Loewner chains and quasiconformal extension of holomorphic mappings

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Abstract. Let \( f(z,t) \) be a Loewner chain on the Euclidean unit ball \( B \) in \( \mathbb{C}^n \). Assume that \( f(z) = f(z,0) \) is quasiconformal. We give a sufficient condition for \( f \) to extend to a quasiconformal homeomorphism of \( \mathbb{R}^{2n} \) onto itself.

1. Introduction. Becker [Be] showed that if a holomorphic function \( f \) on the unit disc \( U \) satisfies
\[
\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{c}{1 - |z|^2} \quad (0 < c < 1),
\]
then \( f \) is univalent on \( U \) and extends to a quasiconformal homeomorphism of \( \mathbb{C} \) onto itself. Pfaltzgra [Pf2] generalized the above result to quasiregular locally biholomorphic mappings on the Euclidean unit ball \( B \) in \( \mathbb{C}^n \). He showed that if a quasiregular locally biholomorphic mapping \( f \) on \( B \) satisfies
\[
\|([Df(z)]^{-1}D^2f(z)(\cdot,\cdot))\| \leq \frac{c}{1 - \|z\|^2} \quad (0 < c < 1),
\]
then \( f \) is biholomorphic on \( B \) and extends to a quasiconformal homeomorphism of \( \mathbb{R}^{2n} \) onto itself. Chuaqui [Ch] obtained a quasiconformal extension of a quasiconformal strongly starlike mapping \( f \) with \( \|([Df(z)]^{-1}f(z))\| \) uniformly bounded on the Euclidean unit ball \( B \) in \( \mathbb{C}^n \). The first author [Ha] extended this result to a bounded balanced domain \( \Omega \) in \( \mathbb{C}^n \) with \( C^1 \) plurisubharmonic defining functions, and the authors [Ha-Ko3] generalized it to the unit ball with respect to an arbitrary norm on \( \mathbb{C}^n \). The authors [Ha-Ko2] also gave a quasiconformal extension of a quasiconformal strongly spirallike mapping \( f \) of type \( \alpha \) with \( \|([Df(z)]^{-1}f(z))\| \) uniformly bounded on a bounded balanced domain \( \Omega \) in \( \mathbb{C}^n \) with \( C^1 \) plurisubharmonic defining functions. To prove the above results, they imbed \( f \) in a Loewner chain.

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On the other hand, Graham–Hamada–Kohr [Gr-Ha-Ko] investigated a family of normalized biholomorphic mappings on the unit ball with respect to an arbitrary norm on $\mathbb{C}^n$ which arises in the study of Loewner chains, namely the family of mappings which have a parametric representation. For the case of the unit polydisc, these maps were studied by Poreda [Por1], [Por2]; on the Euclidean ball, they were studied extensively by Kohr [Ko]. In [Gr-Ha-Ko], growth and covering theorems for these mappings as well as coefficient estimates were obtained.

In this paper, we will give a sufficient condition for a normalized quasiconformal biholomorphic mapping $f$ on $B$ which can be imbedded in a Loewner chain to extend to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself. We also show that the results in [Ch], [Ha-Ko2], [Ha-Ko3], [Pf2] and [Re-Ma] can be reduced to ours.

2. Preliminaries. Let $\mathbb{C}^n$ denote the space of $n$ complex variables $z = (z_1, \ldots, z_n)'$ with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\|z\| = \sqrt{\langle z, z \rangle}$. The symbol $'$ means the transpose of vectors and matrices. The origin $(0, \ldots, 0)'$ is denoted by $0$, and $L(\mathbb{C}^n, \mathbb{C}^m)$ is the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. Let $I$ denote the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$.

Let $B_r = \{z \in \mathbb{C}^n : \|z\| < r\}$ and let $B = B_1$. In the case of one complex variable, $B_r$ is denoted by $U_r$ and $U_1$ by $U$. If $G \subseteq \mathbb{C}^n$ is an open set, let $H(G)$ denote the set of holomorphic mappings from $G$ into $\mathbb{C}^n$. If $f \in H(B_r)$, we say that $f$ is normalized if $f(0) = 0$ and $DF(0) = I$.

We recall that a mapping $F : B \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if $F(\cdot, t)$ is biholomorphic on $B$, $F(0, t) = 0$, $DF(0, t) = e^t I$ for $t \geq 0$ and

$$F(z, s) \prec F(z, t), \quad z \in B, \quad 0 \leq s \leq t < \infty,$$

where the symbol $\prec$ means the usual subordination.

It is known that starlikeness can be characterized in terms of Loewner chains: $f$ is a normalized starlike mapping if and only if $f(z, t) = e^t f(z)$, $z \in B$, $t \geq 0$, is a Loewner chain [Pf-Su]. For the analytical characterization of starlikeness, the reader may consult [Su1] and [Su2] (cf. [Ha]).

Now, we recall the notion of spirallikeness due to Gurganus [Gu] and Suffridge [Su3]. Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $m(A) > 0$, where

$$m(A) = \inf \{ \Re \langle Az, z \rangle : z \in \mathbb{C}^n, \|z\| = 1 \}.$$

A normalized locally biholomorphic mapping $f \in H(B)$ is called spirallike relative to $A$ if $f$ is biholomorphic on $B$ and $f(B)$ is a spirallike domain with respect to $A$, that is,

$$e^{-sA} f(B) \subset f(B), \quad s \geq 0,$$
where

\[ e^{-sA} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} s^k A^k. \]

Suffridge [Su3] showed that if \( f \) is a normalized locally biholomorphic mapping on \( B \), then \( f \) is spirallike relative to a linear operator \( A \) with \( m(A) > 0 \) if and only if

\[ \Re\langle [Df(z)]^{-1}Af(z), z \rangle > 0, \quad z \in B \setminus \{0\}. \]

If \( A = e^{-i\alpha}I \), where \( \alpha \in \mathbb{R} \), \( |\alpha| < \pi/2 \), and \( f \) is spirallike relative to \( A \), we say that \( f \) is **spirallike of type \( \alpha \)** ([Ha-Ko1]).

The authors [Ha-Ko1] showed that spirallikeness of type \( \alpha \) has the following characterization in terms of Loewner chains: \( f \) is spirallike of type \( \alpha \) if and only if

\[ f(z,t) = e^{(1-ia)t}f(e^{iat}z), \quad z \in B, \quad t \geq 0, \]

is a Loewner chain, where \( a = \tan \alpha \).

Let

\[ h(z,t) = iaz + (1 - ia)e^{-iat}[Df(e^{iat}z)]^{-1}f(e^{iat}z). \]

Then we have

\[ \Re\langle h(z,t), z \rangle = \frac{1}{\cos \alpha} \Re\{e^{-i\alpha}[Df(e^{iat}z)]^{-1}f(e^{iat}z), e^{iat}z\} > 0 \]

for any \( z \in B \setminus \{0\} \). Let \( z \in \partial B \) and let \( \zeta \in U \setminus \{0\} \). Then

\[ \Re\left( \frac{h(\zeta z,t)}{\zeta}, z \right) = \frac{1}{|\zeta|^2} \Re\langle h(\zeta z,t), \zeta z \rangle > 0. \]

Let

\[ \phi_z(\zeta,t) = \left( \frac{h(\zeta z,t)}{\zeta}, z \right) \]

for \( \zeta \neq 0 \) and \( \phi_z(0,t) = 1 \). Since \( h(0,t) = 0 \) and \( Dh(0,t) = I \), \( \phi_z(\cdot,t) \) is a holomorphic function on \( U \) and \( \Re\phi_z(\zeta,t) > 0 \) for \( \zeta \in U \) from (2.1).

If we put

\[ \sigma_z(\zeta,t) = \frac{\phi_z(\zeta,t) - 1}{\phi_z(\zeta,t) + 1}, \]

then \( \sigma_z(\cdot,t) \) is a holomorphic function on \( U \) such that \( \sigma_z(0,t) = 0 \) and \( |\sigma_z(\zeta,t)| < 1 \) for \( \zeta \in U \).

**Definition 2.1.** \( f \) is said to be **strongly spirallike of type \( \alpha \)** if \( \phi_z(U,0) \) is contained in a compact subset of the right half-plane independent of \( z \in \partial B \). Or, equivalently, there exists a \( c \) with \( 0 < c < 1 \) such that \( |\sigma_z(\zeta,0)| \leq c \) uniformly for \( z \in \partial B, \zeta \in U \).

When \( \alpha = 0 \), the above definition coincides with the definition of strongly starlike mappings due to Chuaqui [Ch] (cf. [Ha]).
Let $\mathcal{M}$ denote the well known set of mappings with “positive real part on $B$”, that is, 
$\mathcal{M} = \{p \in H(B): p(0) = 0, Dp(0) = I, \Re(p(z), z) > 0, z \in B \setminus \{0\}\}$. The following lemma is proved in Graham–Hamada–Kohr [Gr-Ha-Ko].

**Lemma 2.2.** For each $r \in (0, 1)$ there exists a constant $M = M(r)$, independent of $p$, such that $\|p(z)\| \leq M(r)$ for $\|z\| \leq r, p \in \mathcal{M}$.

Using Lemma 2.2 and [Por3, Theorem 6], the authors of [Gr-Ha-Ko] obtained the following lemma.

**Lemma 2.3.** Let $h_t(z) = h(z, t) : B \times [0, \infty) \to \mathbb{C}^n$ satisfy the following conditions:

(i) for each $t \geq 0$, $h_t(\cdot) \in \mathcal{M};$

(ii) for each $z \in B$, $h(z, t)$ is a measurable function of $t \in [0, \infty)$.

Let $f_t(z) = f(z, t) : B \times [0, \infty) \to \mathbb{C}^n$ be such that $f(\cdot, t) \in H(B), f(0, t) = 0, Df(0, t) = e^t I$ for each $t \geq 0$ and $f(z, \cdot)$ is a locally Lipschitz continuous function of $t \in [0, \infty)$ locally uniformly with respect to $z \in B$. Suppose that 
$$\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t) \quad \text{a.e. } t \geq 0,$$

for all $z \in B$. Further, assume that there exists an increasing sequence $\{t_m\}$ such that $t_m > 0, t_m \to \infty$ and
$$\lim_{m \to \infty} e^{-t_m} f(z, t_m) = F(z)$$
locally uniformly on $B$. Then $f(z, t)$ is a Loewner chain and
$$\lim_{t \to \infty} e^t v(z, s, t) = f(z, s)$$
locally uniformly on $B$ for each $s \geq 0$, where $v = v(z, s, t)$ is the solution of the initial value problem

$$(2.2) \quad \frac{\partial v}{\partial t} = -h(v, t) \quad \text{a.e. } t \geq s, \quad v(s) = z.$$

**Remark 2.4.** (i) In the case of one complex variable, every Loewner chain satisfies the assumptions of Lemma 2.3 (cf. [Pom, Theorem 6.2]).

(ii) Let $f(z, t)$ be a Loewner chain which is locally Lipschitz continuous in $t$ locally uniformly with respect to $z \in B$. In [Gr-Ha-Ko, Theorem 1.10], it was shown that there exists a mapping $h = h(z, t)$ such that $h(\cdot, t) \in \mathcal{M}$ for each $t \geq 0$, $h(z, t)$ is measurable in $t$ for each $z \in B$, and for almost all $t \geq 0,$
$$\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t) \quad \text{for all } z \in B.$$

Therefore, if $\{e^{-t} f(z, t)\}_{t \geq 0}$ is a normal family, then $f(z, t)$ satisfies the assumptions of Lemma 2.3.
Let \( \Omega, \Omega' \) be domains in \( \mathbb{R}^m \). Let \( \| \cdot \| \) be the Euclidean norm on \( \mathbb{R}^m \) and \( K > 0 \) be a constant. A homeomorphism \( f : \Omega \to \Omega' \) is said to be \( K \)-quasiconformal if it is differentiable a.e., ACL (absolutely continuous on lines) and

\[
\|D(f; x)\|^m \leq K|\det D(f; x)| \quad \text{a.e. in } \Omega,
\]

where \( D(f; x) \) denotes the (real) Jacobian matrix of \( f \) and

\[
\|D(f; x)\| = \sup\{\|D(f; x)(a)\| : \|a\| = 1\}.
\]

Let \( G \) be a domain in \( \mathbb{C}^n \) and let \( K > 0 \) be a constant. A holomorphic mapping \( f : G \to \mathbb{C}^n \) is said to be \( K \)-quasiregular if

\[
\|Df(z)\|^n \leq K|\det Df(z)|, \quad z \in G,
\]

where

\[
\|Df(z)\| = \sup\{\|Df(z)(a)\| : \|a\| = 1\}.
\]

We remark that a \( K \)-quasiregular biholomorphic mapping is \( K^2 \)-quasiconformal.

3. Main results

**Lemma 3.1.** Let \( f(z, t) \) be a Loewner chain which satisfies the assumptions of Lemma 2.3. If there exists a constant \( c_1 > 0 \) such that

\[
c_1\|z\|^2 \leq \Re \langle h(z, t), z \rangle
\]

for \( z \in B \setminus \{0\}, \ t \geq 0 \), then there exists a constant \( d \) such that

\[
\|f(z, t)\| \leq de^t\|z\|
\]

for \( z \in B, \ t \geq 0 \).

**Proof.** Since \( h(\cdot, t) \in \mathcal{M} \),

\[
\|z\|^2 \frac{1 - \|z\|}{1 + \|z\|} \leq \Re \langle h(z, t), z \rangle \leq \|z\|^2 \frac{1 + \|z\|}{1 - \|z\|}
\]

for \( z \in B \setminus \{0\} \) by [Pf1, Lemma 2.1]. Fix \( s \geq 0 \) and \( z \in B \setminus \{0\} \) and let \( v(t) = v(z, s, t) \) be the solution of the initial value problem (2.2). Then

\[
\frac{\partial \|v\|}{\partial t} = -\frac{1}{\|v\|} \Re \langle h(v, t), v \rangle < 0 \quad \text{a.e. on } [s, \infty).
\]

Since \( \|v(t)\| \to 0 \) as \( t \to \infty \), there exists a \( t_0 > 0 \) such that \( \|v(t)\| < 1/2 \) for \( t \geq t_0 \). If \( \|z\| > 1/2 \), then, for \( t > t_0 \), we have
\[
\log\left(\frac{\|e^{t-s}v(z,s,t)\|}{\|z\|}\right) = -\int_s^t \frac{\|v(\tau)\|}{\Re\langle h(v(\tau)), v(\tau) \rangle} \frac{d\|v(\tau)\|}{d\tau} d\tau - \int \frac{1}{x} dx
\]

\[
\leq \int \frac{1/2}{x(1-x)} dx + \int \frac{1}{c_1 x} dx - \int \frac{1}{x} dx
\]

\[
\leq \int \frac{2}{1-x} dx + \int \frac{1}{c_1 x} dx \leq \left(2 + \frac{1}{c_1}\right) \log 2
\]

by the assumption and (3.1). If \(\|z\| \leq 1/2\), then, for \(t > t_0\), we have

\[
\log\left(\frac{\|e^{t-s}v(z,s,t)\|}{\|z\|}\right) = -\int_s^t \frac{\|v(\tau)\|}{\Re\langle h(v(\tau)), v(\tau) \rangle} \frac{d\|v(\tau)\|}{d\tau} d\tau - \int \frac{1}{x} dx
\]

\[
\leq \int \frac{1/2}{x(1-x)} dx + \int \frac{1}{c_1 x} dx - \int \frac{1}{x} dx
\]

\[
\leq \int \frac{2}{1-x} dx \leq 2 \log 2
\]

by (3.1). Letting \(t \to \infty\), we obtain the conclusion.

Let \(f(z,t)\) be a Loewner chain such that \(f(z,0)\) is quasiconformal. The following theorem gives a sufficient condition for \(f(z,0)\) to have a quasiconformal extension to \(\mathbb{R}^{2n}\).

**Theorem 3.2.** Let \(f(z,t)\) be a Loewner chain which satisfies the assumptions of Lemma 2.3. Assume that the following conditions are satisfied:

(i) \[\|Df(z,t)\| \leq \frac{M_1(t)}{(1-\|z\|)^\alpha}, \quad z \in B, \quad t \geq 0,\]

where \(M_1(t)\) is locally bounded with respect to \(t\) and \(\alpha\) is a constant with \(0 \leq \alpha < 1\);

(ii) there exists a constant \(c_1 > 0\) such that \[c_1 \|z\|^2 \leq \Re\langle h(z,t), z \rangle\]

for \(z \in B \setminus \{0\}, \quad t \geq 0;\)

(iii) there exists a constant \(c_2 > 0\) such that \(\|h(z,t)\| \leq c_2\) for \(z \in B, \quad t \geq 0;\)

(iv) \(f(z,t)\) is \(K_1\)-quasiconformal for each \(t\).

Then \(f(z,t)\) has a continuous extension to \(\overline{B}\) (again denoted by \(f(z,t)\))
and
\[ F(z) = \begin{cases} f(z, 0), & z \in \overline{B}, \\ f(z/\|z\|, \log \|z\|), & z \notin \overline{B}, \end{cases} \]
is a quasiconformal homeomorphism of \( \mathbb{R}^{2n} \) onto itself.

**Proof.** By assumption (i) and [Pf2, Lemma 2.2], \( f_t(z) = f(z, t) \) has a continuous extension to \( \overline{B} \) and
\[ \|f(z, t) - f(w, t)\| \leq M_2(t)\|z - w\|^{1-\alpha} \]
for \( z, w \in \overline{B} \), where \( M_2(t) \) is locally bounded on \( [0, \infty) \).

Fix \( s \geq 0 \) and \( z \in B \setminus \{0\} \) and let \( v(t) = v(z, s, t) \) be the solution of the initial value problem (2.2). Then
\[ \frac{\partial\|v\|}{\partial t} = -\frac{1}{\|v\|} \Re(h(v, t), v) \leq -c_1\|v\| \a.e. \text{ on } [s, \infty) \]
by assumption (ii). Then we have
\[ \|v(z, s, t)\| \leq \|z\|e^{-c_1(t-s)} \]
for \( 0 \leq s < t \). Therefore, \( v(B, s, t) \subset B \) for \( 0 \leq s < t \). Since \( f_s(z) = f_t(v(z, s, t)) \) for \( 0 \leq s < t \), \( z \in B \), and \( f_s \) is continuous on \( \overline{B} \), we have \( f_s(\overline{B}) \subset f_t(B) \) for \( 0 \leq s < t \). Then
\[ v(z, s, t) = f_t^{-1}(f_s(z)), \quad z \in \overline{B}, \]
defines a continuous extension of \( v \) to \( \overline{B} \). For \( z \in B \), we have
\[ \|z - v(z, s, t)\| = \left\| \int_{s}^{t} h(v, \tau) d\tau \right\| \leq c_2|t - s| \]
for \( 0 \leq s < t \) by assumption (iii). Since \( v \) is continuous on \( \overline{B} \), this estimate holds for \( z \in \overline{B} \). Suppose that \( f_s(z_1) = f_s(z_2) \) for \( z_1, z_2 \in \overline{B} \). Then for \( t > s \), we have
\[ f_t(v(z_1, s, t)) = f_t(v(z_2, s, t)). \]
Since \( v(\overline{B}, s, t) \subset B \) for \( 0 \leq s < t \) and \( f_t \) is univalent on \( B \), we obtain \( v(z_1, s, t) = v(z_2, s, t) \). Letting \( t \to s \), we have \( z_1 = z_2 \) by (3.4). Therefore, \( f_s \) is univalent on \( \overline{B} \).

For \( 1/2 < r < 1 \), let \( f^r(z, t) = r^{-1}f(rz, t) \) (\( t \geq 0 \)) and let
\[ F^r(z) = \begin{cases} f^r(z, 0), & z \in \overline{B}, \\ f^r(z/\|z\|, \log \|z\|), & z \notin \overline{B}. \end{cases} \]
We will show that \( F^r \) converges uniformly on compact subsets of \( \mathbb{R}^{2n} \) to \( F \) as \( r \to 1 \), is ACL, is differentiable a.e., has outer dilatation bounded a.e. with a bound independent of \( r \), and is a homeomorphism of \( \mathbb{R}^{2n} \) onto itself. Then by [Vä, Theorem 21.7 and Corollary 37.4], \( F \) is a quasiconformal homeomorphism of \( \mathbb{R}^{2n} \) onto itself, since \( F|B = f(z, 0) \) is nonconstant.
By Lemma 3.1 and assumption (ii), we have
\begin{equation}
\|f(z, s)\| \leq c_3 e^s \|z\|
\end{equation}
on $\overline{B} \times [0, \infty)$ for some constant $c_3$. Let $z \in \overline{B}$. Then
\begin{align*}
\|F^r(z) - F(z)\| &= \|r^{-1} f(rz, 0) - f(z, 0)\| \\
&\leq r^{-1} \|f(rz, 0) - f(z, 0)\| + (r^{-1} - 1) \|f(z, 0)\| \\
&\leq r^{-1} M_2(0) \|rz - z\|^{1 - \alpha} + (r^{-1} - 1) c_3 \|z\|
\end{align*}
by (3.2) and (3.5). Let $z \in \mathbb{R}^{2n} \setminus B$. Then
\begin{align*}
\|F^r(z) - F(z)\| &= \left\| r^{-1} \left( f(z, \log \|z\|) - f\left( \frac{z}{\|z\|}, \log \|z\| \right) \right) \right\| \\
&\leq r^{-1} \left\| f\left( \frac{z}{\|z\|}, \log \|z\| \right) - f\left( \frac{z}{\|z\|}, \log \|z\| \right) \right\| \\
&\quad + (r^{-1} - 1) \left\| f\left( \frac{z}{\|z\|}, \log \|z\| \right) \right\| \\
&\leq r^{-1} M_2(\log \|z\|)(1 - r)^{1 - \alpha} + (r^{-1} - 1) c_3 \|z\|
\end{align*}
by (3.2) and (3.5). Thus, $F^r$ converges uniformly on compact subsets of $\mathbb{R}^{2n}$ to $F$.

Since $\|Df^r(z, t)\| = \|Df(rz, t)\| \leq M_1(t)/(1 - r)^{\alpha}$ by assumption (i), we have
\begin{equation}
\|f^r(z, t) - f^r(w, t)\| \leq \frac{M_3(t)}{(1 - r)^{\alpha}} \|z - w\|
\end{equation}
for $z, w \in \overline{B}$, where $M_3(t)$ is locally bounded on $[0, \infty)$. Using the relation $f(z, s) = f(v(z, s, t), t)$ for $0 \leq s < t$, (3.4) and (3.6), we have
\begin{align*}
\|f^r(z, t) - f^r(z, s)\| &= r^{-1} \|f(rz, t) - f(rz, s)\| \\
&= r^{-1} \|f(rz, t) - f(v(rz, s, t), t)\| \\
&= \|f^r(z, t) - f^r(r^{-1} v(rz, s, t), t)\| \\
&\leq \frac{M_3(t)}{(1 - r)^{\alpha}} \|z - v(rz, s, t)\| \\
&\leq \frac{c_2 M_3(t)}{r(1 - r)^{\alpha}} |t - s|
\end{align*}
for $z \in \overline{B}$, $0 \leq s < t$. If $z, w \in \overline{B}$, then
\begin{equation}
\|F^r(z) - F^r(w)\| = \|f^r(z, 0) - f^r(w, 0)\| \leq \frac{M_3(0)}{(1 - r)^{\alpha}} \|z - w\|
\end{equation}
by (3.6). If $z, w \in \mathbb{R}^{2n} \setminus B$ and $\|z\| \geq \|w\|$, then
(3.9) \[ \|F^r(z) - F^r(w)\| = \left\| f^r \left( \frac{z}{\|z\|}, \log \|z\| \right) - f^r \left( \frac{w}{\|w\|}, \log \|w\| \right) \right\| \]
\[ \leq \left\| f^r \left( \frac{z}{\|z\|}, \log \|z\| \right) - f^r \left( \frac{w}{\|w\|}, \log \|z\| \right) \right\| + \left\| f^r \left( \frac{w}{\|w\|}, \log \|w\| \right) - f^r \left( \frac{w}{\|w\|}, \log \|w\| \right) \right\| \]
\[ \leq (2 + r^{-1}c_2) \frac{M_3(\log \|z\|)}{(1-r)^\alpha} \|z - w\| \]

by (3.6) and (3.7). If \( z \in B, w \in \mathbb{R}^{2n} \setminus \overline{B} \), then there exists a real number \( t \) with \( 0 < t < 1 \) such that \( \zeta = (1-t)z + tw \in \partial B \). Therefore,
\[ \|F^r(z) - F^r(w)\| \leq \|F^r(z) - F^r(\zeta)\| + \|F^r(\zeta) - F^r(w)\| \]
\[ \leq \frac{M_3(0)}{(1-r)^\alpha} \|z - \zeta\| + (2 + r^{-1}c_2) \frac{M_3(\log \|w\|)}{(1-r)^\alpha} \|\zeta - w\| \]
\[ \leq \frac{1}{(1-r)^\alpha} \{M_3(0) + (2 + r^{-1}c_2)M_3(\log \|w\|)\} \|z - w\| \]

by (3.8) and (3.9). Thus, \( F^r \) satisfies a local Lipschitz condition (with exponent 1) in \( \mathbb{R}^{2n} \). Hence \( F^r \) is ACL in \( \mathbb{R}^{2n} \) and by a theorem of Rademacher–Stepanov [Sa, p. 311], it is (real) differentiable a.e. in \( \mathbb{R}^{2n} \).

Next, we will show that \( F^r \) has outer dilatation bounded a.e. with a bound independent of \( r \). Let \( f_t = u_t + iv_t \) and let \( F^r = u^r + iv^r \). Then we have
\[ \|D(u^r, v^r; x, y)\|^{2n} \leq K_1 |\det D(u^r, v^r; x, y)| \quad \text{in } \overline{B}, \]

since
\[ D(u^r, v^r; x, y) = D(u_0, v_0; rx, ry) \quad \text{in } \overline{B} \]

and \( f_t \) is \( K_1 \)-quasiconformal in \( B \). For \( z \notin \overline{B} \), let \( \zeta = r\|z\|^{-1}z \in B \setminus \{0\} \) and let \( \zeta = \xi + i\eta \). Then
\[ D(u^r, v^r; x, y) = \|z\|^{-1}D(u_t, v_t; \xi, \eta)(I + M(\xi, \eta)), \]

where \( t = \log \|z\| \) and
\[ M(\xi, \eta) = r^{-1} \begin{pmatrix} \Re(h(\zeta, t) - \zeta) \\ \Im(h(\zeta, t) - \zeta) \end{pmatrix} \grad \|(x, y)\|, \]

as \( \partial f(z, t)/\partial t = Df(z, t)h(z, t) \). Since \( M(\xi, \eta) \) has rank 1,
\[ \det(I + M(\xi, \eta)) = 1 + \tr M(\xi, \eta) = r^{-1}\|\zeta\|^{-1}\Re(h(\zeta, t), \zeta) \geq c_1 r^{-1}\|\zeta\| \geq c_1 \]

by Lemma 3.1 and assumption (ii). Then we have
(3.10) \[ |\det D(u^r, v^r; x, y)| = \|z\|^{-2n}|\det D(u_t, v_t; \xi, \eta)| |\det(I + M(\xi, \eta))| \]
\[ \geq \|z\|^{-2n}c_1 |\det D(u_t, v_t; \xi, \eta)|. \]
Since \( \text{grad} \| (x,y) \| \) is uniformly bounded, \( \| M(\xi,\eta) \| = r^{-1} \| h(\zeta,t) - \zeta \| \cdot \text{grad} \| (x,y) \| \) is uniformly bounded for \( r \) near 1 by assumption (iii). Thus
\[
\| D(u^r, v^r; x, y) \| \leq \| z \|^{-1} \| D(u_t, v_t; \xi, \eta) \| \| I + M(\xi, \eta) \| \\
\leq K_2 \| z \|^{-1} \| D(u_t, v_t; \xi, \eta) \|
\]
for some constant \( K_2 > 0 \).

By (3.10), (3.11) and assumption (iv), we have
\[
\| D(u^r, v^r; x, y) \|^{2n} \leq K_2^{2n} \| z \|^{-2n} \| D(u_t, v_t; \xi, \eta) \|^{2n} \\
\leq K_1 K_2^{2n} \| z \|^{-2n} |\det D(u_t, v_t; \xi, \eta)| \\
\leq K_1 K_2^{2n} c_1^{-1} |\det D(u^r, v^r; x, y)|
\]
on \( \mathbb{R}^{2n} \setminus \overline{B} \).

Finally, we will show that \( F^r \) is a homeomorphism of \( \mathbb{R}^{2n} \) onto itself. It is clear that \( F^r \) is continuous on \( \mathbb{R}^{2n} \). From (3.3), we have
\[
f_s(\overline{B}_r) \subset f_t(B_r) \quad \text{for } 0 < r < 1 \text{ and } 0 \leq s < t.
\]
Using (3.12) and the fact that \( f_t \) is univalent on \( B \), we can show that \( F^r \) is univalent on \( \mathbb{R}^{2n} \). By [Pf1, Lemma 2.2], we have
\[
\| f(z, s) \| = \lim_{t \to \infty} e^t \| v(z, s, t) \| \geq e^s \frac{\| z \|}{(1 + \| z \|)^2}
\]
for all \( z \in B, \ s \geq 0 \). Then
\[
\| F^r(z) \| = r^{-1} \left| f \left( r \frac{z}{\| z \|}, \log \| z \| \right) \right| \geq \frac{R}{(1 + r)^2}
\]
on \( \| z \| = R \) with \( R \geq 1 \). Since
\[
F^r(\{ \| z \| = R \}) = r^{-1} f_{\log R}(\{ \| z \| = r \})
\]
for any \( R > 1 \), \( F^r(\{ \| z \| = R \}) \) is the boundary of the domain \( G_R = r^{-1} f_{\log R}(\{ \| z \| < r \}) \). By (3.12), we have \( F^r(\{ 1 < \| z \| < R \}) \subset G_R \setminus F^r(\overline{B}) \). Hence \( F^r(\{ 1 < \| z \| < R \}) = G_R \setminus F^r(\overline{B}) \), since \( F^r \) is a local diffeomorphism on \( \mathbb{R}^{2n} \setminus \overline{B} \) by (3.10), continuous and univalent on \( \mathbb{R}^{2n} \). Then using (3.13), we can show that \( F^r \) is a surjective map onto \( \mathbb{R}^{2n} \). Let \( \mathbb{R}^{2n} \cup \{ \infty \} = S^{2n} \) be a one-point compactification of \( \mathbb{R}^{2n} \). We extend \( F^r \) to \( S^{2n} \) by \( F^r(\infty) = \infty \). Then \( F^r \) is a continuous bijective mapping from \( S^{2n} \) onto itself by (3.13). Therefore, \( F^r \) is a homeomorphism from \( S^{2n} \) onto itself. This completes the proof.

**Remark 3.3** If there exists an \( E(z,t) : B \times [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n) \) which is holomorphic in \( z \), \( E(0,t) = 0 \) for \( t \geq 0 \) and \( \| E(z,t) \| \leq c < 1 \) for \( z \in B, t \geq 0 \) such that \( h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z) \) for \( z \in B, t \geq 0 \), then we can show that conditions (ii) and (iii) of Theorem 3.2 are satisfied as in [Pf1].
EXAMPLE 3.4. We give an example which shows that condition (ii) in Theorem 3.2 cannot be omitted. Let $B$ be the Euclidean unit ball in $\mathbb{C}^2$. Let
\[ f(z) = (z_1 + az_2^2, z_2)' \]
Then $f$ is starlike if and only if $|a| \leq 3\sqrt{3}/2$ by Example 3 in [Su3]. Put $a = 3\sqrt{3}/2$. Since $f$ is starlike, $f(z, t) = e^t f(z)$ is a Loewner chain which satisfies the assumptions of Lemma 2.3. Since $Df(z, t) = e^t Df(z)$ and $f$ is a polynomial, condition (i) is satisfied. As $(h(z, t) = [Df(z)]^{-1} f(z) = (z_1 - a z_2^2, z_2)'$, condition (iii) is satisfied. Because $\|Df(z)\|$ is uniformly bounded in $B^2$, $\det Df(z) = 1$ and $Df(z, t) = e^t Df(z)$, condition (iv) is satisfied. Since $\langle h(z, t), z \rangle = \|z\|^2 - a z_1 z_2^2$ tends to 0 as $(z_1, z_2) \to (1/\sqrt{3}, \sqrt{2}/3)$, condition (ii) is not satisfied. Further, we show that
\[ F(z) = \begin{cases} f(z, 0), & z \in B, \\ f(z/\|z\|, \log \|z\|), & z \notin B, \end{cases} \]
is not quasiconformal. For $z \notin B$, $F(z) = (z_1 + a z_2^2/\|z\|, z_2)$. By a direct computation, we have $\det DF(x, y) = 0$ for $z_1 = k/\sqrt{3}$, $z_2 = \sqrt{2} k/\sqrt{3}$ with $k > 1$, but $\|DF(x, y)\| \neq 0$. This implies that $F$ is not quasiconformal.

4. Applications. Ren–Ma [Re–Ma, Theorem 2] obtained the following theorem. We prove this result by using Theorem 3.2.

**Theorem 4.1.** Let $f : B \to \mathbb{C}^n$ be a normalized holomorphic mapping on $B$ and let $G(z)$ be a nonsingular $n \times n$ matrix, holomorphic as a function of $z \in B$. Suppose that $G(0) = I$ and the following assumptions hold:

(i) $\|[G(z)]^{-1} Df(z) - I\| \leq c, z \in B$;
(ii) $\|z\|^2 [G(z)]^{-1} Df(z) - I] + (1 - \|z\|^2) [G(z)]^{-1} DG(z)(z, \cdot)\| \leq c, z \in B$, where $0 \leq c < 1$;
(iii) there exists a $K \geq 1$ such that $\|G(z)\|^n \leq K |\det G(z)|, z \in B$.

Then $f$ is univalent and quasiregular on $B$ and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

**Proof.** We will show that
\[ f(z, t) = f(ze^{-t}) + G(ze^{-t})(e^t - e^{-t})z \]
satisfies the conditions of Theorem 3.2.

Indeed, $f(\cdot, t) \in H(B)$, $f(0, t) = 0$, $Df(0, t) = e^t I$, $t \geq 0$, and $f(z, \cdot) \in C^\infty([0, \infty)), z \in B$. On the other hand, it is obvious that $\lim_{t \to \infty} e^{-t} f(z, t) = z$ locally uniformly on $B$. Straightforward computations show that
\[ \frac{\partial f}{\partial t}(z, t) = Df(z, t) h(z, t), \quad z \in B, \quad t \geq 0, \]
where \( h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z) \), and
\[
E(z,t) = -e^{-2t}\left\{ [G(ze^{-t})]^{-1}Df(ze^{-t}) - I \right\} \\
- (1 - e^{-2t})[G(ze^{-t})]^{-1}DG(ze^{-t})(ze^{-t}, \cdot)
\]
for all \( z \in B \) and \( t \geq 0 \). Using condition (i), we see that
\[
\|E(z,0)\| \leq c < 1, \quad z \in B,
\]
and using the maximum modulus theorem for holomorphic mappings into complex Banach spaces and condition (ii), we deduce for \( t > 0 \) that
\[
\|E(z,t)\| \leq \max_{\|z\|=1} \|E(z,t)\| \leq c < 1.
\]
Thus \( \|E(z,t)\| \leq c < 1 \) for \( z \in B \) and \( t \geq 0 \). Since \( E(0,t) = 0 \) for \( t \geq 0 \), conditions (ii) and (iii) of Theorem 3.2 are satisfied by Remark 3.3. Hence \( f(z,t) \) is a Loewner chain by Lemma 2.3.

Moreover, in view of the second condition in the hypothesis, we deduce that
\[
(1 - \|z\|^2)\|[G(z)]^{-1}DG(z)(z, \cdot)\| \leq 2c.
\]
Then with a similar reasoning to the one in [Pf2, Theorem 2.1], we conclude that there exists an absolute constant \( M > 0 \) such that
\[
\|G(z)\| \leq \frac{M}{(1 - \|z\|)^c}, \quad z \in B.
\]
Hence we obtain
\[
\|Df(z,t)\| = e^t\|G(ze^{-t})(I - E(z,t))\|
\leq e^t\frac{M}{(1 - \|ze^{-t}\|)^c} (1 + c) \leq \frac{M(1 + c)e^t}{(1 - \|z\|)^c},
\]
and therefore the first condition from Theorem 3.2 holds.

It remains to show that \( f(z,t) \) is \( K_1 \)-quasiconformal in \( B \) for all \( t \geq 0 \).
Indeed, from the third condition in the hypothesis, it is easy to deduce that
\[
\|Df(z,t)\|^n = e^{nt}\|G(ze^{-t})(I - E(z,t))\|^n
\leq e^{nt}\|G(ze^{-t})\|^n(1 + c)^n \leq e^{nt}K|\det G(ze^{-t})|(1 + c)^n
= \frac{|\det Df(z,t)|}{|\det[I - E(z,t)]|} (1 + c)^nK
\leq K\left(\frac{1 + c}{1 - c}\right)^n |\det Df(z,t)|.
\]
This completes the proof.

In particular, we obtain the following corollary.

**Corollary 4.2.** Let \( f : B \to \mathbb{C}^n \) be a normalized holomorphic mapping on \( B \) and let \( a : B \to \mathbb{C} \) be a holomorphic function such that \( a(z) \neq 0 \), \( z \in B \), and \( a(0) = 1 \). Suppose that the following assumptions hold:
Loewner chains and quasiconformal extension

(i) \[ |Df(z) - a(z)I| \leq c|a(z)|, \quad z \in B; \]

(ii) \[ \|z\|^2[Df(z) - a(z)I] + (1 - \|z\|^2) \frac{da}{dz}(z)z' \leq c|a(z)|, \]

where \( 0 \leq c < 1. \)

Then \( f \) is univalent and quasiregular on \( B \) and extends to a quasiconformal homeomorphism of \( \mathbb{R}^{2n} \) onto itself.

In particular, for \( a(z) \equiv 1 \), we obtain a result due to Brodskiï [Br].

If \( G(z) = Df(z) \), \( z \in B \), in Theorem 4.1, we get Pfaltzgraff’s quasiconformal extension result [Pf2, Theorem 2.1]. In this case, condition (iii) is equivalent to the condition that \( f \) is quasiregular on \( B \).

We next consider quasiconformal extension of quasiconformal strongly spirallike mappings of type \( \alpha \). The authors [Ha-Ko2] obtained the following theorem. We prove this result by using Theorem 3.2.

**Theorem 4.3.** Let \( f \) be a quasiconformal, strongly spirallike mapping of type \( \alpha \) with \( ||Df(z)||^{-1}f(z)|| \) uniformly bounded on \( B \). Then \( f(z) \) has a continuous extension to \( B \) (again denoted by \( f \)) and

\[
F(z) = \begin{cases} 
  f(z), & z \in B, \\
  \|z\|^{1-ia}f(z/\|z\|^{1-ia}), & z \notin B,
\end{cases}
\]

is a quasiconformal homeomorphism of \( \mathbb{R}^{2n} \) onto itself.

**Proof.** Let

\[
f(z,t) = e^{(1-ia)t}f(e^{iat}z), \quad z \in B, \quad t \geq 0,
\]

where \( a = \tan \alpha \) and

\[
h(z,t) = iaz + (1 - ia)e^{-iat[Df(e^{iat}z)]^{-1}f(e^{iat}z)}.
\]

Then it is easy to show that \( f(z,t) \) is a Loewner chain which satisfies the assumptions of Lemma 2.3. We will show that \( f(z,t) \) satisfies the assumptions of Theorem 3.2. Clearly, \( \|h(z,t)\| \) is uniformly bounded for \( z \in B, \ t \geq 0 \). This implies assumption (iii). Let \( \phi_z \) and \( \sigma_z \) be as in Section 2. There exists a constant \( c \) such that \( |\sigma_z(\zeta,0)| \leq c < 1 \) uniformly for \( z \in \partial B \). Since

\[
\Re\langle h(z,t), z \rangle = \Re \left\{ \frac{1}{e^{iat}} \langle h(e^{iat}z,0), z \rangle \right\} = \|z\|^2\Re\phi_z(\|z\|,0), \quad \bar{z} = e^{iat}z/\|z\|,
\]

for \( z \in B \setminus \{0\} \), we obtain

\[
\|z\|^2 \frac{1-c\|z\|}{1+c\|z\|} \leq \Re\langle h(z,t), z \rangle \leq \|z\|^2 \frac{1+c\|z\|}{1-c\|z\|}, \quad z \in B \setminus \{0\},
\]

by applying the Schwarz lemma to \( \sigma_z(\cdot,0) \) as in Pfaltzgraff [Pf1, Lemma 2.1]. This implies assumption (ii).
Let \( w(z) = [Df(z)]^{-1}f(z) \). Since \( h(z, 0) = iaz + (\cos \alpha)^{-1}e^{-i\alpha}w(z) \), we have
\[
\|z\|^2 \frac{1 - c\|z\|}{1 + c\|z\|} \leq \frac{1}{\cos \alpha} \Re(e^{-i\alpha}w(z), z) \leq \|z\|^2 \frac{1 + c\|z\|}{1 - c\|z\|}
\]
for \( z \in B \setminus \{0\} \) from (4.1). Therefore,
\[
\|z\| \frac{1 - c\|z\|}{1 + c\|z\|} \leq \|w(z)\| \frac{1}{\cos \alpha}.
\]
Also, from (4.1) and Lemma 3.1, we have \( \|f(z, t)\| \leq de^t\|z\| \) for \( z \in B \), \( t \geq 0 \), where \( d \) is a constant. Then, using \( Df(z)w(z) = f(z) \), we have
\[
\left\| Df(z) \left( \frac{w(z)}{\|w(z)\|} \right) \right\| = \frac{\|f(z)\|}{\|w(z)\|} \leq \frac{d(1 + c)}{\cos \alpha (1 - c)}
\]
for \( z \neq 0 \). By the Cauchy–Riemann equations, this implies that
\[
\left\| D(u, v; x, y) \left( \Re \frac{w}{\|w\|}, \Im \frac{w}{\|w\|} \right) \right\| \leq \frac{d(1 + c)}{\cos \alpha (1 - c)}
\]
for \( z \neq 0 \) from (4.2). Thus \( \|Df(z)\| \) is uniformly bounded in \( B \). Since \( Df(z, t) = e^tDf(e^{iat}z) \), we have \( \|Df(z, t)\| \leq Me^t \) for \( z \in B \), \( t \geq 0 \) and \( f(z, t) \) is \( K_1 \)-quasiconformal for \( t \geq 0 \), where \( K_1 \) is a positive constant. This implies assumptions (i) and (iv), and completes the proof.

If we put \( \alpha = 0 \) in Theorem 4.3, we obtain the following corollary, which was obtained in [Ch] (cf. [Ha], [Ha-Ko3]).

**Corollary 4.4.** Let \( f \) be a quasiconformal, strongly starlike mapping with \( \|(Df(z))^{-1}f(z)\| \) uniformly bounded on \( B \). Then \( f(z) \) has a continuous extension to \( \overline{B} \) (again denoted by \( f \)) and
\[
F(z) = \begin{cases} f(z), & z \in \overline{B}, \\ \|z\|f(z/\|z\|), & z \notin \overline{B}, \end{cases}
\]
is a quasiconformal homeomorphism of \( \mathbb{R}^{2n} \) onto itself.

We remark that the mapping in Example 3.4 shows that the assumption of strong starlikeness in the above corollary cannot be omitted.

**Remark 4.5.** We now consider generalizations of our results to the unit ball \( B \) with respect to an arbitrary norm \( \| \cdot \| \) on \( \mathbb{C}^n \). For each \( z \in \mathbb{C}^n \setminus \{0\} \),
let
\[ T(z) = \{ z^* \in L(\mathbb{C}^n, \mathbb{C}) : z^*(z) = \|z\|, \|z^*\| = 1 \}. \]

This set is nonempty, by the well known Hahn–Banach theorem. Let
\[ M = \{ p \in H(B) : p(0) = 0, Dp(0) = I, \Re z^*(p(z)) > 0, \]
\[ z \in B \setminus \{0\}, z^* \in T(z) \} . \]

Then Lemmas 2.2 and 2.3 hold without any change. If we replace the condition
\[ c_1 \|z\|^2 \leq \Re \langle h(z, t), z \rangle \]
by
\[ c_1 \|z\| \leq \Re z^*(h(z, t)) , \]
then Lemma 3.1 and Theorem 3.2 hold. Remark 3.3, Theorem 4.1 and Corollary 4.2 also hold if \( c < 1/3 \). Theorem 4.3 and Corollary 4.4 hold without any change.

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