ANNALES
POLONICI MATHEMATICI
105.1 (2012)

Periodic solutions for second order Hamiltonian systems on an arbitrary energy surface

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Abstract. Two theorems about the existence of periodic solutions with prescribed energy for second order Hamiltonian systems are obtained. One gives existence for almost all energies under very natural conditions. The other yields existence for all energies under a further condition.

1. Introduction. We study the second order Hamiltonian systems

$$-\ddot{q}(t) = V'(q(t)), \tag{1.1}$$

where $\ddot{q}(t)$ is the second derivative of q with respect to t, $V \in C^1(\mathbb{R}^N, \mathbb{R})$, and $V'(x)$ denotes the gradient of V with respect to x. Throughout this paper V is called a *potential function*, and $(\cdot,\cdot)$ and $|\cdot|$ denote the inner product and norm in $\mathbb{R}^N$ respectively.

Define $H(p,q) = \frac{1}{2}|p|^2 + V(q)$. Then it is well-known that H is a first integral of the system (1.1). The Hamiltonian system corresponding to H is

$$-\dot{p} = \frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p}, \tag{1.2}$$

where $p = \dot{q}$. It is natural to consider whether (1.1) has a periodic solution on a fixed energy surface $\{(p,q) : H(p,q) = h\}$. The search of periodic solutions with prescribed energy is a problem with a long history. We refer the readers to [R, MW, S2] for the research on the prescribed energy problems for general Hamiltonian systems (1.2); in the following we restrict ourselves to mentioning some results about the second order Hamiltonian systems (1.1). In the 1980s, Benci [B], Gluck and Ziller [GZ] and Hayashi [H] obtained the following result via totally different methods.

THEOREM 1.1. *Suppose $V \in C^2(\mathbb{R}^N, \mathbb{R})$ and*

(B1) $\Omega_h := \{x \in \mathbb{R}^N : V(x) < h\}$ *is non-empty and bounded.*

2010 *Mathematics Subject Classification*: Primary 34C25; Secondary 58E05.
Key words and phrases: Hamiltonian systems, periodic solutions, prescribed energy problem, monotonicity method.

DOI: 10.4064/ap105-1-1 © Instytut Matematyczny PAN, 2012

Then (1.1) *has a periodic solution of energy* h. *If moreover*

(B2) $V'(q) \neq 0$ *for* $q \in V_h := \{x : V(x) = h\}$,

then the periodic solution is non-constant.

The proofs of Gluck–Ziller and Hayashi used techniques from algebraic topology and differential geometry, while Benci used the singular potential well and an approximation scheme. But in all these proofs, the assumption $V \in C^2(\mathbb{R}^N, \mathbb{R})$ is essential. Recently, Zhang [Z] got the following result under a weaker smoothness assumption:

THEOREM 1.2. *Suppose* $V \in C^1(\mathbb{R}^N, \mathbb{R})$ *satisfies:*

(A1) *there exist positive constants* μ_1 *and* μ_2 *such that*

$$(V'(x), x) \geq \mu_1 V(x) - \mu_2, \quad \forall x \in \mathbb{R}^N;$$

(A2) $V'(x) \to 0$ *as* $|x| \to \infty$;

(A3) *there exist positive constants* a *and* b *such that*

$$V(x) \geq a|x|^{\mu_1} - b, \quad \forall x \in \mathbb{R}^N.$$

Then for any $h > \max\{\mu_2/\mu_1, \limsup_{|x|\to 0} V(x)\}$, *system* (1.1) *has a non-constant periodic solution of energy* h.

Note that (A3) implies that (B1) holds for any $h > \limsup_{|x|\to 0} V(x)$. Then it is natural to ask whether Theorem 1.1 holds with a weaker smoothness assumption, or, in other words, whether Theorem 1.2 remains true when $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and (B1) holds. The purpose of this paper is to answer this question. Our main result is:

THEOREM 1.3. *Suppose* $V \in C^1(\mathbb{R}^N, \mathbb{R})$ *satisfies the following conditions:*

(V1) V *achieves a global minimum* V_0 *at* x_0;

(V2) $\mathcal{H} := \liminf_{|x|\to\infty} V(x) > V_0$.

Then for almost all $h \in (V_0, \mathcal{H})$, *there exists a non-constant periodic solution of energy* h.

REMARK. Note that (V1) and (V2) combined are equivalent to (B1). In fact, when (B1) holds, V achieves a minimum V_0 at $x_0 \in \Omega_h$. Since $V(x) \geq h$ for $x \in \Omega_h^c$, V_0 is a global minimum. The boundedness of Ω_h implies that $\liminf_{|x|\to\infty} V(x) > h > V_0$. So (V1) and (V2) hold. Conversely, if (V1) and (V2) hold, it is easy to see that for any $h \in (V_0, \mathcal{H})$, Ω_h is non-empty and bounded.

There is a gap between Theorems 1.1 and 1.3: the latter says nothing about the energy in a zero measure set. Without further information, the conclusion of Theorem 1.3 cannot hold on all energy surfaces, since there exist examples such that (1.1) has no non-constant periodic solution (see

Remark II in [B]). Recalling Theorem 1.1, we conjecture that there exists a non-constant periodic solution on all energy surfaces if $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and (B1), (B2) hold. But up till now we have not been able to prove this. Therefore, we use the stronger assumption (A1).

THEOREM 1.4. *Suppose* $V \in C^1(\mathbb{R}^N, \mathbb{R})$ *satisfies* (V1), (V2) *and* (A1). *Then for any* $h \in (\mu_2/\mu_1, \mathcal{H})$, *there exists a non-constant periodic solution of energy* h.

REMARK. If there exists a critical point x_0 of V on V_h, (A1) becomes $\mu_1 V(x_0) - \mu_2 \le 0$, i.e., $h \le \mu_2/\mu_1$. So (A1) implies (B2) when $h > \mu_2/\mu_1$.

REMARK. Combining the conditions (A1), (A2) with (A3), we have the following inequalities:

$$\mu_1(a|x|^{\mu_1} - b) - \mu_2 \le \mu_1 V(x) - \mu_2 \le (V'(x), x) \le |V'(x)|\,|x| = o(1)|x|.$$

This implies $\mu_1 < 1$. As in Theorem 1.4 we do not need (A2), there is no such restriction on μ_1. Moreover we weaken (A3) to (V2). For these reasons, Theorem 1.4 improves Theorem 1.2 considerably.

In order to prove Theorems 1.3 and 1.4, we use the functional

$$f(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2\,dt \int_0^1 (h - V(u))\,dt, \quad u \in W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N). \tag{1.3}$$

Zhang [Z] also applied this functional and the generalized mountain-pass theorem of Benci–Rabinowitz [BR]. In this paper we adopt a different approach. We consider (1.3) as a family of functionals f_h parameterized by h, and then use the monotonicity method to treat this parameter dependent functional. The monotonicity method was introduced by Struwe in [S2, S1] to solve some specific variational problems. Then Jeanjean [J] developed an abstract version of Struwe's method for a family of functionals $I(\lambda, \cdot)$ of a special form

$$I(\lambda, u) = A(u) - \lambda B(u),$$

under the assumption that $I(\lambda, u)$ has a mountain-pass geometry uniformly for λ in a fixed bounded interval. Moreover if $I(\lambda, u)$ is monotone with respect to λ, then one can obtain a bounded P.S. sequence for almost all λ. Noting that the functional f_h is increasing with respect to h, if there exists a min-max scheme uniformly for h, then we can use the same idea to obtain a bounded P.S. sequence for almost all h without any further condition. That is the main reason we obtain Theorem 1.3 under natural conditions (V1) and (V2). For more applications of the monotonicity technique we refer the readers to [CT], [SZ] and [MS].

2. Proofs of Theorems 1.3 and 1.4. We apply the above mentioned functional

$$f_h(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2\,dt \int_0^1 (h - V(u))\,dt \tag{2.1}$$

defined on $E := W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ to study the system (1.1). E is a Hilbert space with inner product and norm defined by

$$\langle u, v\rangle = \int_0^1 (\dot{u}, \dot{v})\,dt + \int_0^1 (u, v)\,dt, \quad \|u\| := \int_0^1 |\dot{u}|^2\,dt + \int_0^1 |u|^2\,dt.$$

Moreover, E has an orthogonal decomposition $E = E_1 \oplus E_2$, where $E_1 = \mathbb{R}^N$ and $E_2 = \{y \in E : \int_0^T y(t)\,dt = 0\}$, and it is compactly embedded into $C(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$. We recall two well-known inequalities (see Proposition 1.3 in [MW]): Sobolev's inequality

$$\|y\|_\infty := \max_{t\in[0,1]} |y(t)| \le \frac{1}{12^{1/2}}\|\dot{y}\|_2 \quad \text{for every } y \in E_2, \tag{2.2}$$

and Wirtinger's inequality:

$$\|y\|_2 \le \frac{1}{2\pi}\|\dot{y}\|_2 \quad \text{for every } y \in E_2. \tag{2.3}$$

Here $\|\cdot\|_p$ denotes the L^p norm.

By our assumptions, it is easy to see that $f_h \in C^1(E, \mathbb{R})$, and its derivative satisfies

$$\langle f_h'(u), v\rangle = \int_0^1 (\dot{u}, \dot{v})\,dt \int_0^1 (h - V(u))\,dt - \frac{1}{2}\int_0^1 |\dot{u}|^2\,dt \int_0^1 (V'(u), v)\,dt. \tag{2.4}$$

The following lemma establishes the relationship between the critical points of f_h and the periodic solutions of (1.1).

Lemma 2.1. *Let $\tilde{u} \in E$ be a critical point of f_h and $f_h(\tilde{u}) > 0$. Set*

$$T^2 = \frac{\frac{1}{2}\int_0^1 |\dot{\tilde{u}}|^2\,dt}{\int_0^1 (h - V(\tilde{u}))\,dt}. \tag{2.5}$$

Then $\tilde{q}(t) = \tilde{u}(t/T)$ is a non-constant T-periodic solution for (1.1) with energy h.

The proof of this lemma can be found in [AZ].

In the rest of this paper, without loss of generality, we assume that the minimum of V is 0 and that it is achieved at 0. In order to prove Theorem 1.3, it is sufficient to prove that for any given interval $[h_1, h_2] \subset (0, \mathcal{H})$ and for almost all $h \in [h_1, h_2]$, f_h has a critical point. From now on, we will not write the integration interval explicitly.

Next we show f_h has a linking geometry uniformly for $h \in [h_1, h_2]$. Let $S := \partial B_\rho \cap E_2$, where ρ is given in the following lemma.

LEMMA 2.2. *There exist $\rho > 0$ and $\alpha > 0$ such that $f_h|_S \geq \alpha$ for all $h \in [h_1, h_2]$.*

Proof. By the continuity of V and $V(0) = 0$, there exists a constant ρ_1 such that

$$V(x) < h_1/2 \quad \text{for } |x| \leq \rho_1.$$

Let $\rho = \sqrt{12}\,\rho_1$. It follows from (2.2) that

$$f_h(u) \geq f_{h_1}(u) = \frac{1}{2}\int |\dot{u}|^2\, dt \int (h_1 - V(u))\, dt \geq \frac{h_1\rho^2}{2} =: \alpha$$

for $u \in S$. ∎

Next let $w_0 = \sin(2\pi t)e$, where e is a unit vector in $\mathbb{R}^N$. Then it can be verified that $w_0 \in E_2$. Set $W := E_1 \oplus \operatorname{span}\{w_0\}$.

LEMMA 2.3. *Suppose $\{y_j\}$ is a sequence in W with $\|y_j\| \to \infty$ as $j \to \infty$. Then for any $\theta \in (0,1)$ there exists a set $U \subset [0,1]$ with* meas$(U) > \theta$ *and a subsequence $\{y_{j_k}\}$ such that $y_{j_k}(t) \to \infty$ uniformly for $t \in U$.*

Proof. Since the dimension of W is finite, all norms on W are equivalent. We may assume $\|y_j\| \neq 0$ and define a bounded sequence by putting $z_j = y_j/\|y_j\|$. Then there exist a subsequence of $\{z_{j_k}\}$ (not relabeled) and $z \in W$ such that $z_{j_k} \to z$ uniformly, with $\|z\| = 1$. Let $U = \{t \in [0,1] : |z(t)| > 2\epsilon\}$. Since $\{t \in [0,1] : |z(t)| = 0\}$ is a finite set and w is continuous, we can choose ϵ sufficiently small such that meas$(U) > \theta$. Then

$$\begin{aligned} |y_{j_k}(t)| = \|y_{j_k}\|\,|z_{j_k}(t)| &\geq \|y_{j_k}\|(|z(t)| - |z(t) - z_{j_k}(t)|) \\ &\geq \|y_{j_k}\|(2\epsilon - |z(t) - z_{j_k}(t)|) \to \infty. \end{aligned}$$

That is, $y_{j_k}(t) \to \infty$ uniformly for $t \in U$. ∎

Given a positive constant R, we define

$$Q_R = \{v + rw_0 : v \in E_1,\ r \geq 0 \text{ and } \|v + rw_0\| \leq R\}.$$

LEMMA 2.4. *There exists $R > \rho$ such that $f_h|_{\partial Q_R} \leq 0$ for all $h \in [h_1, h_2]$.*

Proof. If this is not true, then there exist sequences $\{R_n\}$ and $\{y_n\} \subset \partial Q_{R_n}$ such that

$$R_n \to \infty, \quad f_{h_2}(y_n) > 0.$$

For simplicity, we write Q_n for Q_{R_n} in the following. The boundary of Q_n consists two parts:

$$\begin{aligned} \partial Q_n &= \partial Q_n^1 \cup \partial Q_n^2 \\ &= \{v \in E_1 : \|v\| \leq R_n\} \cup \{v + \lambda w_0 : \lambda > 0,\ |v + rw_0| = R_n\}. \end{aligned}$$

If $u \in \partial Q_n^1 \subset E_1$, then obviously $f_h(u) = 0$. Since $f_{h_2}(y_n) > 0$, the only possibility is $y_n \in \partial Q_n^2$, i.e., $\|y_n\| = R_n \to \infty$. Choose $\theta \in (0,1)$ such that $(2-\theta)/\theta < \mathcal{H}/h_2$. Then according to Lemma 2.3, there exist a set $U \subset [0,1]$ with $\mathrm{meas}(U) > \theta$ and a subsequence $\{y_{n_k}\}$ such that $y_{n_k}(t) \to \infty$ uniformly for $t \in U$. By (V2) there exists a positive constant D such that

$$V(x) \geq \mathcal{H} - \frac{\mathcal{H} - h_2}{2} \quad \text{for } |x| \geq D.$$

Then it follows from $f_{h_2}(y_n) > 0$ and $\|y_{n_k}\| = R_n \to \infty$ that

$$\begin{aligned} 0 < \int (h_2 - V(y_{n_k}))\,dt &= \int_U (h_2 - V(y_{n_k}))\,dt + \int_{U^c} (h_2 - V(y_{n_k}))\,dt \\ &\leq -\theta \frac{\mathcal{H} - h_2}{2} + (1-\theta)h_2 < 0, \end{aligned}$$

a contradiction. ∎

For brevity, we write Q for Q_R in the rest of this paper. We use the above constructed S and Q to define a family of maps,

$$\Gamma = \{\gamma \in \mathbf{C}(\bar{Q}, X) : \gamma|_{\partial Q} = I\},$$

and a family of numbers,

$$c_h = \inf_{\gamma \in \Gamma} \sup_{u \in Q} f_h(\gamma(u)).$$

Here $\bar{Q}$ is the closure of Q, ∂Q the boundary of Q relative to $E_1 \oplus \mathrm{span}\{w_0\}$, and I is the identity operator. The numbers c_h yield a map $c : (h_1, h_2) \to \mathbb{R}$ given by $c(h) := c_h$.

Lemma 2.5. *$c(h)$ has the following two properties:*

(i) *$c(h)$ is increasing on $[h_1, h_2]$;*
(ii) *there exists a positive constant β such that $\alpha \leq c(h) \leq \beta$ for all $h \in [h_1, h_2]$.*

Proof. (i) Obvious.

(ii) According to [R, Proposition 5.9], S and Q are linked, i.e. $\gamma(Q) \cap S \neq \emptyset$ for all $\gamma \in \Gamma$; this implies

$$\sup_{u \in Q} f_{h_1}(\gamma(u)) \geq \inf_{u \in S} f_{h_1}(u) \geq \alpha, \quad \forall \gamma \in \Gamma.$$

Hence $c_h \geq c_{h_1} \geq \alpha$ for all $h \in [h_1, h_2]$. On the other hand

$$c_h \leq \sup_{u \in Q} f_h(u) \leq \sup_{u \in Q} f_{h_2}(u) =: \beta. \ ∎$$

Since $c(h)$ is increasing, the derivative $c'_h = dc(h)/dh$ exists for almost all $h \in [h_1, h_2]$. Fix a point h_0 in $[h_1, h_2]$ where c'_{h_0} exists and let $\{h_n\} \subset [h_1, h_2]$ be a strictly decreasing sequence approaching h_0.

LEMMA 2.6. *There exists a bounded P.S. sequence for f_{h_0} at level c_{h_0}, i.e. there exist a constant K and a sequence $\{u_n\}$ such that*

$$f_{h_0}(u_n) \to c_{h_0}, \; f'_{h_0}(u_n) \to 0, \; \textit{as } n \to \infty, \quad \textit{and} \quad \|u_n\| \le K + 1.$$

Proof. The proof will be divided into two steps.

STEP 1. First, we show that there exists a sequence of maps $\{\gamma_n\} \subset \Gamma$ such that

(i) $\sup_{u \in Q} f_{h_0}(\gamma_n(u)) \le c_{h_0} + (c'_{h_0} + 2)(h_n - h_0)$;
(ii) $A_n = \{\gamma_n(u) : f_{h_0}(\gamma_n(u)) \ge c_{h_0} - (h_n - h_0), \; y \in Q\}$ is uniformly bounded.

By the definition of c_{h_n}, there exists $\gamma_n \in \Gamma$ such that

$$\sup_{u \in Q} f_{h_n}(\gamma_n(u)) \le c_{h_n} + (h_n - h_0).$$

Since f_h is increasing with respect to h, we have

$$\begin{aligned} f_{h_0}(\gamma_n(u)) \le f_{h_n}(\gamma_n(u)) &\le c_{h_n} + (h_n - h_0) \\ &= c_{h_0} + (c_{h_n} - c_{h_0}) + (h_n - h_0) \\ &= c_{h_0} + (c'_{h_0} + o(1) + 1)(h_n - h_0) \\ &\le c_{h_0} + (c'_{h_0} + 2)(h_n - h_0), \end{aligned}$$

where we have used $c_{h_n} - c_{h_0} = (c'_{h_0} + o(1))(h_n - h_0)$. This proves (i).

If $z = \gamma_n(u) \in A_n$, by definition we have $f_{h_0}(\gamma_n(u)) \ge c_{h_0} - (h_n - h_0)$. This implies

$$\frac{f_{h_n}(z) - f_{h_0}(z)}{h_n - h_0} \le \frac{c_{h_n} + (h_n - h_0) - c_{h_0} + (h_n - h_0)}{h_n - h_0} \le c'_{h_0} + 3.$$

On the other hand

$$\frac{f_{h_n}(z) - f_{h_0}(z)}{h_n - h_0} = \frac{\frac{h_n - h_0}{2} \int |\dot{z}(t)|^2 \, dt}{h_n - h_0} = \frac{1}{2} \int |\dot{z}(t)|^2 \, dt.$$

Thus

$$\int \frac{1}{2} |\dot{z}(t)|^2 \, dt \le c'_{h_0} + 3. \tag{2.6}$$

If A_n is not uniformly bounded, then there exists a sequence $\{z_n\}$ such that

$$z_n \in A_n, \quad \|z_n\| \to \infty \quad \text{as } n \to \infty.$$

Let $z_n = v_n + w_n$ where $v_n \in E_1$ and $w_n \in E_2$. From $z_n \in A_n$ and (2.6) we have $\|\dot{w}_n\|_2^2 \le 2(c'_{h_0} + 3)$, so the only possibility is $|v_n| \to \infty$. Then from

$$|z_n(t)| \ge |v_n| - \|w_n\|_\infty \ge |v_n| - 2c(c'_{h_0} + 3),$$

it follows that $|z_n(t)| \to \infty$ uniformly. Using (V2) we find that

$$\int (h_0 - V(z_n)) \, dt \le \int (h_2 - V(z_n)) \, dt < 0$$

for n large enough. This contradicts the fact that $f_{h_0}(z_n) \geq \alpha/2$ for large n. So there exists a constant K such that

$$\|z\| \leq K \quad \text{for any } z \in \bigcup_{n=1}^{\infty} A_n.$$

STEP 2. Define $\mathcal{N}_\epsilon = \{z : \|z\| \leq K+1, |f_{h_0}(z) - c_{h_0}| \leq \epsilon\}$. Then $A_n \subset \mathcal{N}_\epsilon$ for large n and $\mathcal{N}_\epsilon$ is non-empty. We will show that for any $\epsilon > 0$, $\inf_{z\in\mathcal{N}_\epsilon} \|f'_{h_0}(z)\| = 0$. If this is not true, then there exists ϵ_0 such that

$$\inf_{z\in\mathcal{N}_{\epsilon_0}} \|f'_{h_0}(z)\| \geq \epsilon_0.$$

Without loss of generality, we can assume $\epsilon_0 < \frac{1}{2}(c_{h_0} - \alpha)$, where α is the constant obtained in Lemma 2.2. A classical deformation argument shows that there exist $\epsilon \in (0, \epsilon_0)$ and a homeomorphism $\eta : X \to X$ such that

- $\eta(z) = z$ if $|f_{h_0}(z) - c_{h_0}| > \epsilon_0$;
- $f_{h_0}(\eta(z)) \leq f_{h_0}(z)$ for all $z \in X$;
- $f_{h_0}(\eta(z)) \leq c_{h_0} - \epsilon$ for all $z \in \mathcal{N} := \{z : \|z\| \leq K, |f_{h_0}(z) - c_{h_0}| \leq \epsilon\}$.

We choose a sufficiently large n such that $(c'_{h_0}+2)(h_n-h_0) \leq \epsilon$ and let γ_n be the corresponding maps. Then $\eta\circ\gamma_n \in \Gamma$. If $\gamma_n(u) \in \{z : f_{h_0}(z) < c_{h_0}-\epsilon\}$, then

$$f_{h_0}(\eta \circ \gamma_n(u)) < f_{h_0}(\gamma_n(u)) < c_{h_0} - \epsilon. \tag{2.7}$$

If $\gamma_n(u) \in \{z : f_{h_0}(z) > c_{h_0} - \epsilon\}$, by Step 1 we know $\|\gamma_n(u)\| \leq K$, and $|f_{h_0}(\gamma_n(u)) - c_{h_0}| \leq \epsilon$; then

$$f_{h_0}(\eta \circ \gamma_n(u)) < f_{h_0}(\gamma_n(u)) - \epsilon. \tag{2.8}$$

From (2.7) and (2.8) we know

$$\sup_{u\in Q} f_{h_0}(\eta \circ \gamma_n(u)) \leq c_{h_0} - \epsilon.$$

This contradicts the definition of c_{h_0}. Thus for any $\epsilon > 0$, $\inf_{z\in\mathcal{N}_\epsilon} \|f'_{h_0}(z)\| = 0$. By the definition of $\inf_{z\in\mathcal{N}_\epsilon} \|f'_{h_0}(z)\|$ there exists a bounded P.S. sequence $\{u_n\}$. ∎

Now we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3. Since the P.S. sequence $\{u_n\}$ obtained in Lemma 2.6 is bounded, there exists a subsequence, still denoted by u_n, which is weakly convergent to a point $u \in E$. By the embedding theorems, it is also uniformly convergent to u. Then we have

$$V(u_n) \to V(u), \quad (V'(u_n), u_n) \to (V'(u), u) \quad \text{uniformly as } n \to \infty. \tag{2.9}$$

Since $\{u_n\}$ is a P.S. sequence at level c_{h_0}, we have $f_{h_0}(u_n) \geq \alpha/2$ for large n. From (V1) and $\min V(x) = 0$ we have

$$\begin{aligned} 0 < \alpha/2 \leq f_{h_0}(u_n) &= \frac{1}{2}\int |\dot{u}_n|^2\, dt \int (h_0 - V(u_n))\, dt \\ &\leq \frac{h_0}{2}\int |\dot{u}_n|^2. \end{aligned}$$

That is,

$$\|\dot{u}_n\|_2^2 \geq \alpha/h_0 > 0 \tag{2.10}$$

for large n. From (2.4) we have

$$\langle f'_{h_0}(u_n), u_n\rangle = \int |\dot{u}_n|^2\, dt \int \left(h_0 - V(u_n) - \frac{1}{2}(V'(u_n), u_n)\right) dt.$$

Hence

$$\int (h_0 - V(u_n))\, dt = \frac{1}{2}\int (V'(u_n), u_n)\, dt + \frac{\langle f'_{h_0}(u_n), u_n\rangle}{\|\dot{u}_n\|_2^2}. \tag{2.11}$$

Since $f'_{h_0}(u_n) \to 0$ and $\{u_n\}$ is bounded, we deduce that $\langle f'_{h_0}(u_n), u_n\rangle \to 0$ as $n \to \infty$. Letting $n \to \infty$ on both sides of (2.11), it follows from (2.9) and (2.10) that

$$\int (h_0 - V(u))\, dt = \frac{1}{2}\int (V'(u), u)\, dt. \tag{2.12}$$

Since $f_{h_0}(u_n) \geq \alpha/2$ for large n, we know $\int (h_0 - V(u_n))\, dt \geq 0$ for large n. We claim that there exists a positive constant δ such that

$$\int (h_0 - V(u_n))\, dt \geq \delta \tag{2.13}$$

for all large n. If this is not true, there exists a subsequence of $\{u_n\}$ (not relabeled) such that $\int (h_0 - V(u_n))\, dt \to 0$ as $n \to \infty$. We know that $\{u_n\}$ is bounded, so

$$f_{h_0}(u_n) = \frac{1}{2}\int |\dot{u}_n|^2\, dt \int (h_0 - V(u_n))\, dt \to 0.$$

This contradicts the fact that $f_{h_0}(u_n) \geq \alpha/2 > 0$. Then from the combination of (2.9), (2.12) and (2.13), we have

$$\frac{1}{2}\int (V'(u), u)\, dt = \int (h_0 - V(u))\, dt \geq \delta > 0. \tag{2.14}$$

As $f'_{h_0}(u_n) \to 0$ we have $\langle f'_{h_0}(u_n), z\rangle \to 0$ for any $z \in E$. Taking $z = u$ and using (2.14) we get

$$\lim_{n\to\infty} \int (\dot{u}_n, \dot{u})\, dt = \lim_{n\to\infty} \int |\dot{u}_n|^2\, dt. \tag{2.15}$$

Since u_n weakly converges to u, we have

$$\begin{aligned}\int(\dot{u}_n,\dot{u})\,dt+\int(u_n,u)\,dt &= \langle u_n,u\rangle\\ &\to \langle u,u\rangle=\int(\dot{u},\dot{u})\,dt+\int(u,u)\,dt.\end{aligned}$$

Combining (2.15), (2.16) and the Sobolev embedding theorem, we have

$$\begin{aligned}\|u_n-u\|^2 &= \int|\dot{u}_n-\dot{u}|^2\,dt+\int|u_n-u|^2\,dt\\ &= \Big(\int|\dot{u}_n|^2\,dt-2\int(\dot{u}_n,\dot{u})\,dt+\int|\dot{u}|^2\,dt\Big)+\int|u_n-u|^2\,dt\\ &\to (\|\dot{u}\|_2^2-2\|\dot{u}\|_2^2+\|\dot{u}\|_2^2)+0=0,\end{aligned}$$

that is, $u_n\to u$ strongly in E. Then from the continuity of f'_{h_0} we have $f'_{h_0}(u)=0$. Applying Lemma 2.1, we get a non-constant periodic solution $q(t):=u(t/T)$ with energy h_0. Since h_0 is arbitrary in a full-measure subset of $[h_1,h_2]$ and the interval $[h_1,h_2]$ is chosen arbitrarily, the theorem follows.

Proof of Theorem 1.4. We show that for any given $h\in[h_1,h_2]\subset(\mu_2/\mu_1,\mathcal{H})$, f_h has a critical point. Applying Theorem 1.3, we get sequences $\{h_n\}\subset(h_1,h_2)$ and $\{u_n\}$ such that

$$h_n\to h,\quad \alpha\le f_{h_n}(u_n)\le\beta,\quad f'_{h_n}(u_n)=0.$$

First, we need to show $\{\|u_n\|\}$ is bounded. From $f'_{h_n}(u_n)=0$ we obtain

$$2\int(h_n-V(u_n))\,dt=\int(V'(u_n),u_n)\,dt.$$

Combining this with (A1) we have

$$\int(h_n-V(u_n))\,dt\ge h_n-\frac{2h_n+\mu_2}{\mu_1+2}=\frac{\mu_1h_n-\mu_2}{\mu_1+2}\ge\frac{\mu_1h_1-\mu_2}{\mu_1+2}>0.$$

Therefore,

$$0<\int|\dot{u}_n|^2\,dt=\frac{f_{h_n}(u_n)}{\int(h_n-V(u_n))\,dt}\le\frac{\beta(\mu_1+2)}{\mu_1h_1-\mu_2}=:C_1.$$

Let $u_n=v_n+w_n$ where $v_n\in E_1$ and $w_n\in E_2$. Since $\|\dot{w}_n\|_2^2=\|\dot{u}_n\|_2^2\le C_1$, if $\|u_n\|$ is unbounded, the only possibility is $|v_n|\to\infty$. In this case $u_n(t)\to\infty$ uniformly. Using (V2) we find that

$$\int(h_n-V(u_n))\,dt\le\int(h_2-V(u_n))\,dt<0$$

for large n. This contradicts the fact that $f_{h_n}(u_n)\ge\alpha$. So $\{\|u_n\|\}$ is bounded. Applying a similar argument to those used in the proof of Theorem 1.3, we deduce that there exists a subsequence of $\{u_n\}$ converging strongly to some point u_0 in E. Then by the continuity of $f_h(u)$ and $f'_h(u)$ with respect to h and u, we know that the limit point u_0 of $\{u_n\}$ satisfies

$$f_h(u_0)\ge\alpha,\quad f'_h(u_0)=0.$$

Let $q_0(t) := u_0(t/T_0)$, where $T_0^2 = \frac{1}{2}\int_0^1 |\dot{u}_0|^2\,dt/\int_0^1 (h - V(u_0))\,dt$. By Lemma 2.1 we know q_0 is a non-constant periodic solution of energy h. Since the interval $[h_1, h_2]$ is chosen arbitrarily, the theorem follows.

Acknowledgements. The authors sincerely thanks the editors and the referee for their many valuable comments which helped improving the paper. This research was partly supported by NNSF of China (Grant no. 10971043), by the Heilongjiang Province Foundation for Distinguished Young Scholars (Grant no. JC200810), and by the Excellent Team Program at Harbin Institute of Technology.

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Received 17.4.2011
and in final form 9.11.2011 (2436)