\mathrm{JC}(l^{(\Phi_s)})$ and $\mathrm{JC}(l^{\Phi_s})$

DEFINITION 3.1. Let Φ be an $\mathcal{N}$-function. Let $\Phi_0(u) = u^2$, $0 < s \le 1$, and let Φ_s be the inverse of

$$\Phi_s^{-1}(u) = [\Phi^{-1}(u)]^{1-s}[\Phi_0^{-1}(u)]^s \quad (u \ge 0). \tag{55}$$

LEMMA 3.2. *Let Φ be an $\mathcal{N}$-function.*

(i) *For any $0 < s \le 1$, $\Phi_s \in \Delta_2(0) \cap \nabla_2(0)$.*

(ii) *Let $N(l^{(\Phi_s)})$ and $N(l^{\Phi_s})$ be the normal structure coefficients of $l^{(\Phi_s)}$ and l^{Φ_s}. Then*

$$2^{s/2} \le N(l^{(\Phi_s)}), \tag{56}$$

$$2^{s/2} \le N(l^{\Phi_s}). \tag{57}$$

THEOREM 3.3. *Let Φ be an $\mathcal{N}$-function. Let Ψ_s^+ be the complementary $\mathcal{N}$-function to Φ_s. Then*

$$\max\{\mathrm{JC}(l^{(\Phi_s)}), \mathrm{JC}(l^{\Phi_s})\} \le 2^{-s/2}, \tag{58}$$

$$\max\{\mathrm{JC}(l^{(\Psi_s^+)}), \mathrm{JC}(l^{\Psi_s^+})\} \le 2^{-s/2}. \tag{59}$$

Proof. We get (58) from (56), (57) and $\mathrm{JC}(X) \le \widetilde{N}(X) = 1/N(X)$. We now prove (59). We first show

$$\mathrm{JC}(l^{\Psi_s^+}) \le 2^{-s/2}. \tag{60}$$

By Lemma 3.2, $l^{\Psi_s^+}$ is reflexive, and of course it is a separable dual space. Let $\{z_i : i \ge 1\}$ be a dense set in $l^{\Psi_s^+}$, and $X_n = \mathrm{span}\{z_i : 1 \le i \le n\}$. For any given bounded closed convex set $A \subset X_n$ with Chebyshev radius $r(A, X_n)$, there always exists a Chebyshev center x of A. In view of Lemma 1.3, there exist an integer $N \le n$, $\{x_i : i \le N\} \subset l^{\Psi_s^+}$, $\{y_i : i \le N\} \subset S((l^{\Psi_s^+})^*) = S(l^{(\Phi_s)})$ and $\{c_i \ge 0 : i \le N\}$ and $\sum_{i=1}^N c_i = 1$, which satisfy conditions (a)–(c) of Lemma 1.3. Putting $\lambda = 2/(2-s)$ in (3), we have

$$\begin{aligned}\frac{2^{\frac{2}{2-s}}[r(A,X_n)]^{\frac{2}{2-s}}}{\left(\frac{n}{n+1}\right)^{\frac{2}{2-s}-1}} &\le [d(A)]^{\frac{2}{2-s}} \sum_{i=1}^{N}\sum_{j=1}^{N} c_i c_j \|y_i - y_j\|_{(\Phi_s)}^{\frac{2}{2-s}} \\ &\le [d(A)]^{\frac{2}{2-s}} 2 \sum_{i=1}^{N} c_i \|y_i\|_{(\Phi_s)}^{\frac{2}{2-s}} = 2[d(A)]^{\frac{2}{2-s}}.\end{aligned}$$

Hence

$$\frac{r(A,X_n)}{d(A)} \le 2^{-s/2}\left(\frac{n}{n+1}\right)^{s/2}.$$

Since A is arbitrary, we obtain

$$\mathrm{JC}(X_n) \le 2^{-s/2}\left(\frac{n}{n+1}\right)^{s/2}.$$

Hence, by using Lemmas 1.4 and 3.2(i) we can prove (60):

$$\mathrm{JC}(l^{\Psi_s^+}) \le \liminf_{n\to\infty} 2^{-s/2}\left(\frac{n}{n+1}\right)^{s/2} = 2^{-s/2}.$$

Corollary 3.4 (Ivanov and Pichugov [6]). *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then*

$$\mathrm{JC}(l^p) = \mathrm{JC}(l^q) = \max(2^{1/p-1}, 2^{-1/p}). \tag{61}$$

Proof. In fact, putting $\Phi(u) = |u|^p$, we have $l^{(\Phi)} = l^p$, $\|\cdot\|_{(\Phi)} = \|\cdot\|_p$, $l^\Psi = l^q$ and $\|\cdot\|_\Psi = \|\cdot\|_q$, where

$$\Psi(v) = \frac{(q-1)^{q-1}}{q^q}|v|^q$$

is the complementary $\mathcal{N}$-function to $\Phi(u)$. By Example 2.10, we need only prove

$$\max\{\mathrm{JC}(l^p), \mathrm{JC}(l^q)\} \le \max(2^{1/p-1}, 2^{-1/p}). \tag{62}$$

If $1 < p \le 2$, we choose $1 < a < p \le 2$. Put $\Phi(u) = |u|^a$, $\Phi_0(u) = u^2$ and $s = 2(p-a)/(p(2-a))$. We have $0 < s \le 1$ and for $u \ge 0$,

$$\Phi_s^{-1}(u) = u^{(1-s)/a+s/2} = u^{1/p},$$

i.e., $\Phi_s(u) = |u|^p$. Since $l^{(\Phi_s)} = l^p$, $l^{\Psi_s^+} = l^q$ and $\lim_{a\searrow 1}(-s/2) = 1/p-1$ we get

$$\max\{\mathrm{JC}(l^p), \mathrm{JC}(l^q)\} \le 2^{1/p-1}, \quad 1 < p \le 2. \tag{63}$$

If $2 \le p < \infty$, we choose $2 \le p < b < \infty$. Let $\Phi(u) = |u|^b$ and $s = 2(b-p)/(p(b-2))$. Again we have $0 < s \le 1$ and $\Phi_s(u) = |u|^p$. Noting that $\lim_{b\nearrow\infty}(-s/2) = -1/p$, we get

$$\max\{\mathrm{JC}(l^p), \mathrm{JC}(l^q)\} \le 2^{-1/p}, \quad 2 \le p < \infty. \tag{64}$$

Finally, (62) follows from (63) and (64).

Corollary 3.5. *Let Φ_s, Ψ_s^+ be as in Theorem 3.3 with $0 < s \le 1$. Then*

$$\max\left(\beta^0_{\Phi_s}, \frac{1}{2\alpha^0_{\Phi_s}}\right) \le \{\mathrm{JC}(l^{(\Phi_s)}), \mathrm{JC}(l^{\Phi_s}), \mathrm{JC}(l^{\Psi_s^+}), \mathrm{JC}(l^{(\Psi_s^+)})\} \le 2^{-s/2}, \tag{65}$$

where

$$\alpha^0_{\Phi_s} = \liminf_{u\to 0} \frac{\Phi_s^{-1}(u)}{\Phi_s^{-1}(2u)} = (\alpha^0_\Phi)^{1-s}\left(\frac{1}{2}\right)^{s/2},$$

$$\beta^0_{\Phi_s} = \limsup_{u\to 0} \frac{\Phi_s^{-1}(u)}{\Phi_s^{-1}(2u)} = (\beta^0_\Phi)^{1-s}\left(\frac{1}{2}\right)^{s/2}.$$

Example 3.6. Let $\Phi_r(u) = e^{|u|^r} - 1$ $(1 < r < \infty)$. For $u \ge 0$, $\Phi_r^{-1}(u) = [\ln(1+u)]^{1/r}$. If $0 < s \le 1$, then

$$\Phi_s^{-1}(u, r) = [\ln(1+u)]^{(1-s)/r} u^{s/2} \quad (u \ge 0).$$

Note that $C^0_\Phi = \lim_{t\to 0} \frac{t\Phi_r'(t)}{\Phi_r(t)} = r$ and $\gamma^0_\Phi = 2^{-1/C^0_{\Phi_r}} = 2^{-1/r}$. Hence

$$\alpha^0_{\Phi_s} = \beta^0_{\Phi_s} = \gamma^0_{\Phi_s} = \lim_{u\to 0} \frac{\Phi_s^{-1}(u,r)}{\Phi_s^{-1}(2u,r)} = (\gamma^0_\Phi)^{1-s}\left(\frac{1}{2}\right)^{s/2} = 2^{-(1-s)/r-s/2}.$$

By (65) we get

$$\begin{aligned} 2^{-s/2} &\ge \{\mathrm{JC}(l^{(\Phi_s)}), \mathrm{JC}(l^{\Phi_s}), \mathrm{JC}(l^{(\Psi_s^+)}), \mathrm{JC}(l^{\Psi_s^+})\} \\ &\ge \begin{cases} 2^{(1-s)/r+s/2-1}, & 1 < r \le 2, \\ 2^{-(1-s)/r-s/2}, & 2 \le r < \infty. \end{cases} \end{aligned}$$

Since

$$\lim_{r\searrow 1} \mathrm{JC}(l^{(\Phi_s)}) = \lim_{r\nearrow\infty} \mathrm{JC}(l^{\Phi_s}) = 2^{-s/2},$$

for $s = 1$ we get $\mathrm{JC}(l^2) = 1/\sqrt{2}$.

EXAMPLE 3.7. Let $1 < p < \infty$ and $\Phi(u) = |u|^{2p} + 2|u|^p$. Then $\Phi \in \Delta_2(0) \cap \nabla_2(0)$ and $C_\Phi^0 = \lim_{t\to 0} \frac{t\Phi'(t)}{\Phi(t)} = p$. For $u \ge 0$,

$$\Phi^{-1}(u) = (\sqrt{1+u} - 1)^{1/p}.$$

For $0 < s \le 1$,

$$\Phi_s^{-1}(u) = (\sqrt{1+u} - 1)^{(1-s)/p} u^{s/2} \quad (u \ge 0).$$

Hence

$$\gamma_{\Phi_s}^0 = (\gamma_\Phi^0)^{1-s}\left(\frac{1}{2}\right)^{s/2} = \left(\frac{1}{2}\right)^{(1-s)/p+s/2}.$$

By (65) we get

$$\begin{aligned} 2^{-s/2} &\ge \{\mathrm{JC}(l^{(\Phi_s)}), \mathrm{JC}(l^{\Phi_s}), \mathrm{JC}(l^{(\Psi_s^+)}), \mathrm{JC}(l^{\Psi_s^+})\} \\ &\ge \begin{cases} 2^{(1-s)/p+s/2-1}, & 1 < p \le 2, \\ 2^{-(1-s)/p-s/2}, & 2 \le p < \infty. \end{cases} \end{aligned}$$

THEOREM 3.8. *Let Φ be an $\mathcal{N}$-function*, $0 < s \le 1$, *and let Ψ_s^+ be the complementary $\mathcal{N}$-function to Φ_s. If $\Phi \notin \Delta_2(0) \cap \nabla_2(0)$, then*

$$\mathrm{JC}(l^{(\Phi_s)}) = \mathrm{JC}(l^{\Psi_s^+}) = \mathrm{JC}(l^{\Phi_s}) = \mathrm{JC}(l^{(\Psi_s^+)}) = 2^{-s/2}. \tag{66}$$

Proof. By Corollary 3.5 we only have to prove that for $\Phi \notin \Delta_2(0) \cap \nabla_2(0)$,

$$\max\left(\beta_{\Phi_s}^0, \frac{1}{2\alpha_{\Phi_s}^0}\right) = 2^{-s/2}, \tag{67}$$

where

$$\beta_{\Phi_s}^0 = (\beta_\Phi^0)^{1-s}\left(\frac{1}{\sqrt{2}}\right)^s, \quad \alpha_{\Phi_s}^0 = (\alpha_\Phi^0)^{1-s}\left(\frac{1}{\sqrt{2}}\right)^s. \tag{68}$$

Since $\Phi \notin \Delta_2(0) \cap \nabla_2(0)$, we have either $\Phi \notin \Delta_2(0)$, or $\Phi \notin \nabla_2(0)$.

(i) If $\Phi \notin \Delta_2(0)$, then $\beta_\Phi^0 = 1$ and $1/2 \le \alpha_\Phi^0$. By (),

$$\beta_{\Phi_s}^0 = 2^{-s/2}, \quad 2\alpha_{\Phi_s}^0 \ge 2\left(\frac{1}{2}\right)^{1-s}\left(\frac{1}{\sqrt{2}}\right)^s = 2^{s/2}.$$

Hence

$$\max\left(\frac{1}{2\alpha_{\Phi_s}^0}, \beta_{\Phi_s}^0\right) = \beta_\Phi^0 = 2^{-s/2}. \tag{69}$$

(ii) If $\Phi \notin \nabla_2(0)$, then $\alpha_\Phi^0 = 1/2$ and $\beta_\Phi^0 \le 1$. By (),

$$\beta_{\Phi_s}^0 \le 2^{-s/2}, \quad 2\alpha_{\Phi_s}^0 = 2\left(\frac{1}{2}\right)^{1-s}\left(\frac{1}{\sqrt{2}}\right)^s = 2^{s/2}.$$

Hence

$$\max\left(\beta^0_{\Phi_s}, \frac{1}{2\alpha^0_{\Phi_s}}\right) = \frac{1}{2\alpha^0_{\Phi_s}} = 2^{-s/2}. \tag{70}$$

Now (67) follows from () and (). Finally, (66) follows from (65) and (67).

EXAMPLE 3.9. Let

$$M^{-1}(u) = \begin{cases} 0, & u = 0, \\ (-\ln u)^{-1/2}u^{1/4}, & 0 < u \le e^{-2}, \\ (e/2)^{1/2}u^{1/2}, & e^{-2} \le u < \infty. \end{cases}$$

Then

$$\mathrm{JC}(l^{(M)}) = \mathrm{JC}(l^M) = \mathrm{JC}(l^{(N)}) = \mathrm{JC}(l^N) = \frac{1}{\sqrt[4]{2}}. \tag{71}$$

In fact (see Kamińska [8, p. 304]), define the $\mathcal{N}$-function

$$\Phi(u) = \begin{cases} 0, & u = 0, \\ e^{-1/|u|}, & |u| \in (0, 1/2], \\ 4e^{-2}u^2, & |u| \in [1/2, \infty). \end{cases}$$

Since

$$C^0_\Phi = \lim_{t\to 0} \frac{t\phi(t)}{\Phi(t)} = \lim_{t\to 0} \frac{1}{t} = \infty,$$

we have $\Phi \notin \Delta_2(0)$. For $u \ge 0$,

$$\Phi^{-1}(u) = \begin{cases} 0, & u = 0, \\ (-\ln u)^{-1}, & 0 < u \le e^{-2}, \\ (e/2)u^{1/2}, & e^{-2} \le u < \infty. \end{cases}$$

We get $M^{-1}(u) = [\Phi_s^{-1}(u)]_{s=1/2}$. By Theorem 3.8 we get ().

EXAMPLE 3.10. Rao and Ren [13] define an $\mathcal{N}$-function Φ to be the inverse of

$$\Phi^{-1}(u) = \begin{cases} 0, & u = 0, \\ u\ln(1/u), & 0 < u \le e^{-2}, \\ (2/e)u^{1/2}, & e^{-2} \le u < \infty. \end{cases}$$

Since $\gamma^0_\Phi = \lim_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = \frac{1}{2}$, by Theorem 1.5 we have $\Phi \notin \nabla_2(0)$. For $0 < s \le 1$,

$$\Phi_s^{-1}(u) = [\Phi^{-1}(u)]^{1-s}\left[\frac{1}{\sqrt{2u}}\right]^s = \begin{cases} 0, & u = 0, \\ (u\ln u)^{1-s}u^{s/2}, & 0 < u \le e^{-2}, \\ (2/e)^{1-s}u^{1/2}, & e^{-2} \le u < \infty. \end{cases}$$

If Ψ_s^+ is the complementary $\mathcal{N}$-function to Φ_s, then by Theorem 3.8 we get

$$\mathrm{JC}(l^{(\Phi_s)}) = \mathrm{JC}(l^{\Psi_s^+}) = \mathrm{JC}(l^{\Phi_s}) = \mathrm{JC}(l^{(\Psi_s^+)}) = 2^{-s/2}.$$

References

[1] D. Amir, *On Jung's constant and related constants in normed linear spaces*, Pacific J. Math. 118 (1985), 1–15.

[2] W. L. Bynum, *Normal structure coefficients for Banach space*, Pacific J. Math. 86 (1980), 427–435.

[3] S. T. Chen, *Geometry of Orlicz spaces*, Dissertationes Math., 356 (1996).

[4] C. E. Cleaver, *On the extension of Lipschitz–Hölder maps on Orlicz spaces*, Studia Math. 42 (1972), 195–204.

[5] S. W. Golomb and L. D. Baumert, *The search for Hadamard matrices*, Amer. Math. Monthly 70 (1963), 12–17.

[6] V. I. Ivanov and S. A. Pichugov, *Jung constants of the l_p^n-spaces*, Math. Notes 48 (1990), 997–1004.

[7] H. W. E. Jung, *Über die kleinste Kugel, die eine raümliche Figur einschliesst*, J. Reine Angew. Math. 123 (1901), 241–257.

[8] A. Kamińska, *Strict convexity of sequence Musielak–Orlicz spaces with Orlicz norm*, J. Funct. Anal. 50 (1983), 285–305.

[9] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen, 1961.

[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, *I and II*, Springer, Berlin, 1977 and 1979.

[11] E. Maluta, *Uniformly normal structure and related coefficients*, Pacific J. Math. 111 (1984), 357–369.

[12] S. A. Pichugov, *Jung's constant for the space L^p*, Math. Notes 43 (1988), 348–353.

[13] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Dekker, New York, 1991.

Department of Mathematics
TongJi University
Shanghai 200092, China
E-mail: abc7212@163.com

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