ANNALES
POLONICI MATHEMATICI
85.3 (2005)

Connectedness of the Carathéodory discs for doubly connected domains

by Leonhard Frerick (Wuppertal) and
Gerald Schmieder (Oldenburg)

Abstract. We prove that the Carathéodory discs for doubly connected domains in the complex plane are connected.

Let $G \subsetneq \overline{\mathbb{C}}$ be a domain and assume that no boundary component is a point. The Carathéodory distance $c(z_0, z_1)$ between two points $z_0, z_1 \in G$ is defined as $\sup |f(z_0)|$, where the supremum is taken over all f holomorphic in G whose modulus is bounded by 1 and which vanish at z_1. The following theorem proves a conjecture of Pflug and Jarnicki ([1, p. 42]) in the doubly connected case:

Theorem 1. *If the boundary of G consists of two continua and $z_0 \in G$, then the Carathéodory discs $\{z \in G : c(z, z_0) < r\}$, $r \in (0,1)$, are connected.*

Proof. It is enough to show that for an arbitrary $z_1 \in G$ there is a curve connecting z_0 and z_1 on which $c(\cdot, z_0)$ is increasing. Since c is invariant under conformal maps, we may replace G by its image under the Möbius transformation $z \mapsto (z - z_0)/(z - z_1)$, and therefore we may assume that $z_0 = 0$, $z_1 = \infty$. Moreover there is some conformal mapping onto some radial slit domain which leaves 0 and ∞ fixed (see e.g. [2, Theorem IX.24]). Since rotations also fix 0 and ∞ we can arrange that ∂G is contained in $\{z \in \mathbb{C} : \Re(z) \leq 0\}$.

Using the continuity of c it is enough to show that $c(\cdot, 0) : (0, \infty) \to (0, 1)$, $t \mapsto c(t, 0)$, is increasing. Let therefore $t_1 > t_0 > 0$ be arbitrary. Note that $\varphi(z) = (z - t_1)/(z - t_0)$ maps the closed left halfplane onto the closed disc bounded by the circle through $1 = \varphi(\infty)$ and $t_1/t_0 = \varphi(0) > 1$, which is symmetrical about the real axis. This shows that $1 < |\varphi(z)| < t_1/t_0$ for each

2000 *Mathematics Subject Classification*: 30C75, 30C80.
Key words and phrases: Carathéodory distance.

$z \in \partial G$ and $m = \max\{|\varphi(z)| : z \in \partial G\} < \varphi(0)$. We set $h(z) := \varphi(z)/m$ and obtain $h(0) > 1$.

Let f be the unique holomorphic function in G, whith modulus bounded by 1, which vanishes at t_0 and fulfills $c(t_0, 0) = c(0, t_0) = f(0)$. If $g := fh$, then g is holomorphic on G, its modulus is bounded by 1 and it vanishes at t_1. Therefore $c(t_0, 0) = f(0) < g(0) \leq c(0, t_1) = c(t_1, 0)$ and the proof is complete.

References

[1] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter, Berlin, 1993.

[2] M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea, New York, 1975.

Fachbereich 7 – Mathematik
Bergische Universität – Gesamthochschule Wuppertal
Gaußstraße 20
42119 Wuppertal, FRG
E-mail: Leonhard.Frerick@math.uni-wuppertal.de

Fakultät V
Institut für Mathematik
Universität Oldenburg
26111 Oldenburg, FRG
E-mail: schmieder@mathematik.uni-oldenburg.de

Reçu par la Rédaction le 19.4.2005 (1574)