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Dirichlet problems without convexity assumption

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Abstract. We deal with the existence of solutions of the Dirichlet problem for sublinear and superlinear partial differential inclusions considered as generalizations of the Euler–Lagrange equation for a certain integral functional without convexity assumption. We develop a duality theory and variational principles for this problem. As a consequence of the duality theory we give a numerical version of the variational principles which enables approximation of the solution for our problem.

1. Introduction. Let us state our notations and hypotheses.

Hypothesis (H). Let Ω be a bounded domain in $\mathbb{R}^n$ having a locally Lipschitz boundary. Assume that the functions $G : \Omega \times \mathbb{R} \to \mathbb{R}$ and $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfy the Carathéodory condition and $H(y, \cdot)$ is Gateaux differentiable and convex for a.e. $y \in \Omega$. Suppose additionally that there exist constants $b_1, b_2, b_3, b_4 > 0$, $r > 1$, $q \geq 3$, $q > n + 1$ and functions $k_1, k_2 \in L^1(\Omega, \mathbb{R})$, $k_3, k_4 \in L^{q-1}(\Omega, \mathbb{R})$ such that

$$\frac{b_1}{r}|x|^r + k_1(y) \leq G(y, x) \leq \frac{b_2}{q}|x|^q + k_2(y),$$
$$\frac{b_3}{2}|z|^2 + k_3(y) \leq H(y, z) \leq \frac{b_4}{2}|z|^2 + k_4(y)$$

for a.e. $y \in \Omega$ and all $x \in \mathbb{R}$, $z \in \mathbb{R}^n$.

Let $H_z(y, z) = \left[\frac{d}{dz_1}H(y, z), \ldots, \frac{d}{dz_n}H(y, z)\right]$ for $z = [z_1, \ldots, z_n] \in \mathbb{R}^n$ and let $\partial_x G(y, x)$ denote the subdifferential of the function $G(y, \cdot)$ for $y \in \Omega$. We shall consider the Dirichlet problem for the partial differential inclusion

$$-\operatorname{div} H_z(y, \nabla x(y)) \in \partial_x G(y, x(y)) \quad \text{for a.e. } y \in \Omega, \tag{1.1}$$

which is a generalization of the membrane equation. We are looking for a nonzero weak solution $x \in W_0^{1,2}(\Omega, \mathbb{R})$ of this problem such that $H_z(\cdot, \nabla x(\cdot))$ has a distributional divergence that is an element of $L^2(\Omega, \mathbb{R})$.

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There have been numerous papers concerning similar problems. If we assume the differentiability of G with respect to the second variable, inclusion (1.1) becomes an elliptic partial differential equation in divergence form discussed e.g. in [5], where $G \in C(\overline{\Omega} \times \mathbb{R})$, in [2], or in [4], where the right-hand side is independent of x and Ω is a bounded n-dimensional polyhedral domain. In [6] N. Grenon has proved the existence of a solution $x \in W_0^{1,p}(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$, $p > 1$, for the PDE

$$-\operatorname{div} A(y, x, Dx) = \mathbf{H}(y, x, Dx) \quad \text{in } \Omega, \tag{1.2}$$

where Ω is an open set in $\mathbb{R}^n$, $n \geq 1$. This follows from the existence of a solution of an associated symmetrized semilinear problem. The basic assumptions in [6] are the following:

(a) $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{H} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions such that for a.e. $y \in \Omega$, all $x \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,

$$|A(y, x, \xi)| \leq \beta(|x|)|\xi|^{p-1} + b(y), \tag{1.3}$$

$$|\mathbf{H}(y, x, \xi)| \leq \gamma(|x|)\{|\xi|^p + d(y)\}, \tag{1.4}$$

where β, γ are positive and locally bounded, b is a positive element of $L^{p'}(\Omega, \mathbb{R})$, $p' = p/(p-1)$, and $d \in L^1(\Omega, \mathbb{R})$;

(b) for a.e. $y \in \Omega$ and all $x \in \mathbb{R}$,

$$\langle A(y, x, \xi) - A(y, x, \xi'), \xi - \xi' \rangle > 0 \quad \text{for all } \xi, \xi' \in \mathbb{R}^n \text{ such that } \xi \neq \xi',$$

and there exists $\alpha > 0$ such that for a.e. $y \in \Omega$ and all $x \in \mathbb{R}$,

$$\alpha|\xi|^p \leq \langle A(y, x, \xi), \xi \rangle \quad \text{for all } \xi \in \mathbb{R}^n,$$

(c) there are nondecreasing $k_i, \theta_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, nonnegative $f_i \in L^q(\Omega, \mathbb{R}_+)$, $\max\{n/p, 1\} < q \leq \infty$, $i = 1, 2$, with $\theta_i(0) > 0$ such that

$$\mathbf{H}(y, x, \xi) \leq \begin{cases} \alpha\{k_1(x)|\xi|^p + \theta_1(x) f_1(y)\} & \text{for all } x \geq 0, \\ \alpha\{-k_2(-x)|\xi|^p - \theta_2(-x) f_2(y)\} & \text{for all } x \leq 0, \end{cases}$$

for $y \in \Omega$ and $\xi \in \mathbb{R}^n$.

Let us note that for $A(y, x, \xi) = H_z(y, \xi)$ and $\mathbf{H}(y, x, \xi) = G_x(y, x)$, (1.2) gives (1.1). In spite of this fact, we cannot use the results of [6]. In the general case described by hypothesis (H), G satisfies the Carathéodory condition only, so that $G_x(y, \cdot)$ is not necessarily continuous. We also do not assume any additional estimate on G_x (see (1.3), (1.4) and (c)).

There are a lot of results concerning the case when H has the special form $H(y, z) = \frac{1}{2}|z|^2$ for $y \in \Omega$ and $z \in \mathbb{R}^n$ (see e.g., [7], [8], [14]). In [17] and [13] the existence of a classical solution of (1.1) is discussed under the following assumptions: $G_x(\cdot, \cdot) \in C(\Omega \times \mathbb{R}, \mathbb{R})$, G_x satisfies an additional estimate on

$\Omega \times \mathbb{R}$ and the following relation between G and G_x holds: there exist $\mu > 0$ and $r \geq 0$ such that for $|x| \geq r$,

$$0 < \mu G(y,x) \leq x G_x(y,x). \tag{1.5}$$

A condition similar to (1.5) is also used in [3]. Numerous papers concern similar problems for G being a polynomial with respect to x (see [15], [19]). In many papers the right-hand side of the equation is continuous ([3], [10], [18]) or convex with respect to x (see [16]). Here we point out that weaker assumptions made on G ($\mathbb{R} \ni x \mapsto G(y,x)$ is not necessarily convex and continuous) are still sufficient to conclude the existence of a solution for (1.1). To this end (1.1) is considered as the generalized Euler–Lagrange equation for the functional J given by

$$J(x) = \int_{\Omega} \{H(y, \nabla x(y)) - G(y, x(y))\}\, dy. \tag{1.6}$$

We see that under hypothesis (H), J is not, in general, bounded on $W_0^{1,q}(\Omega,\mathbb{R})$, so that we must look for critical points of (1.6) of "minmax" type or find subsets X and X^d, on which the action functional J or its dual J_D is bounded. We shall apply another approach and choose special sets over which we will calculate the minimum of J and J_D. The main difficulty in this approach is to show that $X \neq \emptyset$. Of course, we have at our disposal the Morse theory and its generalizations, saddle points theorems, mountain pass theorems (see e.g. [10], [11], [13], [17], [15]), but none of these methods exhausts all critical points of J. Moreover our assumptions are not strong enough to use, for example, the Mountain Pass Theorem: G is a Carathéodory function so it is not sufficiently smooth, we assume neither convexity of G nor additional relations concerning G_x and G (see (1.5)), in consequence, J is not C^1 on a sufficiently large subset of $W_0^{1,q}(\Omega,\mathbb{R})$ and it does not satisfy, in general, the (PS)-condition. We shall develop a duality theory that permits us to omit, in our proof of the existence of critical points, the deformation lemmas, the Ekeland variational principle or PS type conditions. Our approach also enables a numerical characterization of solutions of our problem.

For $A > 0$ and $p_1 \in L^{q-1}(\Omega,\mathbb{R}^n)$ let

$$\begin{aligned}\overline{X} := \{x \in W_0^{1,q-1}(\Omega,\mathbb{R});\ \operatorname{div} H_z(\cdot,\nabla x(\cdot)) \in L^{q'}(\Omega,\mathbb{R}) \\ \text{and } \|H_z(\cdot,\nabla x(\cdot)) - p_1\|_{L^{q-1}(\Omega,\mathbb{R}^n)} \leq A\}\end{aligned}$$

with $q' = q/(q-1)$.

Remark 1. By the assumption $n < q-1$ and the Sobolev embedding theorem, we have $W_0^{1,q-1}(\Omega,\mathbb{R}) \subset L^q(\Omega,\mathbb{R})$.

Consider sets $X_0 \subset \overline{X}$ with the following property: for each $x \in X_0$, there exists $\widetilde{x} \in X_0$ such that

$$\int_{\Omega} \{\langle x(y), -\operatorname{div}(H_z(y, \nabla\widetilde{x}(y)))\rangle - G^*(y, -\operatorname{div}(H_z(y, \nabla\widetilde{x}(y))))\}\, dy = \int_{\Omega} G(y, x(y))\, dy. \tag{1.7}$$

What is lacking is the fact that nonempty sets X_0 exist. In Section 5 we shall consider (1.1) for $H(y,z) = \frac{1}{2}k(y)|z|^2$ for $y \in \Omega$, $z \in \mathbb{R}^n$ and $k \in C^1(\Omega, \mathbb{R})$, and formulate a sequence of assumptions concerning G which yield a nonempty set X_0. We will show that, in this case, it is sufficient to assume the convexity of G and the boundedness of G_x in the L^{q-1} norm on a ball only. Since J is C^1 on the ball only and we have the local estimate on G_x, the existence result for (1.1) cannot be derived from the Mountain Pass Theorem. We also give an example of G satisfying all these assumptions.

Throughout the paper we shall assume hypothesis (H) and

HYPOTHESIS (H1). *There exists a nonempty set X_0 satisfying* (1.7).

For any such X_0, we define X to be the union of X_0 and the set of all solutions of problem (1.1) which belong to $\overline{X}$.

REMARK 2. For all $x \in X$,

$$\partial_x G(y, x(y)) \neq \emptyset \quad \text{and} \quad G(y, x(y)) = G^{**}(y, x(y))$$

a.e. on Ω.

Proof. This follows from the definition of X, the properties of subdifferential and the Fenchel formula. ∎

Let

$$X^d := \{p \in L^{q-1}(\Omega, \mathbb{R}^n);\ \text{there exists } x \in X \text{ such that } p(y) = H_z(y, \nabla x(y)) \text{ for a.e. } y \in \Omega\}. \tag{1.8}$$

Since $q - 1 > 2$ and Ω is bounded, $X^d \subset L^2(\Omega, \mathbb{R}^n)$.

REMARK 3. For every $x \in X$, there exists $p \in X^d$ satisfying

$$-\operatorname{div} p(y) \in \partial_x G(y, x(y)) \quad \text{for a.e. } y \in \Omega.$$

Proof. Fix $x \in X$. Then there exists $\widetilde{x} \in X$ such that (1.7) holds. Taking $p(y) = H_z(y, \nabla\widetilde{x}(y))$ for a.e. $y \in \Omega$ we see that $p \in X^d$,

$$\int_{\Omega} \{\langle x(y), -\operatorname{div}(p(y))\rangle - G^*(y, -\operatorname{div} p(y))\}\, dy = \int_{\Omega} G(y, x(y))\, dy,$$

and, in consequence, the required relation is satisfied. ∎

REMARK 4. The definitions of X and X^d imply that there exists $M > 0$ such that for all $p \in X^d$,

$$\|p\|_{L^{q-1}(\Omega,\mathbb{R}^n)} \le M.$$

2. Duality. The aim of this section is to develop a duality which describes the connections between the critical values of J and the infimum of the dual functional $J_D : X^d \to \mathbb{R}$ defined as follows:

$$J_D(p) = \int_\Omega \{-H^*(y, p(y)) + G^*(y, -\operatorname{div} p(y))\}\, dy, \tag{2.1}$$

where $H^*(y,\cdot)$ and $G^*(y,\cdot)$ $(y \in \Omega)$ denote the Fenchel conjugate of $H(y,\cdot)$ and $G(y,\cdot)$, respectively.

To this end we need the perturbation $J_x : L^q(\Omega,\mathbb{R}) \to \mathbb{R}$ of J given by

$$J_x(g) = \int_\Omega \{-H(y, \nabla x(y)) + G(y, g(y) + x(y))\}\, dy.$$

It is clear that $J_x(0) = -J(x)$ for all $x \in X$.

For every $x \in X$ define a conjugate $J_x^\# : X^d \to \mathbb{R}$ of J_x by

$$\begin{aligned} J_x^\#(p) &= \sup_{g \in L^q(\Omega,\mathbb{R})} \int_\Omega \{\langle g(y), \operatorname{div} p(y)\rangle - G(y, g(y) + x(y)) + H(y, \nabla x(y))\}\, dy \\ &= \int_\Omega \{G^*(y, \operatorname{div} p(y)) + H(y, \nabla x(y)) - \langle x(y), \operatorname{div} p(y)\rangle\}\, dy. \end{aligned} \tag{2.2}$$

Now we show that for all $p \in X^d$,

$$\sup_{x \in X} (-J_x^\#(-p)) = -J_D(p). \tag{2.3}$$

Indeed, fix $p \in X^d$. From (1.8) we obtain the existence of $\overline{x} \in X$ satisfying $p(\cdot) = H_z(\cdot, \nabla \overline{x}(\cdot))$ a.e. on Ω and, in consequence,

$$\int_\Omega \{\langle \nabla \overline{x}(y), p(y)\rangle - H(y, \nabla \overline{x}(y))\}\, dy = \int_\Omega H^*(y, p(y))\, dy, \tag{2.4}$$

so that

$$\begin{aligned} &\int_\Omega \{\langle \nabla \overline{x}(y), p(y)\rangle - H(y, \nabla \overline{x}(y))\}\, dy \\ &\quad \le \sup_{x \in X} \int_\Omega \{\langle \nabla x(y), p(y)\rangle - H(y, \nabla x(y))\}\, dy \\ &\quad \le \sup_{v \in L^2(\Omega,\mathbb{R}^n)} \int_\Omega \{\langle v(y), p(y)\rangle - H(y, v(y))\}\, dy \\ &\quad = \int_\Omega H^*(y, p(y))\, dy = \int_\Omega \{\langle \nabla \overline{x}(y), p(y)\rangle - H(y, \nabla \overline{x}(y))\}\, dy. \end{aligned} \tag{2.5}$$

This implies

$$\sup_{x\in X}(-J_x^{\#}(-p)) = \sup_{x\in X}\int_{\Omega}\{\langle\nabla x(y), p(y)\rangle - H(y,\nabla x(y)) - G^*(y,-\operatorname{div} p(y))\}\,dy$$
$$= \int_{\Omega}\{H^*(y,p(y)) - G^*(y,-\operatorname{div} p(y))\}\,dy = -J_D(p),$$

as claimed.

We also need another relation:

$$\sup_{p\in X^d}(-J_x^{\#}(-p)) = -J(x) \tag{2.6}$$

for each $x\in X$. To prove this, fix $x\in X$ and use Remark 3 to find $\overline{p}\in X^d$ such that for a.e. $y\in\Omega$,

$$-\operatorname{div}\overline{p}(y)\in\partial_x G(y,x(y)),$$

and further

$$\int_{\Omega}\{\langle x(y), -\operatorname{div}\overline{p}(y)\rangle - G^*(y,-\operatorname{div}\overline{p}(y))\}\,dy = \int_{\Omega} G(y,x(y))\,dy. \tag{2.7}$$

By arguments similar to those in the proof of (2.5), we obtain

$$\sup_{p\in X^d}\int_{\Omega}\{\langle x(y), -\operatorname{div} p(y)\rangle - G^*(y,-\operatorname{div} p(y))\}\,dy \tag{2.8}$$
$$= \int_{\Omega} G^{**}(y,x(y))\,dy = \int_{\Omega} G(y,x(y))\,dy,$$

where the last equality is due to Remark 2. By (2.8) and (2.2),

$$\sup_{p\in X^d}(-J_x^{\#}(-p))$$
$$= \sup_{p\in X^d}\int_{\Omega}\{\langle x(y), -\operatorname{div} p(y)\rangle - G^*(y,-\operatorname{div} p(y)) - H(y,\nabla x(y))\}$$
$$= \int_{\Omega}\{-H(y,\nabla x(y)) + G(y,x(y))\}\,dy = -J(x).$$

Now we have the following duality principle:

THEOREM 2.1.

$$\inf_{x\in X} J(x) = \inf_{p\in X^d} J_D(p).$$

Proof. From (2.6) and (2.3),

$$\sup_{x\in X}(-J(x)) = \sup_{x\in X}\sup_{p\in X^d}(-J_x^{\#}(-p)) = \sup_{p\in X^d}\sup_{x\in X}(-J_x^{\#}(-p)) = \sup_{p\in X^d}(-J_D(p)),$$

which yields the assertion. ∎

3. Variational principles. Now we use the duality from the previous section to establish relations between the existence of minimizers of J_D and J. We also give a variational principle for minimizing sequences of both functionals. This result enables numerical characterization of minimizing sequences of J_D and approximation of its infimum. To this end we need a kind of perturbation of J_D. For each $p \in X^d$ define the functional $J_{D_p} : L^2(\Omega, \mathbb{R}^n) \to \mathbb{R}$ as

$$J_{D_p}(h) = \int_{\Omega} \{H^*(y, p(y) + h(y)) - G^*(y, -\operatorname{div} p(y))\}\, dy.$$

THEOREM 3.1. *Assume that $\overline{p} \in X^d$ is a minimizer of J_D, i.e., $J_D(\overline{p}) = \inf_{p\in X^d} J_D(p)$. Then there exists $\overline{x} \in X$ which is a minimizer of J,*

$$J(\overline{x}) = \inf_{x\in X} J(x), \tag{3.1}$$

and such that $\nabla\overline{x} \in \partial J_{D_{\overline{p}}}(0)$. Moreover

$$J^{\#}_{\overline{x}}(-\overline{p}) + J_{D_{\overline{p}}}(0) = 0, \tag{3.2}$$

$$J^{\#}_{\overline{x}}(-\overline{p}) - J(\overline{x}) = 0. \tag{3.3}$$

Proof. By (1.8) for some $\overline{x} \in X$,

$$\int_{\Omega} \{\langle \nabla\overline{x}(y), \overline{p}(y)\rangle - H(y, \nabla\overline{x}(y))\}\, dy = \int_{\Omega} H^*(y, \overline{p}(y))\, dy.$$

Adding $\int_{\Omega}\{-G^*(y, -\operatorname{div}\overline{p}(y))\}\, dy$ to both sides we obtain (3.2).

After some calculation, we see that

$$J^*_{D_{\overline{p}}}(\nabla\overline{x}) = J^{\#}_{\overline{x}}(-\overline{p})$$

(where $J^*_{D_{\overline{p}}}$ denotes the Fenchel conjugate of $J_{D_{\overline{p}}}$). Thus, by (3.2) and the properties of the subdifferential, we have the inclusion $\nabla\overline{x} \in \partial J_{D_{\overline{p}}}(0)$.

We now show that $\overline{x}$ is a minimizer of $J : X \to \mathbb{R}$. By Theorem 2.1, to prove (3.1), it is sufficient to show that $J_D(\overline{p}) \ge J(\overline{x})$, and this follows from the equalities $J_{D_{\overline{p}}}(0) = -J_D(\overline{p})$, (3.2) and (2.8):

$$\begin{aligned} J_D(\overline{p}) &= J^{\#}_{\overline{x}}(-\overline{p}) \ge \inf_{p\in X^d}(J^{\#}_{\overline{x}}(-p)) \\ &= -\sup_{p\in X^d}\int_{\Omega}\{\langle\overline{x}(y), -\operatorname{div} p(y)\rangle - G^*(y, -\operatorname{div} p(y)) - H(y, \nabla\overline{x}(y))\}\, dy \\ &= -\int_{\Omega}\{G(y, \overline{x}(y)) - H(y, \nabla\overline{x}(y))\}\, dy = J(\overline{x}). \end{aligned} \tag{3.4}$$

Finally, (3.3) is a simple consequence of (3.2) and the fact that $J_{D_{\overline{p}}}(0) = -J_D(\overline{p}) = -J(\overline{x})$. ∎

COROLLARY 3.2. *If $\overline{p} \in X^d$ satisfies $J_D(\overline{p}) = \inf_{p \in X^d} J_D(p)$ then there exists $\overline{x} \in X$ which is a solution of the Dirichlet problem for* (1.1):

$$-\operatorname{div}[H_z(y, \nabla \overline{x}(y))] \in \partial_x G(y, \overline{x}(y)) \tag{3.5}$$

for a.e. $y \in \Omega$.

Proof. By Theorem 3.1 there exists $\overline{x} \in X$ for which (3.2) and (3.3) hold. Hence

$$\int_\Omega \{H^*(y, \overline{p}(y)) + H(y, \nabla \overline{x}(y)) - \langle \nabla \overline{x}(y), \overline{p}(y) \rangle\}\, dy = 0$$

and

$$\int_\Omega \{G^*(y, -\operatorname{div} \overline{p}(y)) + G(y, \overline{x}(y)) - \langle \overline{x}(y), -\operatorname{div} \overline{p}(y) \rangle\}\, dy = 0.$$

Using the properties of the Fenchel conjugate, we obtain, for a.e. $y \in \Omega$,

$$H^*(y, \overline{p}(y)) + H(y, \nabla \overline{x}(y)) - \langle \nabla \overline{x}(y), \overline{p}(y) \rangle = 0,$$
$$G^*(y, -\operatorname{div} \overline{p}(y)) + G(y, \overline{x}(y)) - \langle \overline{x}(y), -\operatorname{div} \overline{p}(y) \rangle = 0$$

so that

$$\overline{p}(y) = H_z(y, \nabla \overline{x}(y)) \quad \text{and} \quad -\operatorname{div} \overline{p}(y) \in \partial_x G(y, \overline{x}(y))$$

for a.e. $y \in \Omega$. This implies (3.5). ■

Now we prove a numerical version of the above variational principle. We give a result on minimizing sequences that is analogous to the previous theorem.

THEOREM 3.3. *Let $\{p_n\}_{n \in \mathbb{N}} \subset X^d$ be a minimizing sequence for $J_D : X^d \to \mathbb{R}$ such that*

$$c := \inf_{n \in \mathbb{N}} J_D(p_n) > -\infty. \tag{3.6}$$

Then for any $n \in \mathbb{N}$ there exists $x_n \in X$ satisfying

$$\nabla x_n \in \partial J_{D_{p_n}}(0), \qquad \inf_{n \in \mathbb{N}} J(x_n) = \inf_{x \in X} J(x). \tag{3.7}$$

Moreover for all $n \in \mathbb{N}$,

$$J_{Dp_n}(0) + J^{\#}_{x_n}(-p_n) = 0, \tag{3.8}$$

and for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$J^{\#}_{x_n}(-p_n) - J(x_n) \le \varepsilon, \tag{3.9}$$

$$|J_D(p_n) - J(x_n)| \le \varepsilon. \tag{3.10}$$

Proof. As in the proof of Theorem 3.1, for any $n \in \mathbb{N}$ there exists $x_n \in X$ satisfying (3.8) and $\nabla x_n \in \partial J_{Dp_n}(0)$. By Theorem 2.1, to prove that $\{x_n\}_{n \in \mathbb{N}} \subset X$ is a minimizing sequence of J on X, it suffices to show that

$$\inf_{n\in\mathbb{N}} J(x_n) = c, \tag{3.11}$$

because from (3.11) and Theorem 2.1 it follows that

$$\inf_{n\in\mathbb{N}} J(x_n) = c = \inf_{n\in\mathbb{N}} J_D(p_n) = \inf_{p\in X^d} J_D(p) = \inf_{x\in X} J(x).$$

It is clear that by Theorem 2.1 and (3.6), for all $n \in \mathbb{N}$,

$$J(x_n) \geq c. \tag{3.12}$$

Fix $\varepsilon > 0$. From (3.6) there exists $n_0 \in \mathbb{N}$ such that $\varepsilon > J_D(p_n) - c$ for all $n > n_0$. Therefore, the equalities $J_{Dp_n}(0) = -J_D(p_n)$, (3.8) and (2.6) imply that for all $n > n_0$,

$$c + \varepsilon > J_D(p_n) = J^{\#}_{x_n}(-p_n) \geq \inf_{p\in X^d} (J^{\#}_{x_n}(-p)) = J(x_n).$$

Thus, by (3.12), $c = \inf_{n\in\mathbb{N}} J(x_n)$.

Conditions (3.9) and (3.10) are satisfied because of the last assertion and the fact that $J^{\#}_{x_n}(-p_n) \leq c + \varepsilon$ for all $n > n_0$. ∎

As a consequence of the previous theorem we obtain

COROLLARY 3.4. *Suppose that $\{p_n\}_{n\in\mathbb{N}} \subset X^d$ is a minimizing sequence for J_D on X^d and $\inf_{n\in\mathbb{N}} J_D(p_n) = c > -\infty$. Then there exists a minimizing sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ for J with*

$$H_z(y, \nabla x_n(y)) = p_n(y) \tag{3.13}$$

for a.e. $y \in \Omega$ and every $n \in \mathbb{N}$. Moreover

$$\lim_{n\to\infty} \int_{\Omega} \{G^*(y, -\operatorname{div} p_n(y)) + G(y, x_n(y)) + \langle \operatorname{div} p_n(y), x_n(y)\rangle\}\, dy = 0. \tag{3.14}$$

4. The existence of solutions for the Dirichlet problem. This section is devoted to the existence of a solution of (1.1) which is a minimizer of J. First we call a relevant lemma from [12]:

LEMMA 4.1. *Let $\Omega \subset \mathbb{R}^n$ and let $F : \Omega \to \mathbb{R}$ be a convex and lower semicontinuous function such that for each $u \in \Omega$ the following inequalities hold*:

$$-b \leq F(u) \leq a\,\frac{1}{q}\,|u|^q + c,$$

for some constants $a > 0$, $b, c \geq 0$, $q > 1$. Then for all $v \in \partial F(u)$,

$$|v| \leq (q' a^{q'/q}(|u| + b + c) + 1)^{q-1},$$

where $q' = q/(q-1)$.

THEOREM 4.2. *There exists $x_0 \in X$ such that*

$$-\operatorname{div}(H_z(y, \nabla x_0(y)) \in \partial_x G(y, x_0(y))$$

for a.e. $y \in \Omega$. Moreover x_0 is a minimizer of J on X:

$$J(x_0) = \inf_{x \in X} J(x).$$

Proof. By hypothesis (H) for every $p \in X^d$ we obtain

$$\begin{aligned} J_D(p) &= \int_\Omega [-H^*(y, p(y)) + G^*(y, -\operatorname{div} p(y))]\, dy \\ &\geq \frac{b_2^{1-q'}}{q'} \|\operatorname{div} p\|^{q'}_{L^{q'}(\Omega,\mathbb{R})} - \frac{1}{2b_3} \|p\|^2_{L^2(\Omega,\mathbb{R}^n)} + \int_\Omega (k_3(y) - k_2(y))\, dy \\ &\geq \frac{b_2^{1-q'}}{q'} \|\operatorname{div} p\|^{q'}_{L^{q'}(\Omega,\mathbb{R})} - \frac{1}{2b_3} \beta M^2 + \int_\Omega (k_3(y) - k_2(y))\, dy, \end{aligned} \tag{4.1}$$

where $\beta = [\operatorname{vol}(\Omega)]^{1-2/(q-1)}$. This implies that J_D is bounded below on X^d.

Taking into account the growth conditions imposed on G and H we see at once that for $\widetilde{a} \in \mathbb{R}$ large enough the set $P_{\widetilde{a}} = \{p \in X^d;\ \widetilde{a} \geq J_D(p)\}$ is not empty. Now we can choose a minimizing sequence $\{p_m\}_{m\in\mathbb{N}} \subset P_{\widetilde{a}}$ for J_D. By (4.1) the sequence $\{\operatorname{div} p_m\}_{m\in\mathbb{N}} \subset X^d$ is bounded in the norm $\|\cdot\|_{L^{q'}(\Omega,\mathbb{R})}$. Moreover, by the definition of X^d, $\{p_m\}_{m\in\mathbb{N}}$ is bounded in $L^{q-1}(\Omega, \mathbb{R}^n)$. Thus, passing to a subsequence if necessary, we deduce that $p_m \rightharpoonup p_0$ as $m \to \infty$, where $p_0 \in L^{q-1}(\Omega, \mathbb{R}^n)$, and $\operatorname{div} p_m \rightharpoonup z$ as $m \to \infty$, where $z \in L^{q'}(\Omega, \mathbb{R})$ ($\rightharpoonup$ denotes weak convergence). So

$$\begin{aligned} \int_\Omega \langle p_0(y), \nabla h(y)\rangle\, dy &= \lim_{m\to\infty} \int_\Omega \langle p_m(y), \nabla h(y)\rangle\, dy \\ &= -\lim_{m\to\infty} \int_\Omega \langle \operatorname{div} p_m(y), h(y)\rangle\, dy = -\int_\Omega \langle z(y), h(y)\rangle\, dy \end{aligned}$$

for any $h \in C_0^\infty(\Omega, \mathbb{R})$, hence

$$\int_\Omega (\langle p_0(y), \nabla h(y)\rangle + \langle z(y), h(y)\rangle)\, dy = 0$$

for all $h \in C_0^\infty(\Omega, \mathbb{R})$, and finally, by the Euler–Lagrange lemma, $\operatorname{div} p_0(y) = z(y)$ for a.e. $y \in \Omega$.

Let $B(p_1; A) = \{z \in L^{q-1}(\Omega, \mathbb{R}^n);\ \|p_1 - z\|_{L^{q-1}(\Omega,\mathbb{R}^n)} \leq A\}$. Since $\{p_m\}_{m\in\mathbb{N}} \subset B(p_1; A) \subset L^{q-1}(\Omega, \mathbb{R}^n)$ and $p_m \rightharpoonup p_0$, and $B(p_1, A)$ is weakly sequentially closed as a convex, closed subset of $L^{q-1}(\Omega, \mathbb{R}^n)$, we have

$$p_0 \in B(p_1; A). \tag{4.2}$$

Moreover, by (4.1), $\{p_m\}_{m\in\mathbb{N}}$ satisfies the assumptions of Corollary 3.4. Therefore there exists some sequence $\{x_m\}_{m\in\mathbb{N}} \subset X$ minimizing J on X

and such that for all $m \in \mathbb{N}$,

$$H_z(y, \nabla x_m(y)) = p_m(y) \quad \text{for a.e. } y \in \Omega. \tag{4.3}$$

Taking into account hypothesis (H), we can use the Fenchel formula and rewrite (4.3) as follows:

$$\nabla x_m(y) \in \partial_p H^*(y, p_m(y)) \quad \text{for a.e. } y \in \Omega, \tag{4.4a}$$

where $\partial_p H^*(y,p)$ is the subdifferential of the function $\mathbb{R}^n \ni p \mapsto H^*(y,p)$, $y \in \Omega$. By Lemma 4.1 and the boundedness of $\{p_m\}_{m\in\mathbb{N}}$ in $L^{q-1}(\Omega,\mathbb{R}^n)$ we see that $\{\nabla x_m\}_{m\in\mathbb{N}}$ is bounded in $L^{q-1}(\Omega,\mathbb{R}^n)$ and $\{x_m\}_{m\in\mathbb{N}}$ is bounded in $W_0^{1,q-1}(\Omega,\mathbb{R})$. Thus, we may choose a subsequence still denoted by $\{x_m\}_{m\in\mathbb{N}}$ weakly convergent to $x_0 \in W_0^{1,q-1}(\Omega,\mathbb{R})$. Hence, by the Rellich–Kondrashov theorem,

$$\lim_{m\to\infty} \|x_m - x_0\|_{L^q(\Omega,\mathbb{R})} = 0.$$

As $\{x_m\}_{m\in\mathbb{N}}$ tends strongly to x_0 in $L^q(\Omega,\mathbb{R})$ and $\{\operatorname{div} p_m\}_{m\in\mathbb{N}}$ tends weakly to $\operatorname{div} p_0$ in $L^{q'}(\Omega,\mathbb{R})$ we get

$$\lim_{m\to\infty} \int_\Omega \langle \operatorname{div} p_m(y), x_m(y)\rangle\, dy = \int_\Omega \langle \operatorname{div} p_0(y), x_0(y)\rangle\, dy \tag{4.5}$$

and

$$\begin{aligned}\liminf_{m\to\infty} &\int_\Omega [G(y, x_m(y)) + G^*(y, -\operatorname{div} p_m(y))]\, dy \\ &= \lim_{m\to\infty} \int_\Omega G(y, x_m(y))\, dy + \liminf_{m\to\infty} \int_\Omega G^*(y, -\operatorname{div} p_m(y))\, dy \\ &\geq \int_\Omega [G(y, x_0(y)) + G^*(y, -\operatorname{div} p_0(y))]\, dy,\end{aligned} \tag{4.6}$$

which follows from the continuity of $L^q(\Omega,\mathbb{R}) \ni x \mapsto \int_\Omega G(y,x(y))\,dy$ and weak lower semicontinuity of $L^{q'}(\Omega,\mathbb{R}) \ni z \mapsto \int_\Omega G^*(y,z(y))\,dy$. Combining (4.5), (4.6) and (3.14) we see that

$$\int_\Omega \{G^*(y, -\operatorname{div} p_0(y)) + G(y, x_0(y)) + \langle \operatorname{div} p_0(y), x_0(y)\rangle\}\, dy \leq 0. \tag{4.7}$$

Thus, by the properties of the Fenchel transform, we have equality in (4.7), and, in consequence, for a.e. $y \in \Omega$,

$$G^*(y, -\operatorname{div} p_0(y)) + G(y, x_0(y)) + \langle \operatorname{div} p_0(y), x_0(y)\rangle = 0.$$

Finally, we obtain

$$-\operatorname{div} p_0(y) \in \partial_x G(y, x_0(y)) \quad \text{for a.e. } y \in \Omega. \tag{4.8}$$

Now using (4.3) we infer that

$$\int_\Omega \{H^*(y, p_m(y)) + H(y, \nabla x_m(y)) - \langle p_m(y), \nabla x_m(y)\rangle\}\, dy = 0.$$

Analysis similar to that in the proof of (4.8) now shows that

$$p_0(y) = H_z(y, \nabla x_0(y)) \quad \text{for a.e. } y \in \Omega. \tag{4.9}$$

(4.8) and (4.9) imply

$$-\operatorname{div} H_z(y, \nabla x_0(y)) \in \partial_x G(y, x_0(y)) \quad \text{for a.e. } y \in \Omega.$$

From (4.2) and (4.9) we have $x_0 \in \overline{X}$ and further, by the last equality, $x_0 \in X$.

To prove the last assertion, it is sufficient to note that

$$\begin{aligned}\inf_{x\in X} J(x) &= \liminf_{m\to\infty} J(x_m) = \liminf_{m\to\infty} \int_\Omega \{H(y, \nabla x_m(y)) - G(y, x_m(y))\}\, dy \\ &= \liminf_{m\to\infty} \int_\Omega H(y, \nabla x_m(y))\, dy - \lim_{m\to\infty} \int_\Omega G(y, x_m(y))\, dy \\ &\ge \int_\Omega H(y, \nabla x_0(y))\, dy - \int_\Omega G(y, x_0(y))\, dy = J(x_0),\end{aligned}$$

which is due to the continuity of $L^q(\Omega, \mathbb{R}) \ni x \mapsto \int_\Omega G(y, x(y))\, dy$, weak lower semicontinuity of $L^2(\Omega, \mathbb{R}^n) \ni z \mapsto \int_\Omega H(y, z(y))\, dy$ and the facts that $x_m \to x_0$ in $L^q(\Omega, \mathbb{R})$ and $\nabla x_m \rightharpoonup \nabla x_0$ in $L^2(\Omega, \mathbb{R}^n)$ as $m \to \infty$. ∎

REMARK 5. If $0 \notin X^d$ (that is, $\|p_1\|_{L^{q-1}(\Omega,\mathbb{R}^n)} > A$) then the above theorem gives the existence of a nonzero solution of (1.1).

5. Applications. We shall apply our theory to derive the existence of solutions of the Dirichlet problem for a certain class of partial differential equations.

Now we recall the relevant theorems from [5] and [1]:

THEOREM 5.1. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$. If $f \in L^p(\Omega, \mathbb{R})$ with $1 < p < \infty$, and $k \in C^1(\overline{\Omega}, \mathbb{R})$, $\overline{k}_0 \ge k(y) \ge k_0 > 0$ for all $y \in \Omega$, then the Dirichlet problem*

$$\begin{cases} \operatorname{div}(k(y)\nabla u(y)) = f(y) & \text{for a.e. } y \in \Omega, \\ u \in W_0^{1,p}(\Omega, \mathbb{R}), \end{cases}$$

has a unique solution $u \in W^{2,p}(\Omega, \mathbb{R})$.

THEOREM 5.2. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$, $1 < p < \infty$ and $k \in C^1(\overline{\Omega}, \mathbb{R})$ and $\overline{k}_0 \ge k(y) \ge k_0 > 0$ for any $y \in \Omega$. Then there exists a constant $\widetilde{c}$ (independent of u) such that*

$$\|u\|_{W^{2,p}(\Omega,\mathbb{R})} \le \widetilde{c}\|\operatorname{div}(k\nabla u)\|_{L^p(\Omega,\mathbb{R})} \tag{5.1}$$

for all $u \in W_0^{1,p}(\Omega, \mathbb{R}) \cap W^{2,p}(\Omega, \mathbb{R})$.

THEOREM 5.3 (Sobolev Imbedding Theorem). *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $p, m \ge 1$, $k \le m$, with $pk > n$. The following inequality* *holds for* $u \in W_0^{m,p}(\Omega,\mathbb{R})$ *with a constant* c *depending on* m, k *and* p, *but not otherwise on* Ω *or on* u:

$$\max_{0\le|\alpha|\le 2-k}\ \sup_{y\in\Omega}|D^{\alpha}u(y)| \le c\|u\|_{W_0^{m,p}(\Omega,\mathbb{R})}.$$

THEOREM 5.4. *Assume that*

1. $q \in [3,\infty)$, $\Omega \in C^{1,1}$ *is a bounded domain in* $\mathbb{R}^n$, *with* $n+1<q$;
2. $k \in C^1(\overline{\Omega},\mathbb{R})$, $\overline{k}_0 \ge k(y) \ge k_0$ *for all* $y \in \Omega$;
3. $G : \Omega \times \mathbb{R} \to \mathbb{R}$ *is differentiable with respect to the second variable on* $\mathbb{R}$ *for a.e.* $y \in \Omega$;
4. *there exists* $z \in L^{q-1}(\Omega,\mathbb{R})$ *with the following properties. Let* $\overline{z} \in W^{2,q-1}(\Omega,\mathbb{R})$ *denote the solution of the equation*

$$-\operatorname{div}(k(y)\nabla\overline{z}(y)) = G_x(y,z(y)) \quad \text{for a.e. } y\in\Omega$$

(*the existence of* $\overline{z}$ *follows from Theorem* 5.1) *and let* $\widetilde{c}$ *be the constant introduced in Theorem* 5.2 *for* $W_0^{1,q-1}(\Omega,\mathbb{R}) \cap W^{2,q-1}(\Omega,\mathbb{R})$. *Then there exist constants* $S_1>0$, $b_1,b_2>0$, $1<r<q$ *and functions* $k_1,k_2 \in L^1(\Omega,\mathbb{R})$ *such that*

(a) *for all* $x\in\mathbb{R}$ *and a.e.* $y\in\Omega$,

$$\frac{b_1}{r}|x|^r + k_1(y) \le G(y,x) \le \frac{b_2}{q}|x|^q + k_2(y), \tag{5.2}$$

(b) $B(\overline{z};\widetilde{c}S_1) \ni x \mapsto \int_{\Omega} G(y,x(y))\,dy$ *is convex, where*

$$B(\overline{z};\widetilde{c}S_1) := \{u \in W_0^{2,q-1}(\Omega,\mathbb{R});\ \|u-\overline{z}\|_{W_0^{2,q-1}(\Omega,\mathbb{R})} < \widetilde{c}S_1\},$$

(c) *for any* $x \in B(\overline{z};\widetilde{c}S_1)$,

$$\|G_x(\cdot,x(\cdot)) - G_x(\cdot,z(\cdot))\|_{L^{q-1}(\Omega,\mathbb{R})} < S_1. \tag{5.3}$$

Then there exists a solution

$$x_0 \in \{u \in W_0^{1,q-1}(\Omega,\mathbb{R}); \|k\nabla u - k\nabla\overline{z}\|_{L^{q-1}(\Omega,\mathbb{R}^n)} \le \overline{k}_0\widetilde{c}S_1\}$$

of the Dirichlet problem for the PDE

$$-\operatorname{div}(k(y)\nabla x(y)) = G_x(y,x(y)) \quad \text{for a.e. } y\in\Omega \tag{5.4}$$

such that $\operatorname{div}(k(\cdot)\nabla x_0(\cdot)) \in L^{q'}(\Omega,\mathbb{R})$.

Proof. First we recall the definition of $\overline{X}$:

$$\overline{X} = \{x \in W_0^{1,q-1}(\Omega,\mathbb{R});\ \operatorname{div}(k\nabla x) \in L^{q'}(\Omega,\mathbb{R}) \text{ and } \|k\nabla x - k\nabla\overline{z}\|_{L^{q-1}(\Omega,\mathbb{R}^n)} \le \overline{k}_0\widetilde{c}S_1\}.$$

Let

$$X_0 := \{x \in W_0^{2,q-1}(\Omega,\mathbb{R});\ \|\operatorname{div}(k\nabla x) + G_x(\cdot,z(\cdot))\|_{L^{q-1}(\Omega,\mathbb{R})} < S_1\}.$$

We show the inclusion $X_0 \subset B(\overline{z}; \widetilde{c}S_1)$. Fix $x \in X_0$. Using Theorem 5.2, we have

$$\begin{aligned} (5.5)\quad \|x-\overline{z}\|_{W_0^{2,q-1}(\Omega,\mathbb{R})} &\le \widetilde{c}\|\operatorname{div}(k\nabla x - k\nabla \overline{z})\|_{L^{q-1}(\Omega,\mathbb{R})} \\ &\le \widetilde{c}\|\operatorname{div} k\nabla x + G_x(\cdot, z(\cdot))\|_{L^{q-1}(\Omega,\mathbb{R})} < \widetilde{c}S_1, \end{aligned}$$

so that $x \in B(\overline{z}; \widetilde{c}S_1)$.

Now we prove that X_0 has the following property: for every $x \in X_0$, there exists $\overline{x} \in X_0$ such that

$$(5.6)\quad \int_\Omega \{\langle x(y), -\operatorname{div}(k(y)\nabla \overline{x}(y))\rangle - G^*(y, -\operatorname{div}(k(y)\nabla\overline{x}(y))\}\, dy = \int_\Omega G(y, x(y))\, dy.$$

To this end fix $x \in X_0$. Since $G_x(\cdot, x(\cdot)) \in L^{q-1}(\Omega, \mathbb{R})$, by Theorem 5.1 there exists a unique solution $x_0 \in W_0^{1,q-1}(\Omega,\mathbb{R}) \cap W^{2,q-1}(\Omega,\mathbb{R})$ of the Dirichlet problem for the equation

$$(5.7)\qquad -\operatorname{div}(k(y)\nabla x_0(y)) = G_x(y, x(y)) \quad \text{a.e. on } \Omega.$$

Thus, by (5.3), (5.7) and the inclusion $X_0 \subset B(\overline{z}; \widetilde{c}S_1)$, we obtain

$$\begin{aligned} &\|\operatorname{div}(k(y)\nabla x_0(y)) + G_x(\cdot, z(\cdot))\|_{L^{q-1}(\Omega,\mathbb{R})} \\ &\qquad = \|G_x(\cdot, x(\cdot))) - G_x(\cdot, z(\cdot))\|_{L^{q-1}(\Omega,\mathbb{R})} < S_1. \end{aligned}$$

This implies $x_0 \in X_0$ and, in consequence, $x_0 \in B(\overline{z}; \widetilde{c}S_1)$. Thus, by (5.7) and the convexity of the functional

$$\phi(u) = \begin{cases} \int_\Omega G(y, u(y))\, dy & \text{for } u \in B(\overline{z}; \widetilde{c}S_1), \\ +\infty & \text{for } u \in W_0^{2,q-1}(\Omega,\mathbb{R}) \setminus B(\overline{z}; \widetilde{c}S_1), \end{cases}$$

we obtain

$$-\operatorname{div}(k\nabla x_0) \in \partial\phi(x)$$

and the properties of the subdifferential yield (5.6).

Moreover, from (5.5), for all $x \in X_0$, we have

$$(5.8)\qquad \|k\nabla x - k\nabla\overline{z}\|_{L^{q-1}(\Omega,\mathbb{R}^n)} \le \overline{k}_0\|x-\overline{z}\|_{W_0^{2,q-1}(\Omega,\mathbb{R})} \le \overline{k}_0\widetilde{c}S_1.$$

Summarizing, $X_0 \ne \emptyset$ ($\overline{z} \in X_0$), $X_0 \subset \overline{X}$ and X_0 has the required property.

Now we define X to be the union of X_0 and the set of all solutions of problem (5.4) which belongs to $\overline{X}$. Let X^d be given by

$$X^d := \{p \in L^{q-1}(\Omega, \mathbb{R}^n);\ \text{there exists } x \in X \text{ such that } p(y) = k(y)\nabla x(y) \text{ for a.e. } y \in \Omega\}.$$

Now Theorem 4.2 yields the existence of a solution $x_0 \in W_0^{1,q-1}(\Omega,\mathbb{R})$, with $\operatorname{div}(k(\cdot)\nabla x_0(\cdot)) \in L^{q'}(\Omega,\mathbb{R})$, of the PDE

$$\operatorname{div}(k(y)\nabla x(y)) + G_x(y, x(y)) = 0. \ \blacksquare$$

REMARK 6. It worth noting that if $\|k\nabla\overline{z}\|_{L^{q-1}(\Omega,\mathbb{R}^n)} > \overline{k}_0\widetilde{c}S_1$ then the above theorem states the existence of a nonzero solution of (5.4).

Proof. If the solution described by Theorem 5.4 is the zero function, then $0 \in \overline{X}$, which implies the estimate

$$\|k\nabla\overline{z}\|_{L^q(\Omega,\mathbb{R})} \le \overline{k}_0\widetilde{c}S_1.$$

But it contradicts our assumption. ∎

As a consequence of Theorem 5.4 we obtain

THEOREM 5.5. *Suppose that*

1. $q \in [3,\infty)$ *and* $\Omega \in C^{1,1}$ *is a bounded domain in* $\mathbb{R}^n$, *with* $n+1 < q$;
2. $k \in C^1(\overline{\Omega},\mathbb{R})$, $\overline{k}_0 \ge k(y) \ge k_0$ *for all* $y \in \Omega$;
3. $G : \Omega\times\mathbb{R} \to \mathbb{R}$ *is differentiable with respect to the second variable on* $\mathbb{R}$ *for a.e.* $y \in \Omega$;
4. *there exist constants* $b_1, b_2 > 0$, $1 < r < q$ *and functions* $k_1, k_2 \in L^1(\Omega,\mathbb{R})$ *such that for all* $x \in \mathbb{R}$ *and a.a.* $y \in \Omega$,
$$\frac{b_1}{r}|x|^r + k_1(y) \le G(y,x) \le \frac{b_2}{q}|x|^q + k_2(y);$$
5. I *is a subset of* $\mathbb{N}$ *such that for each* $i \in I$ *there exists* $z_i \in L^{q-1}(\Omega,\mathbb{R})$ *with the following properties*:
   (a) *there exists* $S_1^i > 0$ *such that*
   $$\|G_x(\cdot,x(\cdot)) - G_x(\cdot,z_i(\cdot))\|_{L^{q-1}(\Omega,\mathbb{R})} < S_1^i$$
   for any $x \in B(\overline{z}_i; S_1^i\widetilde{c})$
   (b) $B(\overline{z}_i; S_1^i\widetilde{c}) \ni x \mapsto \int_\Omega G(y,x(y))\,dy$ *is convex*,

   where $\overline{z}_i \in W_0^{2,q-1}(\Omega,\mathbb{R})$ *satisfies the equality*
   $$-\operatorname{div}(k(y)\nabla\overline{z}_i(y)) = G_x(y,z_i(y)) \quad \textit{for a.e. } y \in \Omega$$
   (*the existence of* $\overline{z}_i$ *follows from Theorem* 5.1).

Then for all $i \in I$ *there exists a solution* $\overline{x}_i$ *of the Dirichlet problem for the PDE*

$$-\operatorname{div}(k(y)\nabla x(y)) = G_x(y,x(y)) \quad \textit{for a.e. } y \in \Omega \tag{5.9}$$

such that $\operatorname{div}(k(\cdot)\nabla\overline{x}_i(\cdot)) \in L^{q'}(\Omega,\mathbb{R})$ *and* $\overline{x}_i \in B_i$, *where*

$$B_i = \{u \in W_0^{1,q-1}(\Omega,\mathbb{R});\ \|k\nabla u - k\nabla\overline{z}_i\|_{L^{q-1}(\Omega,\mathbb{R}^n)} \le \overline{k}_0\widetilde{c}S_1^i\}.$$

If we assume additionally that $B_i \cap B_j = \emptyset$ *for all* $i,j \in I$, $i \ne j$, *then* $\overline{x}_i \ne \overline{x}_j$ *for all* $i,j \in I$, $i \ne j$ *and, in consequence*, $\#S \ge \#I$, *where* S *denotes the set of solutions for* (5.9).

Now we shall give an explicit example of G satisfying the assumptions of Theorem 5.4.

EXAMPLE 5.6. *Assume that*

1. $n = 4$, $q = 6$, $\Omega \subset \mathbb{R}^4$;
2. $k \in C^1(\overline{\Omega}, \mathbb{R})$, $\overline{k}_0 \geq k(y) \geq k_0$ *for all* $y \in \Omega$;
3. z *is any element of* $C_0(\Omega, \mathbb{R})$ *such that* $\|z^5 - 6z^3 + 8z + 1\|_{L^5(\Omega,\mathbb{R})} < \frac{1}{10}$;
4. $b > 0$, $a \in L^\infty(\Omega, \mathbb{R})$ *and*

$$b < \|a\|_{L^\infty(\Omega,\mathbb{R})} < \min\left\{1, \frac{3}{40\sqrt[5]{|\Omega|} + 5,5}\right\}.$$

Then there exists a nonzero solution

$$x_0 \in \left\{u \in W_0^{1,5}(\Omega, \mathbb{R});\ \|k\nabla u - k\nabla \overline{z}\|_{L^5(\Omega,\mathbb{R}^n)} \leq \overline{k}_0 \frac{3}{5(1+c)}\right\}$$

of the Dirichlet problem for the PDE

$$-\operatorname{div}(k(y)\nabla x(y)) = \frac{a(y)}{(1+c)(1+\widetilde{c})}\left((x(y))^5 - 6(x(y))^3 + 8x(y) + 1\right) \tag{5.10}$$

for a.e. $y \in \Omega$, *such that* $\operatorname{div}(k(\cdot)\nabla x_0(\cdot)) \in L^{6/5}(\Omega, \mathbb{R})$, *with* c *being the Sobolev constant* (*Theorem* 5.3 *for* $W_0^{2,5}(\Omega, \mathbb{R})$ *and* $k = 2$) *and* $\widetilde{c}$ *described in Theorem* 5.2 *for* $p = 5$.

Proof. It is clear that the right-hand side of (5.10) is the derivative of $G : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$G(y, x) = \frac{a(y)}{6(1+c)(1+\widetilde{c})}\left((x^2 - 4)^2(x^2 - 1) + 6x\right)$$

with respect to the second variable. For a.e. $y \in \Omega$ the function $G(y, \cdot)$ is differentiable on $\mathbb{R}$ and convex in the interval $(-7/10, 7/10)$. Moreover for all $x \in \mathbb{R}$ and a.e. $y \in \Omega$ we have

$$\frac{a(y)}{6(1+c)(1+\widetilde{c})}(3x^4 - 200) \leq G(y, x) \leq \frac{a(y)}{6(1+c)(1+\widetilde{c})}(26x^6 + 36),$$

so that G satisfies the required growth conditions. Since $G_x(\cdot, z(\cdot)) \in L^5(\Omega, \mathbb{R})$ Theorem 5.1 leads to the existence of $\overline{z} \in W_0^{1,5}(\Omega, \mathbb{R}) \cap W^{2,5}(\Omega, \mathbb{R})$ which is a solution of the PDE

$$-\operatorname{div}(k(y)\nabla u(y)) = \frac{a(y)}{(1+c)(1+\widetilde{c})}\left((z(y))^5 - 6(z(y))^3 + 8z(y) + 1\right).$$

From 5.2 we know that

$$\begin{aligned} \|\overline{z}\|_{C_0(\Omega,\mathbb{R})} &\leq c\|\overline{z}\|_{W^{2,5}(\Omega,\mathbb{R})} \leq c\widetilde{c}\|\operatorname{div}(k(y)\nabla \overline{z}(y))\|_{L^5(\Omega,\mathbb{R})} \\ &\leq (1+c)(1+\widetilde{c})\|\operatorname{div}(k(y)\nabla \overline{z}(y))\|_{L^5(\Omega,\mathbb{R})} \\ &= (1+c)(1+\widetilde{c})\left\|\frac{a(y)}{(1+c)(1+\widetilde{c})}(z^5 - 6z^3 + 8z + 1)\right\|_{L^5(\Omega,\mathbb{R})} \\ &\leq \|a\|_{L^\infty(\Omega,\mathbb{R})}\|z^5 - 6z^3 + 8z + 1\|_{L^5(\Omega,\mathbb{R})} \leq \frac{1}{10}. \end{aligned} \tag{5.11}$$

Put

$$S_1 := \frac{3}{5(1+c)\widetilde{c}},$$

$$B(\overline{z};\widetilde{c}S_1) := \left\{u \in W_0^{2,5}(\Omega,\mathbb{R});\ \|u-\overline{z}\|_{W_0^{2,5}(\Omega,\mathbb{R})} < \frac{3}{5(1+c)}\right\}.$$

Using (5.11) we have the inclusion $B(\overline{z};\widetilde{c}S_1) \subset \{u \in C_0(\Omega,\mathbb{R});\ \|u\|_{C_0} < 7/10\}$, so that $x \mapsto \int_\Omega G(y,x(y))\,dy$ is convex in $B(\overline{z};\widetilde{c}S_1)$.

Now we shall show that for all $x \in B(\overline{z};\widetilde{c}S_1)$,

$$\|G_x(\cdot,x(\cdot)) - G_x(\cdot,z(\cdot))\|_{L^5(\Omega,\mathbb{R})} < \frac{3}{5(1+c)\widetilde{c}}.$$

Indeed,

$$\begin{aligned}
&\|G_x(\cdot,x(\cdot)) - G_x(\cdot,z(\cdot))\|_{L^5(\Omega,\mathbb{R})} \\
&\quad \le \frac{\|a\|_{L^\infty(\Omega,\mathbb{R})}}{(1+c)\widetilde{c}}\{(x(y))^5 - 6(x(y))^3 + 8x(y)\|_{L^5(\Omega,\mathbb{R})} \\
&\qquad + \|(z(y))^5 - 6(z(y))^3 + 8z(y)\|_{L^5(\Omega,\mathbb{R})}\} \\
&\quad \le \frac{\|a\|_{L^\infty(\Omega,\mathbb{R})}}{(1+c)\widetilde{c}}\left[\sqrt[5]{|\Omega|}\left(\left(\frac{7}{10}\right)^5 + 6\left(\frac{7}{10}\right)^3 + 8\left(\frac{7}{10}\right)\right) + \frac{11}{10}\right] \\
&\quad \le \frac{\|a\|_{L^\infty(\Omega,\mathbb{R})}}{(1+c)\widetilde{c}}\left[8\sqrt[5]{|\Omega|} + \frac{11}{10}\right] \le \frac{3}{5(1+c)\widetilde{c}},
\end{aligned}$$

as claimed.

We have just shown that all the assumptions of Theorem 5.4 are satisfied. Since zero does not satisfy the equation below, there exists a nonzero solution $x_0 \in W_0^{1,5}(\Omega,\mathbb{R})$ of

$$-\operatorname{div}(k(y)\nabla x(y)) = \frac{a(y)}{(1+c)(1+\widetilde{c})}\left((x(y))^5 - 6(x(y))^3 + 8x(y) + 1\right)$$

for a.e. $y \in \Omega$ with $\operatorname{div}(k(\cdot)\nabla x_0(\cdot)) \in L^{6/5}(\Omega,\mathbb{R})$. ∎

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