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Universal sequences for Zalcman's Lemma and Q_m-normality

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Abstract. We prove the existence of sequences $\{\varrho_n\}_{n=1}^{\infty}$, $\varrho_n \to 0^+$, and $\{z_n\}_{n=1}^{\infty}$, $|z_n| = 1/2$, such that for every $\alpha \in \mathbb{R}$ and for every meromorphic function $G(z)$ on $\mathbb{C}$, there exists a meromorphic function $F(z) = F_{G,\alpha}(z)$ on $\mathbb{C}$ such that $\varrho_n^{\alpha} F(nz_n + n\varrho_n\zeta)$ converges to $G(\zeta)$ uniformly on compact subsets of $\mathbb{C}$ in the spherical metric. As a result, we construct a family of functions meromorphic on the unit disk that is Q_m-normal for no $m \geq 1$ and on which an extension of Zalcman's Lemma holds.

1. Introduction. First we set some notations and conventions. We denote by Δ the open unit disk in $\mathbb{C}$. For $z \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) = \{|z - z_0| < r\}$, $\Delta'(z_0, r) = \{0 < |z - z_0| < r\}$ and $\overline{\Delta}(z_0, r) = \{|z - z_0| \leq r\}$. We write $f_n \overset{\chi}{\Rightarrow} f$ on D to indicate that the sequence $\{f_n\}$ of meromorphic functions on D converges to f uniformly on compact subsets of D in the spherical metric χ, and $f_n \Rightarrow f$ on D if the convergence is in the Euclidean metric. For a function f meromorphic on $\mathbb{C}$, $\Pi(f)$ is the family $\{f(nz) : n \in \mathbb{N}\}$, considered as a family of functions on Δ. If D is a domain and $E \subset D$, then the derived set of E with respect to D, denoted by $E_D^{(1)}$, is the set of accumulation points of E in D. For $k \geq 2$ the derived set of order k of E with respect to D is defined inductively by $E_D^{(k)} = (E_D^{(k-1)})_D^{(1)}$. The family $\Pi(f)$ is not normal for a nonconstant f meromorphic on $\mathbb{C}$. Normality properties of $\Pi(f)$ were studied from various angles, as will be explained in what follows.

An important and very useful criterion for normality is the following known lemma of L. Zalcman.

ZALCMAN'S LEMMA ([Za]). *A family $\mathcal{F}$ of functions meromorphic* (*analytic*) *on the unit disk Δ is* not *normal if and only if there exist*

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(a) *a number* $0 < r < 1$;
(b) *points* z_n, $|z_n| < r$;
(c) *functions* $f_n \in \mathcal{F}$; *and*
(d) *numbers* $\varrho_n \to 0^+$,

such that

$$f_n(z_n + \varrho_n\zeta) \overset{\chi}{\Rightarrow} g(\zeta) \quad \text{on } \mathbb{C}, \tag{1}$$

where g is a nonconstant meromorphic (entire) function on $\mathbb{C}$. Moreover, $g(\zeta)$ can be taken to satisfy the normalization $g^{\#}(\zeta) \le g^{\#}(0) = 1$ for $\zeta \in \mathbb{C}$.

Here $g^{\#}(\zeta)$ is the spherical derivative,

$$g^{\#}(\zeta) = \frac{|g'(\zeta)|}{1 + |g(\zeta)|^2}.$$

Later X. C. Pang extended this result to a criterion of (non)normality by replacing (1) by

$$\varrho_n^{\alpha} f_n(z_n + \varrho_n\zeta) \overset{\chi}{\Rightarrow} g(\zeta) \quad \text{on } \mathbb{C},$$

where α is any real number satisfying $-1 < \alpha < 1$. This generalization is very useful to deal with conditions for normality that involve derivatives (see [Pa1], [Pa2]). The interested reader is referred to [PZ] for a modification of Zalcman's Lemma, dealing with families of functions having only multiple zeros.

In [Ne1], we studied the collection of functions g that are limits in (1) for members of the family $\Pi(f)$, where $f(z)$ is a given nonconstant meromorphic function on $\mathbb{C}$.

We have the following result which will be proved in Section 2.

Theorem A. *There exist sequences $\{\varrho_n\}_{n=1}^{\infty}$, $\varrho_n \to 0^+$, and $\{z_n\}_{n=1}^{\infty}$, $|z_n| = 1/2$, such that for every $\alpha \in \mathbb{R}$ and for every function G meromorphic on $\mathbb{C}$, there is a meromorphic function $F(z) = F_{G,\alpha}(z)$ on $\mathbb{C}$ such that*

$$\varrho_n^{\alpha} F(nz_n + n\varrho_n\zeta) \overset{\chi}{\Rightarrow} G(\zeta) \quad \text{on } \mathbb{C} \tag{2}$$

and

$$\overline{\{z_n : n \ge 1\}} = \{|z| = 1/2\}. \tag{3}$$

These sequences may be said to be universal with respect to Zalcman's Lemma (or its extensions) for the families $\Pi(F)$.

Q_m-normality and the family $\Pi(f)$. Let m be a positive integer. A family $\mathcal{F}$ of functions meromorphic on a domain D is called *Q_m-normal* on D if each sequence $S = \{f_n\}$ in $\mathcal{F}$ has a subsequence $S' = \{f_{n_k}\}$ such that $f_{n_k} \overset{\chi}{\Rightarrow} f$ on $D \setminus E$, where f is a function on $D \setminus E$ (which happens to be meromorphic or $f \equiv \infty$), and $E \subset D$ satisfies $E_D^{(m)} = \emptyset$. If $\nu \in \mathbb{N}$, then

a family $\mathcal{F}$ is called *Q_m-normal of order at most ν* on D if in addition S' can always be taken such that $E_D^{(m-1)}$ contains at most ν points.

The theory of Q_m-normal families was developed by C. T. Chuang [Ch]. In [Ne3] it was shown that for every $m \in \mathbb{N}$ and $\nu = 1, 2, 3, \ldots, \infty$ there exists an entire function $f = f_{m,\nu}$ such that $\Pi(f)$ is Q_m-normal of exact order ν (i.e., Q_m-normal of order ν but not of order μ for any $\mu < \nu$). In [Ne4], it was proved that if there exist $a, b \in \widehat{\mathbb{C}}$ such that f attains a and b finitely often each, and f is not a rational function, then $\Pi(f)$ is Q_m-normal for no $m \in \mathbb{N}$. In [Ne2] the following extension to Zalcman's Lemma was introduced.

N Lemma. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and $m \geq 1$. In order that $\mathcal{F}$ not be a Q_m-normal family in D, it is necessary and sufficient that there exist*

(a) *a sequence $S = \{f_n\}_{n=1}^{\infty}$ of functions of $\mathcal{F}$;*
(b) *a set $E \subset D$ satisfying $E_D^{(m)} \neq \emptyset$, and for each point $z \in E$:*
(c) *a sequence $\{\omega_{n,z}\}_{n=1}^{\infty}$ of points in D such that $\omega_{n,z} \to z$;*
(d) *a sequence $\eta_{n,z} \to 0^+$; and*
(e) *a nonconstant function $g_z(\zeta)$ meromorphic on $\mathbb{C}$ such that*
(f) *$f_n(\omega_{n,z} + \eta_{n,z}\zeta) \overset{\chi}{\Rightarrow} g_z(\zeta)$ on $\mathbb{C}$.*

An analogous extension exists for Pang's modification, where for every $-1 < \alpha < 1$ we have instead of (f)

(f_α) $\eta_{n,z}^{\alpha} f_n(\omega_{n,z} + \eta_{n,z}\zeta) \overset{\chi}{\Rightarrow} g_z(\zeta)$ on $\mathbb{C}$.

We shall call this extension the *extended N Lemma.* The "natural" generalization of the N Lemma is not true in the direction ($\Rightarrow$) for a family $\mathcal{F}$ which is not Q_m-normal in D for every $m \in \mathbb{N}$. This means that for such an $\mathcal{F}$, there may not exist $E \subset D$ with $E_D^{(m)} \neq \emptyset$ for every $m \geq 1$ and a sequence S of functions of $\mathcal{F}$, satisfying (c)–(f) of the N Lemma. (The direction ($\Leftarrow$) is true of course in this case.)

However, by the result of Theorem A, we shall construct a family $\mathcal{F}$ which is Q_m-normal for no $m \geq 1$, but satisfies (a)–(f) of the N Lemma, with uncountable set E in (b). This construction is detailed in Theorem B.

Remark. In [Ne5], we introduced a transfinite extension of the notion of Q_m-normality and also obtained a "correct" extension of Zalcman's Lemma (or of the N Lemma) for countable ordinal numbers.

Theorem B. *There exists an entire function F such that $\Pi(F)$ is Q_m-normal for no $m \geq 1$, and $\Pi(F)$ satisfies* (a)–(f) *of the N Lemma with $E = \{|z| = 1/2\}$ in* (b).

The proof of Theorem B is given in Section 3. We also give there an extension of Theorem B in the spirit of condition (f_α) in the extended N Lemma, for every $\alpha \in \mathbb{R}$.

2. Proof of Theorem A

DEFINITION. Let B be a circle in $\mathbb{C}$, centered at z_0, and let L be a ray with origin at z_1, tangent to B at z_2. We say that L is *tangent to B from the right* (resp. *from the left*) if

$$\arg \frac{z_0 - z_1}{z_2 - z_1} > 0 \qquad \left(\text{resp. } \arg \frac{z_0 - z_1}{z_2 - z_1} < 0\right),$$

where we take the argument $-\pi < \arg z \leq \pi$.

We now construct a sequence of closed disks, $\{B_k\}_{k=2}^{\infty}$, together with sequences of tangent rays, $\{R_k\}_{k=2}^{\infty}$ and $\{L_k\}_{k=2}^{\infty}$, all originating at $z = 0$. For $k = 2$, let $B_2 = \overline{\Delta}(1, \log 2)$ and let R_2 (resp. L_2) be the ray originating at $z = 0$ and tangent to B_2 from the right (resp. left). Suppose we have defined B_k, R_k, L_k for $k \geq 2$. Let B_{k+1} be the disk centered on $\{|z| = (k+1)/2\}$ with radius $\log(k+1)$ such that L_k is tangent to B_{k+1} from the right (i.e., $R_{k+1} = L_k$). L_{k+1} will be the ray that originates at $z = 0$ and is tangent to B_{k+1} from the left. It is easy to verify that B_k, R_k and L_k are all well defined. For each $k \geq 2$ denote by

- α_k the angle between R_k and L_k, measured counterclockwise;
- c_k the center of B_k, $c_k = (k/2)e^{i\theta_k}$,

where $\{\theta_k\}_{k=2}^{\infty}$ is defined as follows:

$$\theta_2 = 0, \quad \theta_3 = \frac{\alpha_2 + \alpha_3}{2}, \quad \theta_k = \frac{\alpha_2}{2} + \sum_{j=3}^{k-1} \alpha_j + \frac{\alpha_k}{2}, \qquad k \geq 4, \tag{4}$$

(or $\theta_k = \theta_{k-1} + (\alpha_{k-1} + \alpha_k)/2$, $k \geq 3$). Moreover, denote by

- T_k the arc of the circle $\{|z| = k/2\}$ which subtends the angle α_k;
- $|T_k|$ the length of T_k;
- V_k the infinite angular sector between R_k and L_k with angle α_k, including R_k and L_k;
- x_k, y_k the points of tangency of R_k and L_k to B_k, respectively.

Geometrical considerations yield

$$\frac{k}{2} - |x_k| = \frac{k}{2} - |y_k| \xrightarrow[k\to\infty]{} 0^+. \tag{5}$$

Define

$$A_k := \operatorname{conv}(\{0\} \cup B_k) \quad \text{(convex hull)}.$$

Note that B_k and B_{k+1} are pairwise disjoint as can be deduced from (5) (for large enough k).

We deduce the relations

$$\frac{\log k}{k/2} = \sin\frac{\alpha_k}{2}, \tag{6}$$

$$\frac{|T_k|}{k/2} = \alpha_k. \tag{7}$$

Dividing (7) by (6), we get

$$\frac{|T_k|}{2\log k} = \frac{\alpha_k/2}{\sin(\alpha_k/2)}. \tag{8}$$

From (6) we see that

$$\alpha_k \searrow 0, \tag{9}$$

and

$$\sum_{k=2}^{\infty} \alpha_k = \infty, \tag{10}$$

which means that the sequence $\{e^{i\theta_k}\}_{k=2}^{\infty}$ encircles the origin infinitely many times.

We now show the existence of $N \in \mathbb{N}$ such that the disks $\{B_k : k \geq N\}$ are pairwise disjoint. From (5) we get $B_k \cap B_{k+1} = \emptyset$ for $k \geq N_1$. Let $k \geq N_1$ and denote by j_k the smallest integer that satisfies $j_k > k$ and $L_{j_k} \subset V_k$. By (4) and (9), $\theta_{j_k} < \theta_k + 2\pi$; so it is sufficient to prove that

$$B_k \cap B_{j_k} = \emptyset \tag{11}$$

for large enough k. By (8), $|T_k|/2\log k \searrow 1$ as $k \to \infty$; so there exists some $\beta > 1$ such that $|T_k| < \beta \cdot 2\log k$ for $k \geq 2$. By (9), we conclude that for some $0 < C < 1$ we have, for $k \geq 2$,

$$j_k - k \geq \frac{2\pi C}{\alpha_k} = \frac{\pi C k}{|T_k|} > \frac{\pi C k}{2\beta \log k}. \tag{12}$$

Set

$$\mu = \frac{\pi C}{2\beta}.$$

In order to prove (11), it is sufficient to show that

$$|c_{j_k} - c_k| > \log j_k + \log k. \tag{13}$$

We distinguish two cases.

Case 1. Suppose that $j_k > 2k$. In this case, for some N_2, we have

$$\frac{\log k}{j_k} + \frac{\log j_k}{j_k} < \frac{1}{4} < \frac{1 - k/j_k}{2}, \qquad k \geq N_2. \tag{14}$$

Case 2. Suppose that $j_k \le 2k$. Then (12) implies that there exists N_3 such that for $k \ge N_3$,

$$\frac{j_k - k}{2} \ge \frac{\mu k}{2\log k} > \log 2k + \log k \ge \log j_k + \log k. \tag{15}$$

From (14) and (15), we deduce (13); it follows that B_k, $k \ge N = \max\{N_1, N_2, N_3\}$, are pairwise disjoint as claimed. Now set $G_N = A_N$ and for $k \ge N$ put $G_{k+1} = G_k \cup A_{k+1}$. Then the closed sets G_k satisfy

$$G_N \subset G_{N+1} \subset \cdots, \tag{16}$$

$$\bigcup_{k=N}^{\infty} G_k = \mathbb{C}, \tag{17}$$

$$\operatorname{dist}(G_k, B_{k+1}) > 0, \qquad G_k \cup B_{k+1} \subset G_{k+1}. \tag{18}$$

Define now, for $n \ge 2$,

$$z_n = \frac{1}{2}\, e^{i\theta_n}, \qquad \varrho_n = \frac{\sqrt{\log n}}{n}, \tag{19}$$

and let G be a meromorphic function on $\mathbb{C}$. For $n \ge N$, set

$$h_n(z) = \varrho_n^{-\alpha} G\left(\frac{z - c_n}{\sqrt{\log n}}\right). \tag{20}$$

By the Mittag-Leffler Theorem, there exists a function $h(z)$ meromorphic on $\mathbb{C}$ such that the poles of h are exactly $\bigcup_{h=N}^{\infty} E_n$, where E_n is the set of poles of h_n in B_n, and its singular part at any pole in B_n is the singular part of h_n at that pole. Then for every $n \ge N$, $\widetilde{h}_n = h_n - h$ is holomorphic in B_n.

We define a sequence $\{p_n\}_{n=N}^{\infty}$ of approximating polynomials as follows. We choose p_N to satisfy

$$\max_{z \in B_N} |p_N(z) - \widetilde{h}_N(z)| < 1/2^N. \tag{21}$$

The existence of p_N is ensured by Runge's Theorem ([Ga, pp. 94–96, Corollary 2 to Runge's Theorem]). Assume that we have defined $p_N, p_{N+1}, \ldots, p_n$. Again by (18) and Runge's Theorem, there exists a polynomial p_{n+1} such that

$$\max_{z \in B_{n+1}} |p_{n+1}(z) - \widetilde{h}_{n+1}(z)| < 1/2^{n+1}, \tag{22}$$

and

$$\max_{z \in G_n} |p_{n+1}(z) - p_n(z)| < 1/2^{n+1}. \tag{23}$$

By (16) and (23), $\{p_n\}$ is a uniform Cauchy sequence on each G_n, $n \ge N$, and hence uniformly convergent on G_n for $n \ge N$. Thus, by (17), there exists an entire function $p(z)$ such that $p_n \Rightarrow p$ on $\mathbb{C}$. Now (20) and (22) imply that p is nonconstant. By (23), for $n \geq N$ and $z \in G_n$ we have

$$|p_n(z) - p(z)| < 1/2^n. \tag{24}$$

Set $F = p + h$, and let K be a compact set in $\mathbb{C}$. There exists $N^* \geq N$ such that $K \subset \Delta(0, \sqrt{\log N^*})$. From (19)–(22) and (24) and the equality of the singular parts of F and h_n in B_n, we get for $\zeta \in K$ and $n \geq N^*$,

$$\begin{aligned}
&|\varrho_n^\alpha F(nz_n + n\varrho_n\zeta) - G(\zeta)| \\
&\quad = |\varrho_n^\alpha F(c_n + \sqrt{\log n}\,\zeta) - \varrho_n^\alpha h_n(c_n + \sqrt{\log n}\,\zeta)| \\
&\quad = \varrho_n^\alpha |p(c_n + \sqrt{\log n}\,\zeta) - \widetilde{h}_n(c_n + \sqrt{\log n}\,\zeta)| \\
&\quad \leq \varrho_n^\alpha |p(c_n + \sqrt{\log n}\,\zeta) - p_n(c_n + \sqrt{\log n}\,\zeta)| \\
&\qquad + \varrho_n^\alpha |p_n(c_n + \sqrt{\log n}\,\zeta) - \widetilde{h}_n(c_n + \sqrt{\log n}\,\zeta)| < \frac{\varrho_n^\alpha}{2^{n-1}} \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

and (2) follows. The assertion in (3) can be deduced from (4) and (10). The proof of Theorem A is complete.

3. Proof of Theorem B. Given a nonconstant entire function G, let F be an entire function corresponding to G by Theorem A with $\alpha = 0$. For $k \geq 3$, set

$$F_k(z) = F(kz).$$

Define a sequence $\{k_n\}_{n=1}^\infty$ of natural numbers inductively. Set $k_1 = 2$. Suppose we have chosen k_n. Choose k_{n+1} so that $k_{n+1} > k_n$ and $|\theta_{k_{n+1}} - 2\pi n|$ is minimal. By (4) and (9), we then have $|\theta_{k_{n+1}} - 2\pi n| \to 0$ as $n \to \infty$, so $z_{k_n} \to 1/2$ (k_{n+1} is chosen such that $z_{k_{n+1}}$ is the z_k closest to $z = 1/2$ at the end of the nth lap around the origin by the sequence $\{z_k\}_{k=2}^\infty$). We are now ready to define the ingredients in (a)–(f) of the N Lemma. Let

$$E = \{|z| = 1/2\},$$

and let the sequence $S = \{f_n\}$ of functions of $\Pi(F)$ be defined by

$$f_n = F_{k_n}.$$

Now for $z = 1/2$, define

$$k_{n,1/2} = k_n, \qquad \omega_{n,1/2} = z_{k_{n,1/2}}, \qquad \eta_{n,1/2} = \varrho_{k_{n,1/2}};$$

then by (2), we get

$$f_n(\omega_{n,1/2} + \eta_{n,1/2}\zeta) \Rightarrow G(\zeta) = g_{1/2}(\zeta) \quad \text{on } \mathbb{C}. \tag{25}$$

It remains to find $\{\omega_{n,z}\}$, $\{\varrho_{n,z}\}$, $g_z(\zeta)$ for $z \in E \setminus \{1/2\}$. Let z be such a point. By (9) and (10) there exists an increasing sequence $\{k_{n,z}\}_{n=1}^\infty$ of positive integers such that for $n \geq 1$,

$$k_n \leq k_{n,z} < k_{n+1}, \tag{26}$$

and

$$z_{k_{n,z}} \underset{n\to\infty}{\longrightarrow} z. \tag{27}$$

A sequence $\{k_{n,z}\}_{n=1}^{\infty}$ that satisfies (26) and (27) is of course not unique. Note that the definition of $k_{n,1/2}$ agrees with (26) and (27). We assert that $k_{n,z}/k_n \to 1$ as $n \to \infty$. In fact, we will show that the convergence is uniform on E. For this purpose, it is enough to show that $k_n/k_{n+1} \to 1$ as $n \to \infty$. Indeed, by (9), $\sum_{k=k_n+1}^{k_{n+1}} \alpha_k \to 2\pi$ as $n \to \infty$, and combining it with (6) we get, for large enough n,

$$\frac{(k_{n+1}-k_n)2\log k_{n+1}}{k_{n+1}/2} < (k_{n+1}-k_n)2\arcsin\left(\frac{\log k_{n+1}}{k_{n+1}/2}\right) < \sum_{k=k_n+1}^{k_{n+1}} \alpha_k < 3\pi.$$

So if $\underline{\lim}\, k_n/k_{n+1} < 1$, we get a contradiction. Set

$$\omega_{n,z} = z_{k_{n,z}}\frac{k_{n,z}}{k_n}, \qquad \eta_{n,z} = \varrho_{k_{n,z}}\frac{k_{n,z}}{k_n}.$$

By (26) and (27), we have $\eta_{n,z} \to 0^+$ and $\omega_{n,z} \to z$ as $n \to \infty$. So together with (2), we have

$$\begin{aligned} |f_n(\omega_{n,z}+\eta_{n,z}\zeta) - G(\zeta)| &= \left|F_{k_n}\left(z_{k_{n,z}}\frac{k_{n,z}}{k_n} + \varrho_{k_{n,z}}\frac{k_{n,z}}{k_n}\zeta\right) - G(\zeta)\right| \\ &= |F(z_{k_{n,z}}k_{n,z} + \varrho_{k_{n,z}}k_{n,z}\zeta) - G(\zeta)| \Rightarrow 0 \quad \text{on } \mathbb{C} \end{aligned} \tag{28}$$

(thus, $g_z = G$ for any $z \in E$).

From (25) and (28), it follows that the family $\Pi(F)$ satisfies conditions (a)–(f) of the N Lemma with $E = \{|z| = 1/2\}$.

We shall now give an extension of Theorem B corresponding to the extension of the N Lemma by condition (f_α).

THEOREM B*. *Let $\alpha \in \mathbb{R}$. Then there exists an entire function F such that $\Pi(F)$ is Q_m-normal for no $m \geq 1$ and satisfies* (a)–(e), (f_α) *of the extended N Lemma with $E = \{|z| = 1/2|\}$ in* (b).

We need the following lemma:

POWER LEMMA. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D and let l, m be positive integers. Then $\mathcal{F}$ is Q_m-normal on D if and only if $\mathcal{F}_l := \{f^l : f \in \mathcal{F}\}$ is Q_m-normal on D.*

The direction ($\Rightarrow$) comes from the definition of Q_m-normality. The opposite direction follows by applying (by negation) the N Lemma with (a)–(f).

4. Proof of Theorem B*. We proceed in two steps. The first step is to find an entire function F that satisfies (a)–(e), (f_α) of the extended

N Lemma. The second step is to show that $\Pi(F)$ is Q_m-normal for no $m \geq 1$. For the first step, take any non-constant entire function G, and let $F = F_{G,\alpha}$ be the corresponding entire function from Theorem A. We apply the proof of Theorem B with a few modifications. We have to replace (25) by

$$\eta_{n,1/2}^{\alpha} f_n(\omega_{n,1/2} + \eta_{n,1/2}\zeta) \Rightarrow G(\zeta) = g_{1/2}(\zeta) \quad \text{on } \mathbb{C},$$

and also replace (28) with

$$\begin{aligned} &|\eta_{n,z}^{\alpha} f_n(\omega_{n,z} + \eta_{n,z}\zeta) - G(\zeta)| \\ &\qquad = \left|\eta_{n,z}^{\alpha} F_{k_n}\left(z_{k_{n,z}} \frac{k_{n,z}}{k_n} + \varrho_{k_{n,z}} \frac{k_{n,z}}{k_n}\zeta\right) - G(\zeta)\right| \\ &\qquad = |\eta_{n,z}^{\alpha} F(z_{k_{n,z}} k_{n,z} + \varrho_{k_{n,z}} k_{n,z}\zeta) - G(\zeta)| \Rightarrow 0 \quad \text{on } \mathbb{C}. \end{aligned}$$

The last convergence (to 0) is true since $\eta_{n,z}/\varrho_{k_{n,z}} \to 1$ as $n \to \infty$.

Now for $-1 < \alpha < 1$ the non-Q_m-normality of $\Pi(F)$ for every $m \geq 1$ is ensured by the opposite direction of the extended N Lemma with (a)–(e), (f_α) and step 2 is done.

For $\alpha \geq 1$ or $\alpha \leq -1$, take l large enough such that $-1 < \alpha/l < 1$. Then by the previous discussion, there is an entire function F for which $\Pi(F)$ is Q_m-normal for no $m \geq 1$ and satisfies (a)–(e), $(\mathrm{f}_{\alpha/l})$ of the extended N Lemma. The family $\Pi(F^l)$ satisfies (a)–(e), (f_α) of the extended N Lemma and since $\Pi(F^l) = \Pi(F)_l$ it follows by the Power Lemma that $\Pi(F^l)$ is also Q_m-normal for no $m \geq 1$, as desired. The proof of Theorem B* is complete.

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