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A characterization of bounded plurisubharmonic functions

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Abstract. We give a characterization for boundedness of plurisubharmonic functions in the Cegrell class $\mathcal{F}$.

1. Introduction. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Denote by $\mathrm{PSH}(\Omega)$ the plurisubharmonic (psh) functions on Ω. The complex Monge–Ampère operator $(dd^c)^n$ is well defined over the class of locally bounded psh functions, according to the fundamental work of Bedford–Taylor in [BT1], [BT2]. Recently Cegrell has introduced in [Ce1], [Ce2] new classes of psh functions on which the complex Monge–Ampère operator can be defined and enjoys important properties, e.g. is continuous under monotone sequences. For precise definitions of Cegrell's classes see the next section. The main aim of this note is to give some characterizations for boundedness of functions in the class $\mathcal{F}$. These results are strongly motivated by Theorem 3 in [Xi2] where a characterization for boundedness of psh functions bounded near the boundary is given. The main result of the note is the following

1.1. Main Theorem. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$, and $u \in \mathcal{F}(\Omega)$. Then u is bounded on Ω if only if there exist constants A and B such that for all $k < B$ with $C_n(\{u < k\}) > 0$ there exist $k \le k_1 < \cdots < k_s = B$ with $k_1 < k+1$ and*

$$\sum_{j=2}^{s} \left(\frac{\|(dd^c u)^n\|_{\{u<k_j\}}}{C_n(\{u < k_{j-1} + 0\})} \right)^{1/n} < A$$

where $C_n(\{u < k + 0\}) = \lim_{t\to k+0} C_n(\{u < t\})$.

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The proof of the Main Theorem is presented in Section 4. In Section 2 we recall some elements of pluripotential theory pertaining to our work.

Applying the Main Theorem, as in [Xi2], we obtain

1.2. COROLLARY. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$, and $u \in \mathcal{F}(\Omega)$. If there exist constants $\delta > 1$ and $A > 0$ such that the inequality*

$$\|(dd^c u)^n\|_{\{u<k\}} \le A(C_n(\{u<k\}))^\delta$$

holds for every $k < 0$, then u is bounded on Ω.

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2. Preliminaries. In this section we recall some elements of pluripotential theory that will be used throughout the paper. All this can be found in [BT2], [Xi1], [Ce1], [Ce2], etc.

2.1. Unless otherwise specified, Ω will be a bounded hyperconvex domain in $\mathbb{C}^n$, meaning that there exists a negative exhaustive psh function for Ω.

2.2. Let Ω be a bounded domain in $\mathbb{C}^n$. The C_n*-capacity* in the sense of Bedford and Taylor on Ω is the set function given by

$$C_n(E) = C_n(E,\Omega) = \sup\Big\{ \int_E (dd^c u)^n : u \in \mathrm{PSH}(\Omega),\ -1 \le u \le 0\Big\}$$

for every Borel set E in Ω. It is known [BT2] that

$$C_n(E) = \int_\Omega (dd^c h^*_{E,\Omega})^n$$

where $h^*_{E,\Omega}$ is the relative extremal psh function for E (relative to Ω) defined as the smallest upper semicontinuous majorant of $h_{E,\Omega}$ where

$$h_{E,\Omega}(z) = \sup\{u(z) : u \in \mathrm{PSH}(\Omega),\ -1 \le u \le 0,\ u \le -1 \text{ on } E\}.$$

2.3. The following classes of psh functions were introduced by Cegrell in [Ce1] and [Ce2]:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \Big\{\varphi \in \mathrm{PSH} \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0,\ \int_\Omega (dd^c\varphi)^n < \infty\Big\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \Big\{\varphi \in \mathrm{PSH}(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi,\ \sup_{j\ge1} \int_\Omega (dd^c\varphi_j)^n < \infty\Big\},$$

$$\mathcal{E} = \mathcal{E}(\Omega) = \Big\{\varphi \in \mathrm{PSH}(\Omega) : \forall z_0 \in \Omega\ \exists \text{ a neighbourhood } \omega \ni z_0$$
$$\exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi \text{ on } \omega,\ \sup_{j\ge1} \int_\Omega (dd^c\varphi_j)^n < \infty\Big\}.$$

The following interesting theorem was proved by Cegrell in [Ce2]:

2.4. THEOREM. *The class $\mathcal{E}$ has the following properties*:

(1) *$\mathcal{E}$ is a convex cone.*
(2) *If $u \in \mathcal{E}$ and $v \in \mathrm{PSH}^-(\Omega) = \{\varphi \in \mathrm{PSH}(\Omega) : \varphi \le 0\}$, then* $\max(u,v) \in \mathcal{E}$.
(3) *If $u \in \mathcal{E}$ and $\mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega) \ni u_j \searrow u$, then $(dd^c u_j)^n$ is weakly convergent.*

3. The comparison principle in the class $\mathcal{F}$. The key element in the proof of our main theorem is the following comparison principle of Xing type [Xi1], [Xi2].

3.1. THEOREM. *Let $u \in \mathcal{F}$ and $v \in \mathrm{PSH}^- \cap L^\infty(\Omega)$. Then*

$$\frac{1}{(n!)^2} \int_{\{u<v\}} (v-u)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_{\{u<v\}} (r-w_1)(dd^c v)^n \le \int_{\{u<v\}} (r-w_1)(dd^c u)^n$$

for all $r \ge 1$ and $w_j \in \mathrm{PSH}(\Omega)$ with $0 \le w_j \le 1$ for all $j \ge 1$.

Proof. Let $\mathcal{E}_0 \ni u_j \searrow u$ be as in the definition of $\mathcal{F}$. By the comparison principle of Xing for bounded psh functions (Lemma 1 in [Xi1]) we have

$$\frac{1}{(n!)^2} \int_{\{u_j<v\}} (v-u_j)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_{\{u_j<v\}} (r-w_1)(dd^c v)^n \le \int_{\{u_j<v\}} (r-w_1)(dd^c u_j)^n$$

for $j \ge 1$. Since $\{u_j < v\} \nearrow \{u < v\}$, it follows that

$$\frac{1}{(n!)^2} \int_{\{u<v\}} (v-u)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_{\{u<v\}} (r-w_1)(dd^c v)^n \le \varlimsup_{j\to\infty} \int_{\{u<v\}} (r-w_1)(dd^c u_j)^n.$$

Let $\varepsilon > 0$. Since $\int_\Omega (dd^c u)^n < \infty$, we can find $L \subset\subset \Omega$ such that $\int_{\Omega\setminus L} (dd^c u)^n < \varepsilon$. Let $L \subset\subset K \subset\subset \Omega' \subset\subset \Omega$ and $g \in C^\infty(\mathbb{C}^n)$ be such that $g = 1$ on $\mathbb{C}^n \setminus K$, $g = 0$ on L and $0 \le g \le 1$. We have

$$\begin{aligned}
&\varlimsup_{j\to\infty} \int_{\{u<v\}} (r-w_1)(dd^c u_j)^n \\
&\le \varlimsup_{j\to\infty} \int_{\{u<v\}\cap K} (r-w_1)(dd^c u_j)^n + \varlimsup_{j\to\infty} \int_\Omega g(r-w_1)(dd^c u_j)^n \\
&\le \varlimsup_{j\to\infty} \int_{\{u<v\}\cap K} (r-w_1)(dd^c u_j)^n + r\Big[\varlimsup_{j\to\infty} \int_\Omega (g-1)(dd^c u_j)^n + \varlimsup_{j\to\infty} \int_\Omega (dd^c u_j)^n\Big]
\end{aligned}$$

$$\leq \overline{\lim_{j\to\infty}} \int\limits_{\{u<v\}\cap K} (r-w_1)(dd^c u_j)^n + r\Big[\int\limits_{\Omega}(g-1)(dd^c u)^n + \int\limits_{\Omega}(dd^c u)^n\Big]$$

$$\leq \overline{\lim_{j\to\infty}} \int\limits_{\{u<v\}\cap K} (r-w_1)(dd^c u_j)^n + r\int\limits_{\Omega\setminus L}(dd^c u)^n$$

$$\leq \overline{\lim_{j\to\infty}} \int\limits_{\{u<v\}\cap K} (r-w_1)(dd^c u_j)^n + r\varepsilon.$$

Thus there exists $K \subset\subset \Omega$ such that

$$\overline{\lim_{j\to\infty}} \int\limits_{\{u<v\}} (r-w_1)(dd^c u_j)^n \leq \overline{\lim_{j\to\infty}} \int\limits_{\{u<v\}\cap K} (r-w_1)(dd^c u_j)^n + r\varepsilon.$$

By the quasicontinuity of u and v (Theorem 3.5 in [BT2]) we can find open subsets G_j of Ω' such that

$$C_n(G_j) < \frac{1}{2^j}, \quad u, v \text{ are continuous on } \Omega' \setminus G_j$$

and

$$G_1 \supset G_2 \supset \cdots .$$

Put $h_j = h_{G_j,\Omega}$. Since $G_j \searrow G = \bigcap_{j=1}^{\infty} G_j$ and $C_n(G) = 0$, it follows that $h_j \nearrow 0$ on $\Omega \setminus E$ for some subset E of Ω with $C_n(E) = 0$. Thus $-h_j \searrow \psi$ with $0 \leq \psi \leq 1$ and $\psi = 0$ on $\Omega \setminus E$. Fix $j_0 \geq 1$. By the compactness of $\{u \leq v\} \cap (K \setminus G_{j_0})$ we have

$$\overline{\lim_{j\to\infty}} \int\limits_{\{u<v\}\cap K} (r-w_1)(dd^c u_j)^n$$

$$\leq \overline{\lim_{j\to\infty}} \int\limits_{\{u\leq v\}\cap(K\setminus G_{j_0})} (r-w_1)(dd^c u_j)^n + \overline{\lim_{j\to\infty}} \int\limits_{G_{j_0}} (r-w_1)(dd^c u_j)^n$$

$$\leq \int\limits_{\{u\leq v\}\cap(K\setminus G_{j_0})} (r-w_1)(dd^c u)^n + r\,\overline{\lim_{j\to\infty}} \int\limits_{G_{j_0}} (dd^c u_j)^n - \underline{\lim}_{j\to\infty} \int\limits_{G_{j_0}} w_1(dd^c u_j)^n$$

$$\leq \int\limits_{\{u\leq v\}\cap(K\setminus G_{j_0})} (r-w_1)(dd^c u)^n$$

$$+ r\,\overline{\lim_{j\to\infty}} \int\limits_{\Omega}(-h_{j_0})(dd^c u_j)^n - \int\limits_{G_{j_0}} w_1(dd^c u)^n.$$

On the other hand, since $K \cap \{u = -\infty\} \subset G_{j_0}$ and

$$\lim_{j\to\infty} \int\limits_{\Omega}(-h_{j_0})(dd^c u_j)^n = \int\limits_{\Omega}(-h_{j_0})(dd^c u)^n \quad \text{(Proposition 5.1 in [Ce2])}$$

we have

$$\overline{\lim_{j\to\infty}} \int_{\{u<v\}\cap K} (r-w_1)(dd^c u_j)^n \tag{1}$$

$$\le \int_{(\{u\le v\}\setminus\{u=-\infty\})\cap K} (r-w_1)(dd^c u)^n + r\int_{\Omega}(-h_{j_0})(dd^c u)^n - \int_{G_{j_0}} w_1(dd^c u)^n.$$

Since E is pluripolar, by the decomposition theorem of Cegrell (Theorem 5.11 in [Ce2]) we have

$$\int_{E\setminus\{u=-\infty\}} (dd^c u)^n = 0. \tag{2}$$

Letting $j_0 \to \infty$ in (1), in view of (2), we obtain

$$\begin{aligned}
&\overline{\lim_{j\to\infty}} \int_{\{u<v\}} (r-w_1)(dd^c u_j)^n \\
&\quad\le \int_{(\{u\le v\}\setminus\{u=-\infty\})\cap K} (r-w_1)(dd^c u)^n + r\int_E (dd^c u)^n - \int_G w_1(dd^c u)^n + r\varepsilon \\
&\quad= \int_{(\{u\le v\}\setminus\{u=-\infty\})\cap K} (r-w_1)(dd^c u)^n \\
&\qquad + r\int_{\{u=-\infty\}\cap E} (dd^c u)^n - \int_G w_1(dd^c u)^n + r\varepsilon \\
&\quad\le \int_{(\{u\le v\}\setminus\{u=-\infty\})\cap K} (r-w_1)(dd^c u)^n \\
&\qquad + r\int_{\{u=-\infty\}\cap K} (dd^c u)^n - \int_{\{u=-\infty\}\cap K} w_1(dd^c u)^n + 2r\varepsilon \\
&\quad\le \int_{\{u\le v\}\cap K} (r-w_1)(dd^c u)^n + 2r\varepsilon.
\end{aligned}$$

Thus

$$\frac{1}{(n!)^2}\int_{\{u<v\}} (v-u)^n dd^c w_1\wedge\cdots\wedge dd^c w_n + \int_{\{u<v\}} (r-w_1)(dd^c v)^n \le \int_{\{u\le v\}} (r-w_1)(dd^c u)^n.$$

Applying the above inequality to $u\in\mathcal{F}$ and $v-\varepsilon$ with $\varepsilon>0$ we have

$$\frac{1}{(n!)^2}\int_{\{u<v-\varepsilon\}} (v-\varepsilon-u)^n dd^c w_1\wedge\cdots\wedge dd^c w_n + \int_{\{u<v-\varepsilon\}} (r-w_1)(dd^c v)^n \le \int_{\{u\le v-\varepsilon\}} (r-w_1)(dd^c u)^n.$$

Letting $\varepsilon \searrow 0$, and using the fact $\{u \le v - \varepsilon\} \subset \{u < v\}$ for all $\varepsilon > 0$, we get

$$\frac{1}{(n!)^2} \int\limits_{\{u<v\}} (v-u)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int\limits_{\{u<v\}} (r-w_1)(dd^c v)^n \le \int\limits_{\{u<v\}} (r-w_1)(dd^c u)^n.$$

4. Proof of Theorem 1.1. The necessity is obvious. To see the sufficiency we assume that $C_n(\{u<k\}) \ne 0$. Otherwise we have $u \ge k$ for some constant k. By the comparison principle (Theorem 3.1), we have

$$(k_j - k)^n \int\limits_{\{u<k\}} (dd^c w)^n \le \int\limits_{\{u<k_j\}} (k_j - u)^n (dd^c w)^n \le \int\limits_{\{u<k_j\}} (1-w)(dd^c u)^n$$

for all $w \in \mathrm{PSH}(\Omega), 0 < w < 1$. Letting $k \searrow k_{j-1}$ we have

$$(k_j - k_{j-1})^n C_n(\{u < k_{j-1} + 0\}) \le \|(dd^c u)^n\|_{\{u<k_j\}}.$$

Therefore

$$B - 1 - k < k_s - k_1 = \sum_{j=2}^{s} (k_j - k_{j-1}) \le \sum_{j=2}^{s} \left[\frac{\|(dd^c u)^n\|_{\{u<k_j\}}}{C_n(\{u<k_{j-1}+0\})}\right]^{1/n} \le A.$$

Thus $B - 1 - A < k$ for all $k < B$ satisfying $C_n(\{u<k\}) \ne 0$. This is impossible.

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