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Sum of squares and the Łojasiewicz exponent at infinity

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Abstract. Let $V \subset \mathbb{R}^n$, $n \geq 2$, be an unbounded algebraic set defined by a system of polynomial equations $h_1(x) = \cdots = h_r(x) = 0$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial. It is known that if f is positive on V then $f|_V$ extends to a positive polynomial on the ambient space $\mathbb{R}^n$, provided V is a variety. We give a constructive proof of this fact for an arbitrary algebraic set V. Precisely, if f is positive on V then there exists a polynomial $h(x) = \sum_{i=1}^r h_i^2(x)\sigma_i(x)$, where σ_i are sums of squares of polynomials of degree at most p, such that $f(x)+h(x) > 0$ for $x \in \mathbb{R}^n$. We give an estimate for p in terms of: the degree of f, the degrees of h_i and the Łojasiewicz exponent at infinity of $f|_V$. We prove a version of the above result for polynomials positive on semialgebraic sets. We also obtain a nonnegative extension of some odd power of f which is nonnegative on an irreducible algebraic set.

1. Introduction. Let $f \in \mathbb{R}[x]$, $x = (x_1, \ldots, x_n)$, be a positive semidefinite polynomial, that is, $f(x) \geq 0$ for $x \in \mathbb{R}^n$. Then

(AH) $$fh^2 = h_1^2 + \cdots + h_m^2 \quad \text{for some } h, h_1, \ldots, h_m \in \mathbb{R}[x],\ h \neq 0,$$

i.e., f is a sum of squares of rational functions. We shall denote by $\sum \mathbb{R}(x)^2$ the set of such sums and by $\sum \mathbb{R}[x]^2$ the set of sums of squares of polynomials. The above theorem is E. Artin's [1] solution of Hilbert's 17th problem. Motzkin [16] gave an example of a positive semidefinite polynomial $f(x_1, x_2) = 1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3)$ which is not a sum of squares of polynomials, so the degree of h in (AH) must be positive.

Positive semidefinite polynomials can also be considered on *closed basic semialgebraic sets*, that is, sets $X \subset \mathbb{R}^n$ of the form

$$X = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}, \quad \text{where } g_1, \ldots, g_r \in \mathbb{R}[x].$$

We define the *preordering* in $\mathbb{R}[x]$, generated by $g_1, \ldots, g_r \in \mathbb{R}[x]$, to be the

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set

$$T(g_1,\dots,g_r)=\Big\{\sum_{e=(e_1,\dots,e_r)\in\{0,1\}^r} s_e g_1^{e_1}\cdots g_r^{e_r} : s_e\in\sum\mathbb{R}[x]^2 \text{ for } e\in\{0,1\}^r\Big\}.$$

Let $f\in\mathbb{R}[x]$. The following Stellensätze are natural generalizations of the above Artin theorem (Krivine [11], Dubois [6], Risler [22]; see also [2]).

Real Nullstellensatz. *Let $I\subset\mathbb{R}[x]$ be an ideal. Then $f=0$ on $V(I):=\{x\in\mathbb{R}^n : g(x)=0$ for any $g\in I\}$ if and only if $f^{2N}+u\in I$ for some integer $N>0$ and $u\in\sum\mathbb{R}[x]^2$.*

Positivstellensatz. *$f>0$ on X if and only if $sf=1+t$ for some $s,t\in T(g_1,\dots,g_r)$.*

Nichtnegativstellensatz. *$f\geq 0$ on X if and only if $sf=f^{2N}+t$ for some integer $N>0$ and $s,t\in T(g_1,\dots,g_r)$.*

These issues were studied in [15], [21], [26], [28]. A remarkable result of Schmüdgen [29] asserts that for X compact every strictly positive polynomial on X belongs to $T(g_1,\dots,g_r)$. A challenging problem is effective computation of the polynomials in the Stellensätze, in particular explicit bounds for their degrees. For instance a relevant estimate for the degree of the denominator in (AH) was obtained by Schmid (see Scheiderer [28]), who proved that the degree of h can be bounded by an n tower of exponentials in the degree of g. In a recently posted preprint, Lombardi, Perrucci and Roy [14] obtained a bound as a tower of five exponentials in n and $\deg g$.

An important issue is extension of semidefinite polynomials on an algebraic set to semidefinite polynomials on the ambient space. The existence of such an extension was proved by C. Scheiderer [25, Corollary 5.5] (see also [27]). A partial result on nonnegative extension of polynomials was obtained by D. Plaumann [20, Lemma 3.2]. In the present paper we give a constructive proof of the existence of a positive semidefinite extension onto the space $\mathbb{R}^n$ (or $\mathbb{R}^{n+r}$ for some $r\in\mathbb{N}$) of a semidefinite polynomial f on an algebraic or semialgebraic set $X\subset\mathbb{R}^n$. We estimate the degree of such an extension in terms of the degree of f and the Łojasiewicz exponent at infinity of a suitable mapping.

By the *Łojasiewicz exponent at infinity* of a mapping $F:\mathbb{R}^n\to\mathbb{R}^m$ *on an unbounded set* S we mean the supremum of the set of exponents ν in the following *Łojasiewicz inequality*:

$$|F(x)|\geq C|x|^\nu \quad \text{for all } x\in S \text{ with } |x|\geq R,$$

for some positive constants C, R, where $|\cdot|$ are norms (in $\mathbb{R}^n$ and $\mathbb{R}^m$); we denote it by $\mathcal{L}_\infty(F|S)$. For $S=\mathbb{R}^n$ the exponent $\mathcal{L}_\infty(F|S)$ will be called the *Łojasiewicz exponent at infinity* of F and denoted by $\mathcal{L}_\infty(F)$. The Łojasiewicz exponent does not depend on the chosen norms in $\mathbb{R}^n$ and $\mathbb{R}^m$.

In what follows, we will use the Euclidean norm. The exponent $\mathcal{L}_\infty(F)$ is an important tool in the study of properness and injectivity of polynomial mappings, in the effective Nullstellensatz and in optimization (for references see for instance [19]).

For $k,n,d \in \mathbb{N}$ and $l \in \mathbb{R}$ we put

$$\theta(k,n,d,l) = k(6k-3)^{n-1}(d+2-l).$$

Let $V \subset \mathbb{R}^n$ be an unbounded algebraic set and let $h_1,\dots,h_r \in \mathbb{R}[x_1,\dots,x_n]$ be polynomials such that $V = \{x \in \mathbb{R}^n : h_1(x) = \dots = h_r(x) = 0\}$. Obviously we may assume that $r \geq n$. Let $k \in \mathbb{N}$, $k \geq \max\{\deg h_1,\dots,\deg h_r\}$. For a polynomial function $f : \mathbb{R}^n \to \mathbb{R}$, $\deg f = d$, which is positive on the set V we have

$$f(x) + h(x) > 0, \qquad x \in \mathbb{R}^n,$$

and

$$\mathcal{L}_\infty(f+h) = \mathcal{L}_\infty(f|V)$$

for an effectively computed polynomial $-h \in T(h_1,-h_1,\dots,h_r,-h_r)$, with

$$\deg h < 2 + 2k + d + \theta(2k,n,d,\mathcal{L}_\infty(f|V)),$$

of the form (4.2) (see Theorem 4.1 and Corollary 5.1). We also obtain a version of the above result for $\mathcal{L}_\infty(f+h) = \beta$, where $\beta \leq \mathcal{L}_\infty(f|V)$ is given (see Corollary 4.3). If additionally V is an irreducible algebraic set and $f(x) \geq 0$ for $x \in V$, with $f|_V \neq 0$, then

$$f(x)f^p(x) = -h(x) + \sigma(x),$$

where $\sigma \in \sum \mathbb{R}(x)^2$, and $-h \in T(h_1,-h_1,\dots,h_r,-h_r)$ is of the form (5.4) (see Corollary 5.3). We also have an estimate for the degree of h similar to the above.

For the basic semialgebraic set

$$X = \{x \in \mathbb{R}^n : g_1(x) > 0,\dots,g_j(x) > 0, g_{j+1}(x) \geq 0,\dots,g_r(x) \geq 0\},$$

where $g_1,\dots,g_r \in \mathbb{R}[x_1,\dots,x_n]$ and $1 \leq j \leq r$, we put $h_i(x,y) = g_i(x)y_i^2 - 1$ for $i = 1,\dots,j$ and $h_i(x,y) = g_i(x) - y_i^2$ for $i = j+1,\dots,r$, and

$$Y = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^r : h_1(x,y) = \dots = h_r(x,y) = 0\}.$$

By Theorem 4.1 we obtain the following version of the Positivstellensatz (see Corollary 5.2): if $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial and $f(x) > 0$ for $x \in X$, then

$$f(x) + h(x,y) = \sigma(x,y),$$

where $\sigma \in \sum \mathbb{R}(x,y)^2$, and $-h \in T(h_1,-h_1,\dots,h_r,-h_r)$ is of the form (5.2). The degree of h is estimated similarly to the above in terms of $\deg f$ and the Łojasiewicz exponent at infinity of $f|_V$.

The main role in our considerations will be played by the following result due to K. Kurdyka and S. Spodzieja (see [12, Corollary 10], cf. [3]–[10],

[23]). Let $\operatorname{dist}(x,V)$ be the distance from $x \in \mathbb{R}^n$ to the set $V \subset \mathbb{R}^n$ in the metric induced by the norm $|\cdot|$ (we set $\operatorname{dist}(x,V)=1$ if $V=\emptyset$). By the *degree* of a polynomial mapping $F=(f_1,\dots,f_m):\mathbb{R}^n\to\mathbb{R}^m$ we mean $\deg F=\max\{\deg f_1,\dots,\deg f_m\}$.

THEOREM 1.1 ([12]). *Let $F=(f_1,\dots,f_m):\mathbb{R}^n\to\mathbb{R}^m$ be a polynomial mapping of degree d. Then for some positive constant C,*

$$|F(x)| \ge C\left(\frac{\operatorname{dist}(x,F^{-1}(0))}{1+|x|^2}\right)^{d(6d-3)^{n-1}} \qquad \text{for } x\in\mathbb{R}^n.$$

2. Preliminaries. We denote by $\mathbf{L}(m,k)$ the set of all linear mappings $\mathbb{R}^m\to\mathbb{R}^k$, where for $k=0$ we put $\mathbb{R}^k=\{0\}$.

We will use the following theorem (see [32, Theorem 4], cf. [31]).

THEOREM 2.1. *Let $F:\mathbb{R}^n\to\mathbb{R}^m$ be a polynomial mapping having a compact set of zeros, and let $n\le k\le m$. Then for any $L\in\mathbf{L}(m,k)$ such that $(L\circ F)^{-1}(0)$ is compact, we have*

$$\mathcal{L}_\infty(F)\ge\mathcal{L}_\infty(L\circ F). \tag{2.1}$$

Moreover, for the generic *$L\in\mathbf{L}(m,k)$, i.e., outside a proper algebraic subset of $\mathbf{L}(m,k)$, the set $(L\circ F)^{-1}(0)$ is compact and*

$$\mathcal{L}_\infty(F)=\mathcal{L}_\infty(L\circ F). \tag{2.2}$$

Let $m\ge k$. We denote by $\Delta(m,k)$ the set of all linear mappings $L=(L_1,\dots,L_k)\in\mathbf{L}(m,k)$ of the form

$$L_i(y_1,\dots,y_m)=y_i+\sum_{j=k+1}^{m}\alpha_{i,j}y_j,\qquad i=1,\dots,k,$$

where $\alpha_{i,j}\in\mathbb{R}$.

Theorem 2.1 implies the following corollary (see [32, Corollary 5]).

COROLLARY 2.2. *Under the assumptions of Theorem 2.1, for the generic linear mapping $L=(L_1,\dots,L_k)\in\Delta(m,k)$, the set of zeroes of $L\circ F$ is compact and*

$$\mathcal{L}_\infty(F)=\mathcal{L}_\infty(L\circ F).$$

Moreover, if $d_j=\deg f_j$ and $d_1\ge\dots\ge d_m$, then $\deg(L_j\circ F)=d_j$ for $j=1,\dots,k$.

Let us recall Proposition 2.10 of [19] (see also [18]).

PROPOSITION 2.3. *Let $\beta=p/q$, where $p\in\mathbb{Z}$, $q\in\mathbb{N}$. Then there exists a polynomial mapping $\Psi:\mathbb{R}^2\to\mathbb{R}^2$ such that*

(a) $\mathcal{L}_\infty(\Psi)=\beta$,
(b) $\deg\Psi\le q\cdot(|p|+q)$.

Moreover, the mapping has at most one zero.

Actually the polynomial mapping Ψ in the above proposition is of one of the following forms: $\Psi = (x, xy - 1) : \mathbb{R}^2 \to \mathbb{R}^2$, the gradient of the polynomial $h_1(x,y) = y^{p+q} - (x + y^q)^{p+q}$ if $p, q \geq 1$, or of $h_2(x,y) = y - y^{1+q}x^{-p-q}$ if $-p > q > 1$.

Let $G'_k(\mathbb{R}^n)$, where $0 \leq k \leq n$, denote the set of all k-dimensional affine subspaces of $\mathbb{R}^n$. Let $G_k(\mathbb{R}^n)$, where $0 \leq k \leq n$, be the set of all k-dimensional linear subspaces of $\mathbb{R}^n$ (cf. [13, B.6.11] for complex Grassmann spaces).

From Proposition 2.3 we obtain the following corollary.

COROLLARY 2.4. *Let $\beta = p/q$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$. Let $n > 2$, and let $A \in G'_2(\mathbb{R}^n)$. Then there exists a polynomial $\psi_\beta : \mathbb{R}^n \to \mathbb{R}$, which is a sum of squares of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$, such that*

(a) $\mathcal{L}_\infty(\psi_\beta|_A) = \beta$,
(b) $\deg \psi_\beta \leq 4q(|p| + 2q)$,
(c) $\psi_\beta^{-1}(0) \subset A$ *contains at most one point.*

Proof. Let $E = (E_1, \ldots, E_{n-2}) \in \mathbf{L}(n, n-2)$ be a linear mapping and $z = (z_1, \ldots, z_{n-2}) \in \mathbb{R}^{n-2}$ be a point such that $A = E^{-1}(z)$. By using a translation, we may assume that $z = 0$. By choosing an appropriate coordinate system, we can assume that $A = \mathbb{R}^2 \times \{0\}$.

From Proposition 2.3 there exists a polynomial mapping $\Psi = (\psi_1, \psi_2) : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\mathcal{L}_\infty(\Psi) = \tfrac{1}{2}\beta \quad \text{and} \quad \deg \Psi \leq 2q(|p| + 2q).$$

Let

$$\psi_\beta(x) = \psi_1^2(x_1, x_2) + \psi_2^2(x_1, x_2) + E_1^2(x) + \cdots + E_{n-2}^2(x)$$

for $x = (x_1, \ldots, x_m) \in \mathbb{R}^n$. Then $\mathcal{L}_\infty(\psi_\beta|A) = 2\mathcal{L}_\infty(\Psi) = \beta$ and

$$\deg \psi_\beta \leqslant \max\{2 \deg \psi_1, 2 \deg \psi_2, 2\} = 2 \deg \Psi \leq 4q(|p| + 2q).$$

So, (a) and (b) are proved. Part (c) follows from the definition of ψ_β and the fact that $\Psi^{-1}(0)$ contains at most one point. ∎

Let $V \subset \mathbb{C}^n$ be a complex algebraic set. We denote by $\delta(V)$ the *total degree* of V, i.e. $\delta(V) := \deg V_1 + \cdots + \deg V_s$, where $V = V_1 \cup \cdots \cup V_s$ is the decomposition of V into irreducible components (see [13, p. 419]).

Let $V \subset \mathbb{R}^n$ be a real algebraic set and let $F : \mathbb{R}^n \to \mathbb{R}^m$, where $m \geq n$, be a polynomial mapping. Let $V_\mathbb{C}$ be the Zariski closure of V in $\mathbb{C}^n$; we call it the *complexification* of V. Let $F_\mathbb{C} : \mathbb{C}^n \to \mathbb{C}^m$ denote the complexification of F (i.e., $F_\mathbb{C}$ is a complex polynomial mapping such that $F_\mathbb{C}|_{\mathbb{R}^n} = F$). We write $\delta(V)$ for the total degree of $V_\mathbb{C} \subset \mathbb{C}^n$.

We will need the following fact ([19, Proposition 2.11] or [18, Proposition 4.5]).

Proposition 2.5. *Let $V \subset \mathbb{R}^n$ be a real algebraic set with $0 < \dim_{\mathbb{R}} V < n-2$. Then there exist $A \in G_2'(\mathbb{R}^n)$ and $f \in \mathbb{R}[x_1,\dots,x_n]$ such that $V \cap A = \emptyset$, $f|_V = 0$, $f|_A = 1$ and $\deg f \le \delta(V)$.*

As is shown in the proof of [19, Proposition 2.11], the affine subspace A and the polynomial f in the above assertion can be effectively determined. More precisely, after choosing an appropriate coordinate system (using for instance a Gröbner basis), one can choose a nonzero polynomial $g \in \mathbb{C}[z_1,\dots,z_{n-2}]$, $\deg g \le \delta(V)$, vanishing on $V_{\mathbb{C}}$. Hence there exists $x_0 \in \mathbb{R}^{n-2}$ such that $\operatorname{Re} g(x_0) \ne 0$. Then one can take $A = \{x_0\} \times \mathbb{R}^2$ and $f = u/u(x_0)$, where $g|_{\mathbb{R}^n} = u + iv$ and $u, v \in \mathbb{R}[x_1,\dots,x_n]$.

Let $V \subset \mathbb{R}^n$ be a real algebraic set. We denote by $\kappa(V)$ the infimum of the numbers $k = \max\{\deg h_1,\dots,\deg h_r\}$, where $r \in \mathbb{N}$, $h_1,\dots,h_r \in \mathbb{R}[x_1,\dots,x_n]$ and $V = \{x \in \mathbb{R}^n : h_1(x) = \dots = h_r(x) = 0\}$. From [19, Proposition 2.13] we have

Lemma 2.6. *Let $V \subset \mathbb{R}^n$ be an algebraic set. Then $\kappa(V) \le \delta(V)$.*

3. Auxiliary results. We prove the following generalization of [19, Theorem 1.1]. Let $V \subset \mathbb{R}^n$ be an unbounded algebraic set of the form

$$V = \{x \in \mathbb{R}^n : h_1(x) = \dots = h_r(x) = 0\},$$

where $h_1,\dots,h_r \in \mathbb{R}[x_1,\dots,x_n]$. We can assume that $r \ge n$, defining $h_i = h_1$ for $i \ge r$. Let $k \in \mathbb{N}$ with

$$k \ge \max\{\deg h_1,\dots,\deg h_r\}.$$

Proposition 3.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$, where $m \ge n \ge 2$, be a polynomial mapping of degree $d > 0$ and suppose that the set $F^{-1}(0) \cap V$ is compact. Let p be an integer satisfying*

$$p \ge \mathcal{L}_\infty(F|V) + \theta(k,n,d,\mathcal{L}_\infty(F|V)). \tag{3.1}$$

Let $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$, and $H : \mathbb{R}^n \to \mathbb{R}^{nr}$ be the polynomial mapping defined by

$$H(x) = (h_i(x)(x_j - \xi_j)^p : i = 1,\dots,r,\ j = 1,\dots,n), \qquad x \in \mathbb{R}^n.$$

Then for the generic linear mapping $L \in \mathbf{L}(nr,m)$ we have

$$\mathcal{L}_\infty(F + L \circ H) = \mathcal{L}_\infty(F|V), \tag{3.2}$$

and $\deg L \circ H \le k + p$.

Proof. It suffices to prove the assertion for $\xi = 0 \in \mathbb{R}^n$. Let $F = (f_1,\dots,f_m) : \mathbb{R}^n \to \mathbb{R}^m$. Since $F^{-1}(0) \cap V$ is compact, we have $\mathcal{L}_\infty(F|V) > -\infty$. It is known that there exist constants $C_1, R_1 > 0$ such that

$$|F(x)| \ge C_1 |x|^{\mathcal{L}_\infty(F|V)} \quad \text{for } x \in V \text{ with } |x| \ge R_1. \tag{3.3}$$

Then there exists a positive constant C such that (cf. [19, formula (3.2)])

$$|F(x)| \geq C|w|^{\mathcal{L}_\infty(F|V)} \quad \text{for } x \in V \text{ with } |x| \geq R_1,\ |x-w| \leq 1. \tag{3.4}$$

Diminishing C or C_1, we can assume that (3.4) holds with $C = C_1$.

From the Mean Value Theorem, for every $x, w \in \mathbb{R}^n$ and for any i there is a point t_i on the segment with end points x, w such that

$$|f_i(x) - f_i(w)| \leq |\nabla f_i(t_i)|\,|x-w|. \tag{3.5}$$

Let $M(w) = \sup\{|\nabla f_i(x)| : |x| \leq |w|+1,\ i = 1, \ldots, m\}$. Since $\deg \nabla f_i \leq d-1$, there exist constants $C_2 > 0$ and $R_2 \geq R_1 + 1$ such that $0 \leq M(w) \leq C_2|w|^{d-1}$ for $|w| \geq R_2$. From (3.5) and the above, for $|w| \geq R_2$, $|x-w| \leq 1$ we have

$$|F(x) - F(w)| \leq M(w)|x-w| \leq C_2|w|^{d-1}|x-w|. \tag{3.6}$$

Let

$$W = \left\{ w \in \mathbb{R}^n : \operatorname{dist}(w, V) \leq \min\left\{1, \frac{C_1}{2C_2}|w|^{\mathcal{L}_\infty(F|V)-d+1}\right\}\right\}.$$

By (3.3), (3.5) and (3.6) we obtain (cf. [19, (3.6)])

LEMMA 3.2. *Under the above notations,*

$$|F(w)| \geq \frac{C_1}{2}|w|^{\mathcal{L}_\infty(F|V)} \quad \textit{for } w \in W \textit{ with } |w| \geq R_2. \tag{3.7}$$

Let $\tilde{H} = (h_1, \ldots, h_r) : \mathbb{R}^n \to \mathbb{R}^r$. From Theorem 1.1 there exists a constant $C_3 > 0$ such that

$$|\tilde{H}(w)| \geq C_3 \left(\frac{\operatorname{dist}(w, V)}{1+|w|^2}\right)^{k(6k-3)^{n-1}} \quad \text{for } w \in \mathbb{R}^n \text{ with } |w| \geq R_2. \tag{3.8}$$

Let

$$U = \mathbb{R}^n \setminus W$$

and $\theta = \theta(k, n, d, \mathcal{L}_\infty(F|V))$. We have $\mathcal{L}_\infty(\tilde{H}|U) \geq -\theta$ by the following lemma, which follows from (3.8) (cf. [19, (3.9)]):

LEMMA 3.3. *There exist constants $C_4 > 0$ and $R_3 \geq R_2$ such that*

$$|\tilde{H}(x)|\,|x|^{\theta(k,n,d,\mathcal{L}_\infty(F|V))} \geq C_4 \quad \textit{for } x \in U \textit{ with } |x| \geq R_3. \tag{3.9}$$

It is easy to see that for some $c, c' > 0$ we have

$$c|H(x)| \leq |\tilde{H}(x)|\,|x|^p \leq c'|H(x)| \quad \text{for } x \in \mathbb{R}^n. \tag{3.10}$$

Let

$$\Phi = (F, H) : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^{nr}.$$

Since $\Psi|_V = (F, 0)|_V$, from (3.7), (3.9) and (3.10) we obtain (cf. [19, (3.11)])

$$\mathcal{L}_\infty(\Phi) = \mathcal{L}_\infty(F|V). \tag{3.11}$$

From Corollary 2.2, for the generic linear mapping $\tilde{L} \in \Delta(m+nr,m)$ we have $\mathcal{L}_\infty(\tilde{L}\circ\Phi) = \mathcal{L}_\infty(\Phi)$ and obviously $\tilde{L} = \mathrm{id}_{\mathbb{R}^m} + L$, where $\mathrm{id}_{\mathbb{R}^m}$ is the identity mapping on $\mathbb{R}^m$ and $L \in \mathbf{L}(nr,m)$ is generic. Then $\tilde{L}\circ\Phi|_V = F|_V$. The inequality $\deg L\circ H \le k+p$ is obvious. From the above and (3.11) we obtain the assertion of Proposition 3.1. ∎

Note that the exponent $\mathcal{L}_\infty(F|V)$ may be a negative rational number. Therefore, the use of the exponent in estimating the degree of the mapping $L\circ H$ improves the estimate.

4. Positive polynomials on algebraic sets. By using Proposition 3.1 we obtain the following theorem on extension of a positive polynomial on a given algebraic set to a sum of squares. Let $h_1,\dots,h_r \in \mathbb{R}[x_1,\dots,x_n]$ and let $V\subset\mathbb{R}^n$ be an algebraic set of the form

$$V = \{x\in\mathbb{R}^n : h_1(x) = \cdots = h_r(x) = 0\}. \tag{4.1}$$

Let $k\in\mathbb{N}$, $k \ge \max\{\deg h_1,\dots,\deg h_r\}$. We will assume that the set V is unbounded.

THEOREM 4.1. *Let $f:\mathbb{R}^n\to\mathbb{R}$, $n\ge 2$, be a polynomial of degree $d>0$. Suppose that the set $f^{-1}(0)\cap V$ is compact and there exists an open set $U\subset\mathbb{R}^n$ such that $V\subset U$ and $f(x)>0$ for all $x\in U\setminus V$. Then there exists a polynomial $h:\mathbb{R}^n\to\mathbb{R}$ of the form*

$$h(x) = \sum_{i=1}^{r}\sum_{j=1}^{n}\alpha_{i,j}h_i^2(x)(x_j-\xi_j)^p, \qquad x\in\mathbb{R}^n, \tag{4.2}$$

where $\alpha_{i,j}\in\mathbb{R}$ are positive, $\xi = (\xi_1,\dots,\xi_n)$ is an arbitrary point of V, and p is an even number satisfying

$$\mathcal{L}_\infty(F|V) + \theta(2k,n,d,\mathcal{L}_\infty(F|V)) \le p < d + \theta(2k,n,d,\mathcal{L}_\infty(f|V)) + 2, \tag{4.3}$$

such that

(a) $(f+h)(x)\ge 0$ *for* $x\in\mathbb{R}^n$,
(b) $\mathcal{L}_\infty(f+h) = \mathcal{L}_\infty(f|V)$,
(c) $\deg h \le p+2k$.

Proof. Assertion (c) follows immediately from (4.2). We will prove the remaining assertions.

Let $F = (f_1,\dots,f_n):\mathbb{R}^n\to\mathbb{R}^n$, where $f_i = f$ for $i=1,\dots,n$. Since $F^{-1}(0)\cap V = f^{-1}(0)\cap V$ is compact, we have $\mathcal{L}_\infty(F|V) = \mathcal{L}_\infty(f|V) > -\infty$. Obviously

$$V = \{x\in\mathbb{R}^n : h_1^2(x) = \cdots = h_r^2(x) = 0\}$$

and $2k \ge \max\{\deg h_1^2,\dots,\deg h_r^2\}$. Since $d\ge\mathcal{L}_\infty(F|V)$, on substituting $2k$ for k, the assumption (3.1) in Proposition 3.1 is equivalent to (4.3). So, by Proposition 3.1 for arbitrary $\xi = (\xi_1, \dots, \xi_n) \in V$, an even integer p satisfying (4.3) and the polynomial mapping $H : \mathbb{R}^n \to \mathbb{R}^{nr}$ defined by

$$H(x) = (h_i^2(x)(x_j - \xi_j)^p : i = 1, \dots, r,\ j = 1, \dots, n), \qquad x \in \mathbb{R}^n,$$

for the generic linear mapping $L \in \mathbf{L}(nr, n)$, we have

$$\mathcal{L}_\infty(F + L \circ H) = \mathcal{L}_\infty(F|V). \tag{4.4}$$

In particular, (4.4) holds for the generic $L \in \mathbf{L}(nr, n)$ with positive coefficients. Without loss of generality, we may assume that $\xi = 0 \in V$. Then the mapping H vanishes only on V.

By Lemma 3.2, there exist $C_1, C_2 > 0$ such that for

$$W = \{w \in \mathbb{R}^n : \operatorname{dist}(w, V) \le \min\{1, C_1|w|^{\mathcal{L}_\infty(f|V)-d+1}\}\}$$

we obtain

$$|F(w)| \ge C_2|w|^{\mathcal{L}_\infty(f|V)} \quad \text{for } w \in W \text{ with } |w| \ge R_2.$$

By the assumptions of the theorem, we may assume that $f(x) > 0$ for $x \in V$ with $|x| \ge R_2$, so diminishing C_2 if necessary, we have

$$f(w) \ge C_2|w|^{\mathcal{L}_\infty(f|V)} \quad \text{for } w \in W \text{ with } |w| \ge R_2. \tag{4.5}$$

By Lemma 3.3, there exist constants $C_3 > 0$ and $R_3 \ge R_2$ such that

$$|H(x)| \ge C_3|x|^d \quad \text{for } x \in \mathbb{R}^n \setminus W \text{ with } |x| \ge R_3. \tag{4.6}$$

By the choice of d, increasing R_3 if necessary, for some $C_4 > 0$ we obtain

$$|f(x)| \le C_4|x|^d \quad \text{for } x \in \mathbb{R}^n \text{ with } |x| \ge R_3.$$

Multiplying H by a sufficiently large number, we may assume that $C_3 > C_4$. Then from (4.5), (4.6) and the fact that $L_i \circ H(x) > 0$ for $L_i \in \mathbf{L}(nr, 1)$ with positive coefficients and $x \in \mathbb{R}^n \setminus V$, we see that for some mapping $L = (L_1, \dots, L_n) \in \mathbf{L}(nr, n)$ with positive coefficients, (4.4) holds and

$$f(x) + L_i \circ H(x) \ge 0 \tag{4.7}$$

for $x \in \mathbb{R}^n$ with $|x| \ge R_3$. Moreover, since $f(x) > 0$ for $x \in U \setminus V$, multiplying H by a sufficiently large number, we may assume that (4.7) holds for $x \in \mathbb{R}^n$ with $|x| \le R_3$. Summing up, (4.7) holds for any $x \in \mathbb{R}^n$, and (a) is verified.

Put $L_0 = L_1 + \dots + L_n$, and let

$$L_0(y_1, \dots, y_{nr}) = \alpha_1 y_1 + \dots + \alpha_{nr} y_{nr},$$

where $\alpha_1, \dots, \alpha_{nr} \in \mathbb{R}$ are positive. Then the polynomial $h = L_0 \circ H : \mathbb{R}^n \to \mathbb{R}$ is of the form (4.2). Since the Euclidean and the polycylindric norms in $\mathbb{R}^n$ are equivalent, there exist $c, c' > 0$ such that

$$c[nf(x) + L_0 \circ H(x)] \le |F(x) + L \circ H(x)| \le c'[nf(x) + L_0 \circ H(x)]$$

for $x \in \mathbb{R}^n$. Hence, from (4.4) we easily deduce that (b) holds. ∎

From Theorem 4.1, Lemma 2.6 and Artin's Theorem (see [1, Satz 1], cf. [28, Theorem 1.1.1]) we obtain

COROLLARY 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial satisfying the assumptions of Theorem* 4.1. *Then there exists a polynomial $g : \mathbb{R}^n \to \mathbb{R}$ of the form $g = f + h$, where g is a sum of squares of rational functions and h is a sum of squares of polynomials, such that*

(a) $g|_V = f|_V$,
(b) $\mathcal{L}_\infty(g) = \mathcal{L}_\infty(f|V)$,
(c) $\deg g \le d + 2\delta(V) + 2 + \theta(2\delta(V), n, d, \mathcal{L}_\infty(f|V))$.

With an additional assumption we will show that when extending a positive polynomial on an algebraic set to a sum of squares, we can require the Łojasiewicz exponent at infinity to have a fixed value. Precisely, we assume that $\dim V \le n - 3$. Thus $n \ge 4$. According to Proposition 2.5 there exist $A \in G_2'(\mathbb{R}^n)$ and $g \in \mathbb{R}[x_1, \ldots, x_n]$ such that

$$V \cap A = \emptyset, \quad g|_V = 1, \quad g|_A = 0, \quad \deg g \le \delta(V).$$

Let $E = (E_1, \ldots, E_{n-2}) \in \mathbf{L}(n, n-2)$ be a linear mapping such that $A = L^{-1}(z)$ for some $z \in \mathbb{R}^{n-2}$. By Corollary 2.4 for any $\beta = \frac{p}{q} \in \mathbb{Q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, there exists a polynomial $\psi_\beta : \mathbb{R}^n \to \mathbb{R}$ which is a sum of squares of polynomials, such that

$$\mathcal{L}_\infty(\psi_\beta|A) = \beta \quad \text{and} \quad \deg \psi_\beta \le (|p| + 2q) \cdot 4q,$$

and $\psi_\beta^{-1}(0) \subset A$ contains at most one point.

COROLLARY 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$, where $n \ge 4$, be a polynomial of degree $d > 0$ and suppose that $f(x) > 0$ for $x \in V$. Let $\beta = p/q \in \mathbb{Q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and let $\beta \le \mathcal{L}_\infty(f|V)$. Take an even integer P satisfying*

$$P \ge d + \theta(2k + 2, n, D, \beta), \tag{4.8}$$

where $D = e\delta(V) + \max\{d, (|p| + 2q) \cdot 4q\}$ and $e \ge 2$ is the smallest even number greater than the order of ψ_β at its zero. Let $\xi = (\xi_1, \ldots, \xi_n) \in A$. Then there exists a polynomial $h : \mathbb{R}^n \to \mathbb{R}$ of the form

$$h(x) = \sum_{i=1}^{r} \sum_{l=1}^{n-2} \sum_{j=1}^{n} \alpha_{i,l,j} h_i^2(x) E_l^2(x)(x_j - \xi_j)^P, \quad x \in \mathbb{R}^n,$$

where $\alpha_{i,l,j}$ are positive real numbers, such that

(a) $(g^e f + (1 - g^2)\psi_\beta + h)|_V = f|_V$,
(b) $(g^e f + (1 - g^2)\psi_\beta + h)(x) \ge 0$ *for* $x \in \mathbb{R}^n$,
(c) $\mathcal{L}_\infty(g^e f + (1 - g^2)\psi_\beta + h) = \beta$,
(d) $\deg(g^e f + (1 - g^2)\psi_\beta + h) \le P + 2k + 2$.

Proof. By the definition of the functions ψ_β, g, the choice of e, and the assumption that $f(x) > 0$ for $x \in V$, there exists an open set $U \subset \mathbb{R}^n$ with

$V \cup A \subset U$ such that $g^2(x)f(x) + (1 - g^2(x))\psi_\beta(x) > 0$ for $x \in U \setminus (V \cup A)$. Moreover, the function $g^2(x)f(x) + (1 - g^2(x))\psi_\beta(x)$ has a compact set of zeroes in $V \cup A$ and $\deg[g^2(x)f(x) + (1 - g^2(x))\psi_\beta(x)] \le D$. Then Theorem 4.1 yields the assertion. ∎

EXAMPLE 4.4. The assumption $f > 0$ on V is essential, as shown by the following example. Let $V = \{(x, y) \in \mathbb{R}^n : x^2 - y^3 = 0\}$, and let $f(x, y) = y$. Then $f \ge 0$ on V, but for every $h \in \mathbb{R}[x, y]$ vanishing on V there exists $(x, y) \in \mathbb{R}^2$ such that $f(x, y)+h(x, y) < 0$. Indeed, $h(x, y) = (x^2-y^3)h_1(x, y)$, and $f(0, y) + h(0, y) = y - y^3h_1(0, y)$. Thus $f(0, y) + h(0, y)$ changes sign at 0.

5. Positivstellensatz on algebraic and semialgebraic sets. Let $V \subset \mathbb{R}^n$ be an algebraic set of the form (4.1). Then V can be considered as a basic semialgebraic set, since

$$V = \{x \in \mathbb{R}^n : g_1(x) \ge 0, \dots, g_{2r}(x) \ge 0\},$$

where $g_1 = h_1$, $g_2 = -h_1$, …, $g_{2r-1} = h_r$, $g_{2r} = -h_r$. Then the preordering T generated by $g_1, \dots, g_{2r}$ is of the form

$$T = \Big\{ \sum_{e\in\{0,1\}^{2r}} \sigma_e g_1^{e_1} \cdots g_{2r}^{e_{2r}} : \sigma_e \in \sum \mathbb{R}[x]^2 \text{ for } e = (e_1, \dots, e_{2r}) \in \{0, 1\}^{2r}\Big\}.$$

From Theorem 4.1 and Artin's Theorem we obtain the following version of the Positivstellensatz on algebraic sets.

COROLLARY 5.1. *If $f : \mathbb{R}^n \to \mathbb{R}$ with $n \ge 2$ is a polynomial of degree $d > 0$, and $f(x) > 0$ for $x \in V$, then*

$$f(x) = -h(x) + \sigma(x),$$

where $\sigma \in \sum \mathbb{R}(x)^2$, h is of the form (4.2), and $-h \in T$. If additionally $k = \max\{\deg g_1, \dots, \deg g_r\}$, $d = \deg f$ and $D = \max\{k, d\}$, then

$$\deg h \le d + 2k + 2 + \theta(2k, n, d, -D(6D - 3)^{n-1}). \tag{5.1}$$

Proof. If V is a bounded algebraic set, then the assertion is obvious. Assume that V is unbounded. The first part of the assertion follows immediately from Theorem 4.1. From [32, Corollary 6] (cf. [7]–[10]), we have

$$\mathcal{L}_\infty(f|V) \ge -D(6D - 3)^{n-1},$$

and by Theorem 4.1, we obtain (5.1). ∎

By considering a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ positive on a basic semialgebraic set X as an element of $\mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_r]$, where r is the number of inequalities defining X, we obtain a version of the Positivstellensatz on any basic semialgebraic set (see Corollary 5.2 below). Let us start with some notations.

Consider the basic semialgebraic set

$$X=\{x\in\mathbb{R}^n : g_1(x)>0,\dots,g_j(x)>0,g_{j+1}(x)\ge 0,\dots,g_r(x)\ge 0\},$$

where $g_1,\dots,g_r\in\mathbb{R}[x_1,\dots,x_n]$ and $0\le j\le r$. Put $h_i(x,y)=g_i(x)y_i^2-1$ for $i=1,\dots,j$ and $h_i(x,y)=g_i(x)-y_i^2$ for $i=j+1,\dots,r$, and let

$$Y=\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^r : h_1(x,y)=\dots=h_r(x,y)=0\}.$$

Then we have $\pi(Y)=X$ for the projection $\pi:\mathbb{R}^n\times\mathbb{R}^r\ni(x,y)\mapsto x\in\mathbb{R}^n$. So, any polynomial $f:\mathbb{R}^n\to\mathbb{R}$ can be considered as a polynomial on Y, by identifying $f\circ\pi$ with f. Denote by T_1 the preordering of $\mathbb{R}[x_1,\dots,x_n,y_1,\dots,y_r]$ generated by $h_1,-h_1,\dots,h_r,-h_r$. By Theorem 4.1 we obtain the following version of the Positivstellensatz on basic semialgebraic sets.

Corollary 5.2. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a polynomial, and let $f(x)>0$ for $x\in X$. Then*

$$f(x)=-h(x,y)+\sigma(x,y),$$

where $\sigma\in\sum\mathbb{R}(x,y)^2$, and $-h\in T_1$ is of the form

(5.2)

$$-h(x,y)=\sum_{i=1}^{r}\sum_{j=1}^{n+r}\alpha_{i,j}h_i(x,y)\cdot(-h_i(x,y))(w_j-\xi_j)^p,\quad (x,y)\in\mathbb{R}^n\times\mathbb{R}^r,$$

where $\alpha_{i,j}$ are positive numbers, $(w_1,\dots,w_{n+r})=(x_1,\dots,x_n,y_1,\dots,y_r)$, and $(\xi_1,\dots,\xi_{n+r})$ is an arbitrary point of Y and p is a positive even number such that

$$p\le d+2+\theta(2k+4,n+r,d,-D(6D-3)^{n+r-1}),\tag{5.3}$$

provided $k=\max\{\deg g_1,\dots,\deg g_r\}$, $d=\deg f$ and $D=\max\{k+2,d\}$.

Proof. By [32, Corollary 6], we have $\mathcal{L}_\infty(f|Y)\ge -D(6D-3)^{n+r-1}$. It is easy to see that $\max\{\deg h_1,\dots,\deg h_r\}\le k+2$. So, for the smallest positive even number satisfying

$$p\ge d+\theta(2k+4,n+r,d,-D(6D-3)^{n-1})$$

the inequality (5.3) holds. Moreover, the assumptions of Theorem 4.1 are satisfied. So Theorem 4.1 yields the assertion. ∎

Corollary 5.2 also includes the case when the basic semialgebraic set X is closed or when it is open. It is known that for a basic closed semialgebraic set

$$X=\{x\in\mathbb{R}^n : g_1(x)\ge 0,\dots,g_r(x)\ge 0\},$$

where $g_1,\dots,g_r\in\mathbb{R}[x_1,\dots,x_n]$, there exists an algebraic set

$$Y=\{(x_1,\dots,x_n,y_1,\dots,y_r)\in\mathbb{R}^n\times\mathbb{R}^r : g_1(x)-y_1^2=0,\dots,g_r(x)-y_r^2=0\}$$

such that $\pi(Y) = X$, where $\pi : \mathbb{R}^n \times \mathbb{R}^r \ni (x,y) \mapsto x \in \mathbb{R}^n$. So, any polynomial $f : \mathbb{R}^n \to \mathbb{R}$ can be considered as a polynomial on Y, upon identifying $f \circ \pi$ with f. Then the preordering T_1 is generated by $g_1(x) - y_1^2$, $-g_1(x) + y_1^2, \ldots, g_m(x) - y_m^2, -g_r(x) + y_r^2$. Thus Corollary 5.2 gives the Positivstellensatz on a closed semialgebraic set for $j = 0$.

For $j = r$, Corollary 5.2 gives the Positivstellensatz for an open semialgebraic set. Indeed, for an open basic semialgebraic set $X = \{x \in \mathbb{R}^n : g_1(x) > 0, \ldots, g_r(x) > 0\}$, there exists an algebraic set $Y = \{(x, y_1, \ldots, y_r) \in \mathbb{R}^n \times \mathbb{R}^r : g_1(x)y_1^2 - 1 = 0, \ldots, g_r(x)y_r^2 - 1 = 0\}$ such that $\pi(Y) = X$. Then the preordering T_1 is generated by $g_1(x)y_1^2 - 1, -g_1(x)y_1^2 + 1, \ldots, g_m(x)y_m^2 - 1$, $-g_r(x)y_r^2 + 1$.

Let V be an irreducible algebraic set of the form (4.1).

Corollary 5.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial, and let $f(x) \geq 0$ for $x \in V$, and $f|_V \neq 0$. Then*

$$f^{p+1} = -h + \sigma,$$

where $\sigma \in \sum \mathbb{R}(x)^2$, and $-h \in T$ is of the form

$$(5.4)\qquad -h(x) = \sum_{i=1}^{r}\sum_{j=1}^{n} \alpha_{i,j} f^p(x) h_i(x) \cdot (-h_i(x))(x_j - \xi_j)^p + \sum_{i=1}^{r} \alpha_i h_i(x) \cdot (-h_i(x))(1 - \xi_{n+1} f(x))^p, \qquad x \in \mathbb{R}^n \times \mathbb{R}^r,$$

where $\alpha_{i,j}, \alpha_i$ are positive numbers, $(\xi_1, \ldots, \xi_n)$ is an arbitrary point of V, $\xi_{n+1} \in \mathbb{R}$ and p is a positive even number such that

$$(5.5)\qquad p \leq d + 2 + \theta(2k + 4, n + 1, d, -D(6D - 3)^n),$$

provided $k = \max\{\deg g_1, \ldots, \deg g_r\}$, $d = \deg f$ and $D = \max\{k, d + 1\}$.

Proof. Let $X = V \setminus V(f)$. Then $f(x) > 0$ for $x \in X$ and $X \neq \emptyset$. Let

$$Y = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : x \in V,\ f(x)y - 1 = 0\},$$

and define $h_i(x,y) = h_i(x)$ for $i = 1, \ldots, r$ and $h_{r+1}(x,y) = f(x)y - 1$. Then by Theorem 4.1 for any $(\xi_1, \ldots, \xi_{n+1}) \in Y$, there exist positive numbers $\alpha_{i,j}$, $i = 1, \ldots, r + 1$, $j = 1, \ldots, n$, and $\sigma \in \sum \mathbb{R}(x,y)^2$ such that

$$f(x) = -h(x,y) + \sigma(x,y),$$

where

$$-h(x,y) = \sum_{i=1}^{r+1}\sum_{j=1}^{n+1} \alpha_{i,j} h_i(x,y) \cdot (-h_i(x,y))(w_j - \xi_j)^p, \qquad (x,y) \in \mathbb{R}^n \times \mathbb{R},$$

and $(w_1, \ldots, w_{n+1}) = (x_1, \ldots, x_n, y)$. Setting $y = 1/f$ yields the assertion. ∎

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