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Blow-up results for some reaction-diffusion equations with time delay

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Abstract. We discuss the effect of time delay on blow-up of solutions to initial-boundary value problems for nonlinear reaction-diffusion equations. Firstly, two examples are given, which indicate that the delay can both induce and prevent the blow-up of solutions. Then we show that adding a new term with delay may not change the blow-up character of solutions.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^N, N \geq 1$, with smooth boundary $\partial\Omega$. The problem with time delay

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + f(t, u(x,t), u(x, t-\tau)), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi(x,t), & x \in \Omega,\ t \in [-\tau, 0], \end{cases}$$

can be described by the following abstract semilinear functional differential equation in the Banach space $X = C(\overline{\Omega})$ ([Wu]):

$$(1.2)\qquad \begin{cases} \mathscr{U}'(t) = A\mathscr{U}(t) + F(t, \mathscr{U}, \mathscr{U}_t), & t \geq 0, \\ \mathscr{U}(t) = \varphi(t), & t \in [-\tau, 0], \end{cases}$$

where A is defined by

$$D(A) = C^2(\Omega) \cap C_0^1(\overline{\Omega}), \qquad A\mathscr{U} = \Delta\mathscr{U},$$

and F is an X-valued nonlinear mapping defined on $\mathcal{C} = \mathcal{C}([-\tau, 0]; X)$ and $\mathscr{U}_t$ defined by $\mathscr{U}_t(\theta) = \mathscr{U}(t+\theta)$ for $\theta \in [-\tau, 0]$, τ is a positive constant. It is shown in Theorem 2.2 of [Wu] that if $F : [0,\infty) \times X \times \mathcal{C} \to X$ is continuous and maps bounded subsets of $[0,\infty) \times X \times \mathcal{C}$ into bounded subsets of X, then for each $\varphi \in \mathcal{C}$, the problem (1.2) has a solution $\mathscr{U}$ defined in a maximal

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interval $[0, t_\varphi)$. Moreover, if $t_\varphi < \infty$ then $\limsup_{t\to t_\varphi} \|\mathscr{U}\|_X = \infty$, i.e., the solution blows up in finite time.

Delay differential equations (DDEs) are often used to describe biological or physical systems (see [H, Pao1, Pao2, Wu] and the recent work [d'O, LGHP, MPBF, RM, Wang, YZ]). For a general perspective on applied DDEs, we refer to a book [E] by T. Erneux. From the applications' point of view it is important to know if a solution to a DDE is bounded, exists on the whole positive half-line or exhibits blow-up. Some results on the existence of global solutions to DDEs can be found in [Pao1, Pao2, Peng, Wu], but blow-up phenomena for them are poorly studied. In [B, EJ], the authors discussed the blow-up for some first order ODEs with delay. In [CDV], the blow-up phenomena are analyzed for both the DDE $u'(t) = B'(t)u(t-\tau)$ and the associated associated parabolic PDE $\partial_t u = \Delta u + B'(t)u(x, t-\tau)$, where $B : [0, \tau] \to \mathbb{R}$ is a positive L^1 function which behaves like $1/|t-t^*|^\alpha$ for some $\alpha \in (0,1)$ and $t^* \in (0, \tau)$. Here B' represents its distributional derivative. For some initial functions the authors proved that blow-up takes place as $t \nearrow t^*$ and the behavior of the solution near t^* was given. The extension to some nonlinear equations was also studied there.

The main aim of this short paper is to discuss the effect of a time delay term on blow-up of solutions to the initial-boundary value problems for nonlinear reaction-diffusion equations. We first give two examples. The first indicates that the delay can induce the blow-up of solutions, and the second that the delay can prevent the blow-up. Then we discuss the effect of adding a new term with delay on the blow-up phenomena for some nonlinear reaction-diffusion equations.

The paper is organized as follows. In the next section, we give two examples which show the delay may change completely the dynamics of solutions. On the contrary, Section 3 shows that adding a new term with delay may not change the blow-up character of solutions.

2. Two examples: inducing and preventing blow-up. In this section, we give two examples. The first indicates that the delay can induce the blow up of solutions, while in the second the delay prevents the blow-up.

Throughout this paper, we always assume $\tau > 0$, $\phi \in C(\overline{\Omega} \times [-\tau, 0])$, $\phi(x,t) \geq 0$, $\phi(x,0) = \phi_0(x)$. For convenience, we set $Q_T = \Omega \times (0,T]$, $Q_{-\tau} = \Omega \times [-\tau, 0]$, and write $\overline{Q}_T, \overline{Q}_{-\tau}$ for their respective closures. Moreover, we set $S_T = \partial\Omega \times [0,T]$. The following comparison principle for the parabolic problem is often used.

LEMMA 2.1. *Suppose that $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ satisfies*

$$
(2.1) \qquad \begin{cases} u_t - \Delta u + Cu \geq 0, & (x,t) \in Q_T, \\ u \geq 0, & (x,t) \in S_T, \\ u(x,0) \geq 0, & x \in \Omega, \end{cases}
$$

where $C = C(x,t)$ is bounded from below in Q_T. Then $u(x,t) \geq 0$ on $\overline{Q}_T$.

When $q > p$, it is well known that for any initial data $\phi_0 \geq 0$, the solution u to the initial-boundary value problem

$$
(2.2) \qquad \begin{cases} u_t = \Delta u + u^p - u^q, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \phi_0(x), & x \in \Omega, \end{cases}
$$

exists globally. Now we consider the initial-boundary value problem with time delay corresponding to the problem (2.2):

$$
(2.3) \qquad \begin{cases} u_t(x,t) = \Delta u(x,t) + u^p(x,t) - u^q(x,t-\tau), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi(x,t), & x \in \Omega,\ t \in [-\tau, 0]. \end{cases}
$$

THEOREM 2.2. *Assume that $p > 1$ and $q \geq 0$. Then there exists a function $\phi \geq 0$ on $\Omega \times [-\tau, 0]$ such that the solution of problem* (2.3) *blows up in finite time. This shows that the delay term can induce blow-up.*

Proof. We choose $\phi(x,t) \leq 1$ for $(x,t) \in \Omega \times [-\tau, -\tau/2]$. Let $u(x,t)$ be the corresponding solution of (2.3) and T be the maximal existence time of u. Then $0 \leq u(x, t-\tau) \leq 1$ on $\overline{\Omega} \times [0, \tau/2]$, and hence u satisfies

$$
(2.4) \qquad \begin{cases} u_t(x,t) \geq \Delta u(x,t) + u^p(x,t) - 1, & x \in \Omega,\ t \in (0, \tau/2), \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0, \tau/2), \\ u(x,0) = \phi_0(x), & x \in \Omega. \end{cases}
$$

Let λ be the eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary condition and $s(x)$ be the corresponding eigenfunction satisfying $\int_\Omega s(x)\,dx = 1$. Clearly $\lambda > 0$ and $s(x) > 0$ in Ω. Multiplying both sides of (2.4) by $s(x)$ and integrating over Ω, we have

$$
(2.5) \qquad \begin{aligned} \frac{d}{dt} \int_\Omega u(x,t)s(x)\,dx & \\ &\geq -\lambda \int_\Omega u(x,t)s(x)\,dx + \int_\Omega u^p(x,t)s(x)\,dx - 1 \int_\Omega s(x)\,dx \\ &\geq -\lambda \int_\Omega u(x,t)s(x)\,dx + \Big(\int_\Omega u(x,t)s(x)\,dx\Big)^p - 1. \end{aligned}
$$

Let $b(t) = \int_\Omega u(x,t)s(x)\,dx$. By (2.5), we have

$$b'(t) \ge \Big(-\lambda b(t) - 1 + \frac{1}{2}b^p(t)\Big) + \frac{1}{2}b^p(t).$$

Since $p > 1$, if we take $\phi_0(x)$ large enough such that

$$-\lambda b(0) - 1 + \frac{1}{2}b^p(0) \ge 0, \qquad \frac{2}{p-1}b^{1-p}(0) < \frac{\tau}{2},$$

then we have $b'(t) \ge \frac{1}{2}b^p(t)$. It follows that

$$T \le \frac{2}{p-1}b^{1-p}(0) < \frac{\tau}{2},$$

which implies that u blows up in finite time. ■

Now we give another example which shows that the delay term can prevent blow-up. It is well known that if the initial data $\phi_0(x) \ge 0$ is large enough, then the solution u of the following initial-boundary value problem of reaction-diffusion equation without delay

$$\text{(2.6)} \qquad \begin{cases} u_t = \Delta u + u^2, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \phi_0(x), & x \in \Omega, \end{cases}$$

must blow up in finite time. Now we consider the corresponding problem with time delay:

$$\text{(2.7)} \qquad \begin{cases} u_t(x,t) = \Delta u(x,t) + u(x,t)u(x,t-\tau), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi(x,t), & x \in \Omega,\ t \in [-\tau, 0]. \end{cases}$$

THEOREM 2.3. *For any $\phi \ge 0$ on $\Omega \times [-\tau, 0]$, the solution u of* (2.7) *exists globally.*

Proof. We use the so-called step method. Set $M_0 := \max_{\overline{Q}_{-\tau}} \phi(x,t)$. Then by (2.7) we have

$$\begin{cases} u_t(x,t) \le \Delta u(x,t) + M_0 u(x,t), & x \in \Omega,\ 0 < t \le \tau, \\ u(x,t) = 0, & x \in \partial\Omega,\ 0 < t \le \tau, \\ u(x,0) = \phi_0(x), & x \in \Omega. \end{cases}$$

Using Lemma 2.1, we obtain $u(x,t) \le w_1(x,t)$ in $\overline{Q}_\tau$, where w_1 is the global solution to the problem

$$\text{(2.8)} \qquad \begin{cases} w_t(x,t) = \Delta w(x,t) + M_0 w(x,t), & x \in \Omega,\ t > 0, \\ w(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ w(x,0) = \phi_0(x), & x \in \Omega. \end{cases}$$

Similarly, set $M_1 := \max\{M_0, \max_{\overline{Q}_\tau} w_1(x,t)\}$. By (2.7) again we know that when $0 \le t \le 2\tau$, u satisfies

$$\begin{cases} u_t(x,t) \le \Delta u(x,t) + M_1 u(x,t), & x \in \Omega,\ 0 < t \le 2\tau, \\ u(x,t) = 0, & x \in \partial\Omega,\ 0 < t \le 2\tau, \\ u(x,0) = \phi_0(x), & x \in \Omega. \end{cases}$$

Using Lemma 2.1 again, we conclude $u \le M_2 := \max\{M_1, \max_{\overline{Q}_{2\tau}} w_2(x,t)\}$ for $(x,t) \in \overline{Q}_{2\tau}$, where w_2 is the global solution of (2.8) with M_0 replaced by M_1. Then step by step, we deduce that the solution u to (2.7) exists in $\overline{Q}_{2\tau}, \overline{Q}_{3\tau}, \ldots$, thus completing the proof. ∎

In fact, the solution of the problem

$$\tag{2.9} \begin{cases} u_t(x,t) = \Delta u(x,t) + u^2(x,t-\tau), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi(x,t), & x \in \Omega,\ t \in [-\tau, 0], \end{cases}$$

also exists globally for any nonnegative $\phi(x,t)$. More generally, for any continuous function g satisfying the Lipschitz condition in t, the solution

$$\tag{2.10} \begin{cases} u_t(x,t) = \Delta u(x,t) + g(t, u(x,t-\tau)), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi(x,t), & x \in \Omega,\ t \in [-\tau, 0], \end{cases}$$

exists globally no matter if the solution of the corresponding problem

$$u_t(x,t) = \Delta u(x,t) + g(t, u(x,t))$$

with the same initial and boundary conditions as those in (2.6) exists globally or blows up in finite time. Theorem 3.3 in the next section guarantees these results.

3. Effects of adding a term with delay. The aim of this section is to show that adding some new term with delay may not change the blow-up character of the problem without the delay term. More precisely, assuming that we have information about the blow-up behavior of the problem

$$\tag{3.1} \begin{cases} v_t(x,t) = \Delta v(x,t) + f(x,t,v(x,t)), & x \in \Omega,\ t > 0, \\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ pv(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

we study the blow-up phenomena for the following problems with delay:

$$(3.2)\qquad \begin{cases} u_t(x,t) = \Delta u(x,t) + f(x,t,u(x,t)) \\ \qquad\qquad\qquad + g(x,t,u(x,t-\tau)), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi(x,t), & x \in \Omega,\ t \in [-\tau, 0], \end{cases}$$

$$(3.3)\qquad \begin{cases} u_t(x,t) = \Delta u(x,t) + f(x,t,u(x,t)) \\ \qquad\qquad\qquad - g(x,t,u(x,t-\tau)), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi(x,t), & x \in \Omega,\ t \in [-\tau, 0], \end{cases}$$

$$(3.4)\qquad \begin{cases} u_t(x,t) = \Delta u(x,t) \\ \qquad\qquad + f(x,t,u(x,t))g(x,t,u(x,t-\tau)), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi(x,t), & x \in \Omega,\ t \in [-\tau, 0]. \end{cases}$$

In the following, we always assume that each of the above problems admits a unique classical, nonnegative maximal in time solution. Furthermore, f and g are nonnegative functions and $f(\cdot,\cdot,z)$ is C^1 with respect to $z \geq 0$.

Lemma 3.1. *Assume that for some $v_0 \geq 0$ on Ω, the solution of problem* (3.1) *blows up in a finite time $T_{\max} < \infty$. Then for all $\phi_0 \geq v_0$, the solution of problem* (3.2) *admits a maximal solution defined on an interval $[-\tau, T)$ with $T \leq T_{\max}$.*

Proof. Since g is nonnegative, by (3.2), we have

$$(3.5)\qquad \begin{cases} u_t(x,t) \geq \Delta u(x,t) + f(x,t,u(x,t)), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \phi_0(x), & x \in \Omega. \end{cases}$$

Subtracting (3.1) from (3.5), and setting $z(x,t) = u(x,t) - v(x,t)$, we get

$$(3.6)\qquad \begin{cases} z_t \geq \Delta z + f(x,t,u) - f(x,t,v), & x \in \Omega,\ t > 0, \\ z(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ z(x,0) = \phi_0(x), & x \in \Omega. \end{cases}$$

Since $f(\cdot,\cdot,z)$ is C^1 with respect to z, by the mean value theorem and Lemma 2.1 we conclude that $z(x,t) \geq 0$. This proves the lemma. ∎

With the same methods, we can get the following statement.

LEMMA 3.2. *Assume that for some $v_0 \geq 0$, the solution of problem* (3.1) *exists globally. Then for all $\phi_0 \leq v_0$, the solution of problem* (3.3) *exists globally too.*

The next theorem shows that the term g alone could never produce finite time blow-up.

THEOREM 3.3. *Assume that $f(x,t,v(x,t)) \equiv f(v)$ is independent of t and $\int_{v_0}^{\infty} ds/f(s) = \infty$. Then the solution of* (3.1) *exists globally. Meanwhile the problem* (3.2) *admits a global solution when $\phi_0 \leq v_0$.*

Proof. Osgood's condition $\int_{v_0}^{\infty} ds/f(s) = \infty$ guarantees that the solution of the problem

$$V' = f(V), \qquad V(0) = v_0,$$

exists globally. By Lemma 2.1, the solution of (3.1) exists globally too. On the other hand, by Proposition 4 in [EJ], the solution of

$$V' = f(V) + M, \qquad V(0) = v_0,$$

(denoted by $\widetilde{V}(t)$) also exists globally for any positive constant M. For the same reason, the solution u of the problem

$$\begin{cases} u_t - \Delta u = f(u) + M, & x \in \Omega,\ t \geq 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{3.7}$$

exists globally and $u(x,t) \leq \widetilde{V}(t)$.

For $t \in [0,\tau]$, the problem (3.2) becomes

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + f(u(x,t)) \\ \qquad\qquad + g(t,\phi(x,t-\tau)), & x \in \Omega,\ t \in (0,\tau], \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0,\tau], \\ u(x,0) = \phi_0(x), & x \in \Omega. \end{cases} \tag{3.8}$$

Set $M = \max\{g(x,t,\phi(x,t)) : (x,t) \in \overline{Q}_{-\tau}\}$. Then we have

$$u_t - \Delta u \leq f(u) + M, \qquad x \in \Omega,\ t \in (0,\tau].$$

By Lemma 2.1 and the global existence for (3.7), we can deduce that the solution u of (3.8) exists in $\overline{Q}_{\tau}$. Now with the same method as in the proof of Theorem 2.3, we find that the solution of (3.2) exists in $\overline{Q}_{2\tau}, \overline{Q}_{3\tau}, \ldots$. This completes the proof. ∎

Finally, also for problem (3.4), by Lemma 2.1, we can get blow-up or global existence simultaneously with problem (3.1):

THEOREM 3.4. *Assume that for some* $v_0 \geq 0$*, the solution of problem* (3.1) *blows up in a finite time* $T_{\max} < \infty$ *and that* $\min\{g(x,t,\phi(x,t)) : (x,t) \in \overline{Q}_{-\tau}\} = m > 0$. *Then for all* $\phi_0 \geq v_0/m$*, the solution of problem* (3.4) *also blows up in a finite time* $T < T_{\max}$. *Similarly, assume that for some* $v_0 \geq 0$*, the solution of problem* (3.1) *exists globally and* $\max\{g(x,t,\phi(x,t)) : (x,t) \in \overline{Q}_{-\tau}\} = M > 0$. *Then for all* $\phi_0 \leq v_0/M$*, the solution of problem* (3.4) *exists globally too.*

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