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Remarks on the relative intrinsic pseudo-distance and hyperbolic imbeddability

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Abstract. We prove the invariance of hyperbolic imbeddability under holomorphic fiber bundles with compact hyperbolic fibers. Moreover, we show an example concerning the relation between the Kobayashi relative intrinsic pseudo-distance of a holomorphic fiber bundle and the one in its base.

1. Introduction. Let Δ be the open unit disc in the complex plane $\mathbb{C}$ and ϱ the distance function on Δ defined by the Poincaré metric of Δ. For every complex space Z, denote by $\mathrm{Hol}(\Delta, Z)$ the space of all holomorphic mappings from Δ to Z with the compact-open topology. In [1], S. Kobayashi introduced for every complex space Z a pseudo-distance d_Z on Z as follows. Given two points $p, q \in Z$, we consider a *chain of holomorphic discs* from p to q, that is, a chain of points $p = p_0, p_1, \ldots, p_k = q$ of Z, pairs of points $a_1, b_1, \ldots, a_k, b_k$ of Δ and holomorphic mappings $f_1, \ldots, f_k \in \mathrm{Hol}(\Delta, Z)$ such that

$$f_i(a_i) = p_{i-1}, \quad f_i(b_i) = p_i \quad \text{for } i = 1, \ldots, k.$$

Denoting this chain by α, put

$$\ell(\alpha) = \sum_{i=1}^{k} \varrho(a_i, b_i)$$

and define the *Kobayashi pseudo-distance* d_Z by

$$d_Z(p, q) = \inf_{\alpha} \ell(\alpha)$$

where the infimum is taken over all chains α of holomorphic discs from p to q.

The relative pseudo-distance was introduced by Kobayashi in [2]. Let Z be a complex space and Y be a complex subspace of Z. Put

$$\mathcal{F}_{Y,Z} = \{f \in \mathrm{Hol}(\Delta, Z) : f^{-1}(Z \setminus Y) \text{ is at most a singleton}\}.$$

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The *relative pseudo-distance* $d_{Y,Z}$ on $\overline{Y}$ is defined in the same way as d_Z on Z, but using only chains of holomorphic discs belonging to $\mathcal{F}_{Y,Z}$. Namely, writing $\ell(\alpha)$ for the length of a chain α of holomorphic discs, we set

$$d_{Y,Z}(p,q) = \inf_{\alpha} \ell(\alpha), \qquad p,q \in \overline{Y},$$

where the infimum is taken over all chains α of holomorphic discs from p to q which belong to $\mathcal{F}_{Y,Z}$ (see [3, p. 80]).

We note that $d_Z \leq d_{Y,Z} \leq d_Y$, $d_{Z,Z} = d_Z$, $d_{\Delta^*,\Delta} = d_\Delta$, where d_Δ is the Poincaré distance on Δ, $\Delta^* = \Delta \setminus \{0\}$ and $d_{Y,Z}$ has the distance-decreasing property, i.e. if $Y' \subset Z'$ is another pair of complex spaces and $f : Z \to Z'$ is a holomorphic map such that $f(Y) \subset Y'$, then

$$d_{Y',Z'}(f(p), f(q)) \leq d_{Y,Z}(p,q) \quad \forall p,q \in \overline{Y} \text{ (see [3])}.$$

We say that Y is *hyperbolically imbedded* in Z if Y is relatively compact in Z and for every pair of distinct points p, q in $\overline{Y} \subset Z$, there exist neighbourhoods U_p and U_q of p and q in Z such that $d_Y(U_p \cap Y, U_q \cap Y) > 0$, where d_Y denotes the Kobayashi pseudo-distance on Y (see [4]).

In [5], we proved the following theorem: Let $(\widetilde{Z}, \pi, Z)$ be a holomorphic fiber bundle with hyperbolic compact fiber F, where $\widetilde{Z}$, Z, F are complex manifolds. Let M be a complex subspace of Z. If M is hyperbolically imbedded in Z and $d_{M,Z}$ induces the given topology on $\overline{M}$, then $\widetilde{M} = \pi^{-1}(M)$ is hyperbolically imbedded in $\widetilde{Z}$.

The first aim of this article is to prove the analogous theorem without the topological condition on $d_{M,Z}$.

As is well known, in general it is not possible to find explicit formulas for the pseudo-distance $d_Z(p,q)$ and for the relative intrinsic pseudo-distance $d_{Y,Z}(p,q)$. Nevertheless, S. Kobayashi proved that if Z, $\widetilde{Z}$ are complex manifolds and $\pi : \widetilde{Z} \to Z$ is a covering map, and $p,q \in Z$ and $\widetilde{p}, \widetilde{q} \in \widetilde{Z}$ are such that $\pi(\widetilde{p}) = p$, $\pi(\widetilde{q}) = q$, then

$$d_Z(p,q) = \inf_{\widetilde{q}} d_{\widetilde{Z}}(\widetilde{p}, \widetilde{q}) \tag{1}$$

where the infimum is taken over all $\widetilde{q} \in \pi^{-1}(q)$. The problem whether the infimum is always attained was posed (see [1]). In [7] W. Zwonek gave a negative answer to this problem by producing an example. In [5] we proved the analogous formula for the relative intrinsic pseudo-distance. Namely, let $(\widetilde{Z}, \pi, Z)$ be a holomorphic fiber bundle with hyperbolic fiber F, where $\widetilde{Z}$, Z, F are complex manifolds. Let Y be a complex subspace of Z and $\widetilde{Y} = \pi^{-1}(Y)$, $p,q \in \overline{Y}$, $\widetilde{p} \in \pi^{-1}(p)$, $\widetilde{q} \in \pi^{-1}(q)$. Then

$$d_{Y,Z}(p,q) = \inf_{\widetilde{q} \in \pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z}}(\widetilde{p}, \widetilde{q}). \tag{2}$$

The second aim of this article is to give an example showing that the infimum in (2) is not always attained.

2. Invariance of hyperbolic imbeddability under holomorphic fiber bundles. In this section we prove the following

THEOREM. *Let $(\widetilde{Z},\pi,Z)$ be a holomorphic fiber bundle with hyperbolic compact fiber F, where $\widetilde{Z}$, Z, F are complex spaces. Let M be a relatively compact complex subspace of Z. Then M is hyperbolically imbedded in Z if and only if $\pi^{-1}(M)$ is hyperbolically imbedded in $\widetilde{Z}$.*

Proof. Put $\widetilde{M}=\pi^{-1}(M)$. Obviously, M is relatively compact in Z iff $\widetilde{M}$ is relatively compact in $\widetilde{Z}$.

($\Leftarrow$) Assume that $\widetilde{M}$ is hyperbolically imbedded in $\widetilde{Z}$. This implies that $\mathrm{Hol}(\Delta,\widetilde{M})$ is relatively compact in $\mathrm{Hol}(\Delta,\widetilde{Z})$ (see [6]). Consider the mapping $\varphi:\mathrm{Hol}(\Delta,\widetilde{Z})\to\mathrm{Hol}(\Delta,Z)$, $\widetilde{f}\mapsto f=\pi\circ\widetilde{f}$. Clearly φ is continuous.

Since every $f\in\mathrm{Hol}(\Delta,M)$ can be lifted to $\widetilde{f}\in\mathrm{Hol}(\Delta,\widetilde{M})$ (see [4]), we have

$$\varphi(\mathrm{Hol}(\Delta,\widetilde{M}))=\mathrm{Hol}(\Delta,M).$$

Hence, by the continuity of φ and since $\mathrm{Hol}(\Delta,\widetilde{M})$ is relatively compact in $\mathrm{Hol}(\Delta,\widetilde{Z})$, we see that $\mathrm{Hol}(\Delta,M)$ is relatively compact in $\mathrm{Hol}(\Delta,Z)$. Thus, M is hyperbolically imbedded in Z.

($\Rightarrow$) Let M be hyperbolically imbedded in Z. Then $\mathrm{Hol}(\Delta,M)$ is relatively compact in $\mathrm{Hol}(\Delta,Z)$ (see [6]). It suffices to prove that $\mathrm{Hol}(\Delta,\widetilde{M})$ is relatively compact in $\mathrm{Hol}(\Delta,\widetilde{Z})$.

Take a sequence $\{\widetilde{f}_n\}$ in $\mathrm{Hol}(\Delta,\widetilde{M})$. Then $\{f_n=\pi\circ\widetilde{f}_n\}\subset\mathrm{Hol}(\Delta,M)$. By the relative compactness of $\mathrm{Hol}(\Delta,M)$, there exists a subsequence $\{f_{n_k}\}$ which converges to some f in $\mathrm{Hol}(\Delta,Z)$.

Let z_0 be an arbitrary point in Δ. It is clear that $\{f_{n_k}(z_0)\}$ converges to $f(z_0)\in Z$. Take an open relatively compact neighbourhood U of $f(z_0)$ such that $\pi^{-1}(U)\simeq U\times F$. Then $f_{n_k}(z_0)\in U$ for all $k\ge N$. This implies that $\pi\circ\widetilde{f}_{n_k}(z_0)\in U$, so we have $\widetilde{f}_{n_k}(z_0)\in\pi^{-1}(U)=U\times F$ for all $k\ge N$.

Hence $\widetilde{f}_{n_k}(z_0)=(f_{n_k}(z_0),y_{n_k})$ for all $k\ge N$, where $\{y_{n_k}\}\subset F$. Since F is compact, without loss of generality we may assume that $\{y_{n_k}\}$ converges to $y_0\in F$. Therefore, $\{\widetilde{f}_{n_k}(z_0)\}$ converges to $(f(z_0),y_0)\in U\times F$.

Let V be an open neighbourhood of z_0 in Δ such that $f_{n_k}(V)\subset U$ for all $k\ge N$. We have

$$\widetilde{f}_{n_k}(V)\subset\pi^{-1}(U)=U\times F,\qquad\forall k\ge N.$$

Consider the restricted mappings $\widetilde{f}_{n_k}|_V=(f_{n_k}|_V,g_{n_k}|_V)$, where $g_{n_k}|_V: V\to F$. Since F is compact hyperbolic, F is taut. Therefore, from the above argument, $\{g_{n_k}\}$ converges to a map g in $\mathrm{Hol}(V, \widetilde{Z})$. Since $\{f_{n_k}\}$ converges to f in $\mathrm{Hol}(\Delta, Z)$, it follows that $\{\widetilde{f}_{n_k}\}$ converges to a map $\widetilde{f} = (f, g)$ in $\mathrm{Hol}(V, \widetilde{Z})$.

Consider the family $\mathcal{V}$ of all pairs $(V, \widetilde{f})$, where V is an open nonempty subset of Δ and $\widetilde{f} \in \mathrm{Hol}(V, \widetilde{Z})$ such that a subsequence $\{\widetilde{f}_{n_k}|_V\}$ converges to $\widetilde{f}$ in $\mathrm{Hol}(V, \widetilde{Z})$.

We define an order relation in $\mathcal{V}$ by setting $(V_1, \widetilde{f}_1) \le (V_2, \widetilde{f}_2)$ if $V_1 \subset V_2$ and for every subsequence $\{\widetilde{f}_{n_k}|_{V_1}\}$ of $\{\widetilde{f}_n|_{V_1}\}$ converging to $\widetilde{f}_1$ in $\mathrm{Hol}(V_1, \widetilde{Z})$ uniformly on compact sets, the sequence $\{\widetilde{f}_{n_k}|_{V_2}\}$ contains a subsequence converging to $\widetilde{f}_2$ in $\mathrm{Hol}(V_2, \widetilde{Z})$. Assume that $\{(V_\alpha, \widetilde{f}_\alpha)\}_{\alpha\in I}$ is a well ordered subset of $\mathcal{V}$. Put $V_0 = \bigcup_{\alpha \in I} V_\alpha$. Define $\widetilde{f}_0 \in \mathrm{Hol}(V_0, \widetilde{Z})$ by $\widetilde{f}_0|_{V_\alpha} = \widetilde{f}_\alpha$ for all $\alpha \in I$. Take a sequence $(V_i, \widetilde{f}_i)_{i=1}^{\infty} \subset \{(V_\alpha, \widetilde{f}_\alpha)\}_{\alpha\in I}$ such that

$$(V_1, \widetilde{f}_1) \le (V_2, \widetilde{f}_2) \le \cdots \quad \text{and} \quad V_0 = \bigcup_{i=1}^{\infty} V_i.$$

Then there exists a subsequence $\{\widetilde{f}^1_n|_{V_1}\}$ of $\{\widetilde{f}_n|_{V_1}\}$ converging to $\widetilde{f}_1$ in $\mathrm{Hol}(V_1, \widetilde{Z})$.

Consider $\{\widetilde{f}^1_n|_{V_2}\}$. As above, $\{\widetilde{f}^1_n\}$ contains a subsequence $\{\widetilde{f}^2_n\}$ such that $\{\widetilde{f}^2_n|_{V_2}\}$ converges to $\widetilde{f}_2$ in $\mathrm{Hol}(V_2, \widetilde{Z})$. Continuing this process we obtain subsequences $\{\widetilde{f}^k_n\}$ such that $\{\widetilde{f}^k_n\} \subset \{\widetilde{f}^{k-1}_n\}$ for every $k \ge 2$ and $\{\widetilde{f}^k_n|_{V_k}\}$ converges uniformly on compact sets to $\widetilde{f}_k$ in $\mathrm{Hol}(V_k, \widetilde{Z})$. Therefore, the diagonal subsequence $\{\widetilde{f}^k_k\}$ converges uniformly on compact sets to $\widetilde{f}_0$ in $\mathrm{Hol}(V_0, \widetilde{Z})$.

Thus $(V_0, \widetilde{f}_0) \in \mathcal{V}$ and hence the subset $\{(V_0, \widetilde{f}_\alpha)\}_{\alpha\in I}$ has an upper bound. By Zorn's Lemma, the family $\mathcal{V}$ has a maximal element $(V, \widetilde{f})$. It suffices to prove that $V = \Delta$.

By definition of $\mathcal{V}$, V is an open nonempty subset of Δ and there exists a subsequence $\{\widetilde{f}_{n_k}\}$ of $\{\widetilde{f}_n\}$ such that $\{\widetilde{f}_{n_k}|_V\} \to \widetilde{f} \in \mathrm{Hol}(V, \widetilde{Z})$ in $\mathrm{Hol}(V, \widetilde{Z})$.

Suppose that $V \ne \Delta$; then $\partial V \ne \emptyset$. Take $z' \in \partial V$. Applying the above argument for z' and the sequence $\{\widetilde{f}_{n_k}\}$, we deduce that there exist an open neighbourhood V' of z' in Δ and a subsequence $\{\widetilde{f}_{n_{k_i}}\}$ of $\{\widetilde{f}_{n_k}\}$ such that $\{\widetilde{f}_{n_{k_i}}|_{V'}\}$ converges to $\widetilde{f}' \in \mathrm{Hol}(V', \widetilde{Z})$ in $\mathrm{Hol}(V', \widetilde{Z})$.

It is easy to see that $\widetilde{f}|_{V\cap V'} = \widetilde{f}'|_{V\cap V'}$. Thus, we may define a holomorphic mapping $\widetilde{F} : V \cup V' \to \widetilde{Z}$ by seting $\widetilde{F}|_V = \widetilde{f}$ and $\widetilde{F}|_{V'} = \widetilde{f}'$. Therefore $\{\widetilde{f}_{n_{k_i}}|V \cup V'\} \to \widetilde{F} \in \mathrm{Hol}(V \cup V', \widetilde{Z})$ in $\mathrm{Hol}(V \cup V', \widetilde{Z})$. Hence $(V \cup V', \widetilde{F}) \in \mathcal{V}$. Since $V \ne V \cup V'$, we have $(V, \widetilde{f}) < (V \cup V', \widetilde{F})$, contrary to the maximality of $(V, \widetilde{f})$. Thus $V = \Delta$. The theorem is proved. ∎

3. Remark. W. Zwonek in [7] proved by example that the infimum in (1) is not always attained. Since $d_{Z,Z} = d_Z$, using the example of W. Zwonek, we find that for the relative intrinsic pseudo-distance, the infimum in the formula (2) is also not always attained. We now give a simpler example as follows.

Let $H^+ = \{a + ib \in \mathbb{C} : b > 0\}$ and

$$E = H^+ \times \mathbb{C}, \qquad G = \{k + il : k, l \text{ are integers}\}.$$

Then G is a discrete complex Lie group.

Consider the action

$$G \times E \to E, \qquad (k + il, (z, w)) \mapsto (z + k + \sqrt{2}\, l, w + l).$$

Put $B = E/G$ and let $\pi : E \to B$ be the canonical projection. Then (E, π, B) is a holomorphic principal fiber bundle with structural group G.

Consider the elements $\alpha = (\sqrt{3}+i, 0) \in E$, $\beta = (i, 0) \in E$. Put $p = \pi(\alpha)$, $q = \pi(\beta)$, $p, q \in B$. We have

$$d_B(p, q) = \inf_{\widetilde{p} \in \pi^{-1}(p)} d_E(\widetilde{p}, \beta).$$

We prove that the infimum in the above formula cannot be attained. In fact, since $\widetilde{p} \in \pi^{-1}(p)$ there exists $k + il \in G$ such that $\widetilde{p} = (\sqrt{3} + i + k + \sqrt{2}\, l, l)$. We have

$$\begin{aligned} d_{H^+}(\sqrt{3} + k + \sqrt{2}\, l + i, i) &= \max(d_{H^+}(\sqrt{3} + k + \sqrt{2}\, l + i, i), d_{\mathbb{C}}(l, 0)) \\ &= d_E(\widetilde{p}, \beta). \end{aligned}$$

It is known that for $z_1, z_2 \in H^+$,

$$d_{H^+}(z_1, z_2) = \ln \frac{1 + \left|\dfrac{z_1 - z_2}{z_1 - \overline{z}_2}\right|}{1 - \left|\dfrac{z_1 - z_2}{z_1 - \overline{z}_1}\right|}.$$

Thus

$$\begin{aligned} d_E(\widetilde{p}, \beta) &= \ln \frac{1 + \left|\dfrac{\sqrt{3} + k + \sqrt{2}\, l}{\sqrt{3} + k + \sqrt{2}\, l + 2i}\right|}{1 - \left|\dfrac{\sqrt{3} + k + \sqrt{2}\, l}{\sqrt{3} + k + \sqrt{2}\, l + 2i}\right|} \\ &= \ln \frac{\sqrt{t^2 + 4} + t}{\sqrt{t^2 + 4} - t} \qquad \text{for } t = |\sqrt{3} + k + \sqrt{2}\, l|. \end{aligned}$$

It is clear that $\sqrt{3} + k + \sqrt{2}\, l \neq 0$ for any integers k, l. Thus $t > 0$ for all k, l and $d_E(\widetilde{p}, \beta) > 0$ for all $\widetilde{p} \in \pi^{-1}(p)$.

On the other hand, in view of the Kronecker theorem, we have

$$\inf\{t\} = \inf_{(k,l)\in\mathbb{Z}^2}\{|\sqrt{3} + k + \sqrt{2}\, l|\} = 0.$$

Thus $\inf_{\widetilde{p}\in\pi^{-1}(p)} d_E(\widetilde{p}, \beta) = 0$. This completes the proof of our statement.

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