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Multiple solutions for some Dirichlet problems with nonlocal terms

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Abstract. We deal with some Dirichlet problems involving a nonlocal term. The existence of two nonzero, nonnegative solutions is achieved by applying a recent result by Ricceri.

1. Introduction. The interest towards nonlocal boundary value problems of the type

$$(P) \qquad \begin{cases} -\Delta u = h(u, \phi(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ϕ is a *nonlocal term*, has increased in the last decades, motivated by several applications to problems arising from physical and biological phenomena. Indeed problem (P) can be considered as the stationary model of an evolution (parabolic) problem describing fully turbulent behaviour of a real flow ([CLMP]), interaction of self-gravitating particles ([BDEMN], [BLN]), the occurrence of shear banding in metals ([BT]), phenomena of Ohmic heating ([L]), or population dynamics subjected to nonlocal interactions ([AB]). The study of the steady states of the evolution problem is in general rather complicated by the presence of the nonlocal term. For instance, the upper and lower solutions method fails due to the absence of general maximum principles. The usual way to deal with such problems employs fixed point theorems ([ES], [FP], [FPS]) which provide the existence of one solution for problem (P). As far as we know, very few papers treat problems of type (P) with variational methods. In these cases special forms of nonlinearities need to be considered. We recall the paper by Fila ([F]) where the properties of

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Because of a surprising coincidence of names within the same department, we have to point out that the first author was born on August 4, 1968.

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the solution of an evolution problem with a nonlinearity of the type

$$h(u,\phi(u)) = g(\phi(u))f(u), \qquad \phi(u) = \int_\Omega F(u)\,dx$$

(and F is a primitive of f) are investigated, and the work by Gomes and Sanchez ([GS]) where existence and multiplicity results for (P) are given when

$$h(u,\phi(u)) = \frac{f(u)}{\phi(u)^p}, \qquad \phi(u) = \int_\Omega F(u)\,dx,$$

with $p > 0$. In particular, in [GS], the authors study a problem depending on a positive parameter λ and, when f grows like $p(u)e^u$ (with p a polynomial taking values in $[0,+\infty[$), they prove the existence of one solution for every λ via minimization of the corresponding energy functional while two solutions are obtained for $p < 1$ and small values of the parameter by an application of the Mountain Pass Theorem. The cited paper extends the celebrated work of Bebernes and Lacey [BL] dealing with existence and nonexistence results for (P), when Ω is a ball or a star-shaped domain and the nonlinearity is of exponential type. In connection with exponential nonlinearities we also mention the paper [BDEMN] where the special case $F = f$ is treated.

In the present paper we will consider nonlocal problems with Dirichlet boundary conditions

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda g(\int_\Omega F(u)\,dx)f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, λ a positive parameter, $f : [0,+\infty[\, \to \mathbb{R}$ is a continuous function, F is a primitive of f and $g(t) = e^t$ or $g(t) = t^{-p}$ with p a real number, $p \leq 1$. We point out that in the last case, we are, at least formally, in the same setting as in [GS].

The structure of the problems we deal with allows us to employ variational methods: this means that solutions of the above problems will be found as critical points of an associated energy functional. Applying a recent abstract multiplicity result by Ricceri for nonlocal problems [R], we prove the existence of two nonnegative solutions for (P_λ) when λ belongs to a suitable interval.

Before stating our results, we introduce the following notation. For $N \geq 1$ denote by $\mathcal{A}$ the class of continuous functions $f : [0,+\infty[\, \to \mathbb{R}$ with $f(0) = 0$ and such that when $N \geq 2$,

$$(1.1) \qquad \sup_{s\in[0,+\infty[} \frac{|f(s)|}{1+s^{q-1}} < +\infty$$

with $q > 1$ for $N = 2$, $1 < q < 2^\star = 2N/(N-2)$ for $N > 2$.

Let F be a primitive of f. We will assume that F satisfies the conditions

(F_1) $\sup\limits_{[0,+\infty[} F > F(0)$;

(F_2) $\limsup\limits_{s\to 0^+} \dfrac{F(s)-F(0)}{s^2} \leq 0$.

Our first result, which deals with the case when $g(t) = e^t$ in (P_λ), reads as follows:

THEOREM 1.1. *Let $f \in \mathcal{A}$. Assume that F satisfies (F_1)–(F_2), and that*

(i) *F is bounded.*

Then there exist an open interval $A \subseteq]0,+\infty[$ and a number $\rho > 0$ such that, for each $\lambda \in A$, the problem

$$(Q_\lambda) \qquad \begin{cases} -\Delta u = \lambda e^{\int_\Omega F(u)\,dx} f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least two nonzero, nonnegative solutions whose norms in $H_0^1(\Omega)$ are less than ρ.

In the next result we choose $g(t) = t^{-p}$ with p a real number less than 1.

THEOREM 1.2. *Let $f \in \mathcal{A}$ and $p < 1$. Assume that F satisfies (F_1)–(F_2), and*

(j) *$F(s) > 0$ for every $s \in [0,+\infty[$;*

(jj) *$\limsup_{s\to+\infty} F(s)/s^r < +\infty$ for some $r \in]0, 2/(1-p)[$.*

Then there exist an open interval $A \subseteq]0,+\infty[$ and a number $\rho > 0$ such that, for each $\lambda \in A$, the problem

$$(R_\lambda) \qquad \begin{cases} -\Delta u = \lambda \dfrac{f(u)}{\left(\int_\Omega F(u)\,dx\right)^p} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least two nonzero, nonnegative solutions whose norms in $H_0^1(\Omega)$ are less than ρ.

Our last theorem treats the case of $g(t) = t^{-1}$. Denote by $|\Omega|$ the measure of Ω.

THEOREM 1.3. *Let $f \in \mathcal{A}$. Assume that F satisfies (F_1)–(F_2) and*

(l) *$F(s) > 1/|\Omega|$ for every $s \in [0,+\infty[$.*

If $N = 1$ we also require that

(ll) *$\limsup_{s\to+\infty} F(s)/s^r < +\infty$ for some $r \in]0,+\infty[$.*

Then there exist an open interval $A \subseteq\,]0,+\infty[$ and a number $\rho > 0$ such that, for each $\lambda \in A$, the problem

$$(S_\lambda) \qquad \begin{cases} -\Delta u = \lambda \dfrac{f(u)}{\int_\Omega F(u)\,dx} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least two nonzero, nonnegative solutions whose norms in $H_0^1(\Omega)$ are less than ρ.

Example 1.4. Put

$$2^\star = \begin{cases} \frac{2N}{N-2} & \text{if } N > 2, \\ +\infty & \text{if } N \le 2. \end{cases}$$

1. Let $2 < q < 2^\star$ and $f(s) = qs^{q-1}\cos(s^q)$ for $s \ge 0$. The primitive $F(s) = \sin(s^q)$ satisfies the assumptions of Theorem 1.1.
2. Let $0 < r < \min\{2/(1-p), 2^\star\}$, $q > \max\{2, r\}$ and

   $$f(s) = s^{q-1}\frac{q + rs^{q-r}}{(1+s^{q-r})^2}.$$

   The primitive

   $$F(s) = \frac{s^q + s^{q-r} + 1}{1 + s^{q-r}}$$

   satisfies the hypotheses of Theorem 1.2.
3. Let $2 < q < 2^\star$ and $f(s) = qs^{q-1}$. The function $F(s) = s^q + c$ is a primitive of f satisfying the assumptions of Theorem 1.3 for $c > 1/|\Omega|$.

2. Preliminary results. The proofs of our results are based on the following multiplicity theorem for nonlocal problems by Ricceri, which is a consequence of an abstract three critical points theorem where a minimax inequality plays a crucial role. The following statement follows easily from [R, Theorems 1.6, 1.7 and Proposition 1.4].

Theorem 2.1. *Let X be a separable and reflexive real Banach space, $\Psi : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous functional of class C^1 whose derivative admits a continuous inverse on $X^\star$ such that $\Psi(0) = 0$, and $J : X \to \mathbb{R}$ a functional of class C^1 with compact derivative. Assume that there exists $\mu > 0$ such that*

$$\inf_{x\in X}[\Psi(x) - \mu(e^{J(x)-J(0)} - 1)] < 0 \le \inf_{x\in X}[\Psi(x) - \mu(J(x) - J(0))] \tag{2.1}$$

and that, for every $\lambda \in \Lambda \equiv\,]\mu e^{-\sup_X J}, \mu e^{-\inf_X J}[$,

$$\lim_{\|x\|\to\infty}(\Psi(x) - \lambda e^{J(x)}) = +\infty. \tag{2.2}$$

Then there exist an open interval $A \subseteq \Lambda$ and a number $\rho > 0$ such that, for each $\lambda \in A$, the equation

$$\Psi'(x) = \lambda e^{J(x)} J'(x)$$

has at least three solutions whose norms are less than ρ.

We will also make use of the following two corollaries which are direct consequences of Theorem 2.1 with J replaced with $\ln J$ and $\ln(\ln J)$ respectively.

COROLLARY 2.2. *Let X and Ψ be as in Theorem 2.1 and $J : X \to \mathbb{R}$ a functional of class C^1 with compact derivative satisfying $J(x) > 0$ for every $x \in X$. Assume that there exists $\mu > 0$ such that*

$$\inf_{x\in X}\left[\Psi(x) - \mu\left(\frac{J(x)}{J(0)} - 1\right)\right] < 0 \leq \inf_{x\in X}\left[\Psi(x) - \mu\ln\left(\frac{J(x)}{J(0)}\right)\right] \tag{2.3}$$

and that, for every $\lambda \in \Lambda \equiv \,]\mu/\sup_X J, \mu/\inf_X J[$,

$$\lim_{\|x\|\to\infty} (\Psi(x) - \lambda J(x)) = +\infty. \tag{2.4}$$

Then there exist an open interval $A \subseteq \Lambda$ and a number $\rho > 0$ such that, for each $\lambda \in A$, the equation

$$\Psi'(x) = \lambda J'(x)$$

has at least three solutions whose norms are less than ρ.

COROLLARY 2.3. *Let X and Ψ be as in Theorem 2.1 and $J : X \to \mathbb{R}$ a functional of class C^1 with compact derivative satisfying $J(x) > 1$ for every $x \in X$. Assume that there exists $\mu > 0$ such that*

$$\inf_{x\in X}\left[\Psi(x) - \mu\left(\frac{\ln J(x)}{\ln J(0)} - 1\right)\right] < 0 \leq \inf_{x\in X}\left[\Psi(x) - \mu\ln\left(\frac{\ln J(x)}{\ln J(0)}\right)\right] \tag{2.5}$$

and that, for every $\lambda \in \Lambda \equiv \,]\mu/\sup_X \ln J, \mu/\inf_X \ln J[$,

$$\lim_{\|x\|\to\infty} (\Psi(x) - \lambda \ln J(x)) = +\infty. \tag{2.6}$$

Then there exist an open interval $A \subseteq \Lambda$ and a number $\rho > 0$ such that, for each $\lambda \in A$, the equation

$$\Psi'(x) = \lambda\frac{J'(x)}{J(x)}$$

has at least three solutions whose norms are less than ρ.

3. Proofs. In our arguments we will need the following abstract lemma.

LEMMA 3.1. *Let $(X, \|\cdot\|_X)$ be a normed space, $f : X \to \mathbb{R}$ a sequentially weakly upper semicontinuous function with $f(\theta) = 0$, $g : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous function such that $g(\theta) = 0$ and $g(x) > 0$*

for every $x \neq \theta$. Assume also that X is compactly embedded into another space $(Y, \|\cdot\|_Y)$ and

$$\limsup_{x\in X,\, \|x\|_Y\to 0} \frac{f(x)}{g(x)} \leq 0. \tag{3.1}$$

Then the function

$$h(x) = \begin{cases} \dfrac{\max\{f(x), 0\}}{g(x)} & \text{if } x \neq \theta, \\ 0 & \text{if } x = \theta, \end{cases}$$

is sequentially weakly upper semicontinuous in X.

Proof. Let $x_0 \in X$ and $\{x_n\}$ a sequence in $X \setminus \{\theta\}$ weakly converging to x_0. We claim that $\limsup_n h(x_n) \leq h(x_0)$. We distinguish two cases.

If $x_0 \neq \theta$, we can fix a positive number ε with $\varepsilon < g(x_0)$. By the assumptions,

$$\limsup_n f(x_n) \leq f(x_0) \quad \text{and} \quad \liminf_n g(x_n) \geq g(x_0)$$

and so, for n large enough, $f(x_n) < f(x_0) + \varepsilon$ and $g(x_n) > g(x_0) - \varepsilon$. Then

$$\limsup_n h(x_n) = \limsup_n \frac{\max\{f(x_n), 0\}}{g(x_n)} \leq \frac{\max\{f(x_0), 0\} + \varepsilon}{g(x_0) - \varepsilon}$$

and letting ε tend to zero proves the claim.

If $x_0 = \theta$, from the compact embedding of X into Y, we have $\|x_n\|_Y \to 0$ and, for a fixed positive number ε, thanks to assumption (3.1), for n large enough, $f(x_n)/g(x_n) < \varepsilon$. Then

$$\limsup_n h(x_n) = \limsup_n \frac{\max\{f(x_n), 0\}}{g(x_n)} \leq \varepsilon$$

and letting ε tend to zero yields the claim. ■

3.1. Proofs of Theorems 1.1–1.3. We denote by X the space $H_0^1(\Omega)$ endowed with the usual norm $\|u\| = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$. Let $\Psi : X \to \mathbb{R}$ be defined by

$$\Psi(u) = \|u\|^2/2.$$

Then Ψ satisfies all the assumptions of Theorem 2.1, i.e. it is sequentially weakly lower semicontinuous (being continuous and convex), it is of class C^1 and its derivative has a continuous inverse on $X^\star$.

Assume that $f \in \mathcal{A}$ and F satisfies the assumptions (F_1)–(F_2). We extend f and its primitive F to the real axis by putting $f(s) = 0$ and $F(s) = F(0)$ for every $s \leq 0$. Let $I : X \to \mathbb{R}$ be defined by

$$I(u) = \int_\Omega F(u)\,dx.$$

As $f \in \mathcal{A}$, I is of class C^1 with compact derivative.

We also have

$$\sup_X I > I(0). \tag{3.2}$$

Indeed, from (F_1), there exists $s_0 > 0$ such that $F(s_0) > F(0)$. Let $\delta > 0$ be small enough. The function $u_0 : \Omega \to \mathbb{R}$ defined by

$$u_0(x) = \begin{cases} s_0 & \text{if } d(x, \partial\Omega) > \delta, \\ s_0 d(x, \partial\Omega)/\delta & \text{if } d(x, \partial\Omega) \le \delta, \end{cases}$$

belongs to X and

$$\begin{aligned} I(u_0) &= \int_{\{x\in\Omega\,:\,d(x,\partial\Omega)>\delta\}} F(s_0)\,dx + \int_{\{x\in\Omega\,:\,d(x,\partial\Omega)\le\delta\}} F(u_0)\,dx \\ &\ge F(s_0)|\{x \in \Omega : d(x, \partial\Omega) > \delta\}| - c_F|\{x \in \Omega : d(x, \partial\Omega) \le \delta\}| \end{aligned}$$

where $c_F = \max F_{[0,s_0]}$ if $N = 1$, and $c_F = a(s_0 + s_0^q)$ if $N \ge 2$ for some positive constant a.

If we let $\delta \to 0$, the right hand side above tends to

$$F(s_0)|\Omega| > F(0)|\Omega|,$$

and so for δ small enough,

$$I(u_0) > I(0).$$

By (F_2), for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for $|s| \le \delta_\varepsilon$,

$$F(s) - F(0) < \varepsilon s^2.$$

As $f \in \mathcal{A}$, for $N \ge 2$ there exists $q < 2^\star$ for $N > 2$, $q < +\infty$ for $N = 2$ (without loss of generality we can assume $q > 2$) such that (1.1) holds. Hence, there exists $c > 0$ such that for $|s| > \delta_\varepsilon$,

$$F(s) - F(0) \le c|s|^q.$$

For $N = 1$, the latter inequality still holds for some $q \in \mathbb{R}$, as a consequence of assumption (i) of Theorem 1.1, (jj) of Theorem 1.2 or (ll) of Theorem 1.3 (in the last two cases choose $q = r$). In conclusion, there exists $q \in]2, +\infty[$ if $N \le 2$ or $q \in]2, 2^\star[$ if $N > 2$ such that

$$F(s) - F(0) \le \varepsilon s^2 + c|s|^q \quad \text{for every } s \in \mathbb{R}. \tag{3.3}$$

Recall that the space X is compactly embedded into $(L^q(\Omega), \|\cdot\|_q)$ and there exists a constant c_q such that

$$\|u\|_q \le c_q\|u\| \quad \text{for every } u \in X. \tag{3.4}$$

We claim that

$$\limsup_{\|u\|_q\to 0} \frac{I(u) - I(0)}{\|u\|^2} \le 0. \tag{3.5}$$

Assume by contradiction that

$$l \equiv \limsup_{\|u\|_q \to 0} \frac{I(u)-I(0)}{\|u\|^2} > 0.$$

Choose $0 < \varepsilon < l(c_q^2|\Omega|^{(q-2)/q})^{-1}$ in (3.3). Then

$$I(u) - I(0) \leq \varepsilon\|u\|_2^2 + c\|u\|_q^q \leq \varepsilon|\Omega|^{(q-2)/q}\|u\|_q^2 + c\|u\|_q^q$$

for every $u \in X$, and, from (3.4),

$$\frac{I(u)-I(0)}{\|u\|^2} \leq \varepsilon c_q^2|\Omega|^{(q-2)/q} + cc_q^2\|u\|_q^{q-2}.$$

Letting $\|u\|_q \to 0$, we deduce

$$l \leq \varepsilon c_q^2|\Omega|^{(q-2)/q},$$

contrary to the choice of ε.

Proof of Theorem 1.1. In order to apply Theorem 2.1 with $J = I$, let us prove the existence of some positive μ such that inequality (2.1) holds.

For this purpose we define $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \begin{cases} \dfrac{\max\{I(u)-I(0),0\}}{\|u\|^2} & \text{if } u \neq 0, \\ 0 & \text{if } u = 0, \end{cases}$$

and we prove that it has a positive maximum on X.

Notice first that, from (3.2), one has

$$\sup_X \Phi > 0. \tag{3.6}$$

Moreover, since I is bounded,

$$\lim_{\|u\|\to\infty} \Phi(u) = 0. \tag{3.7}$$

From (3.5) and Lemma 3.1 applied to the functions $f(u) = I(u)-I(0)$ and $g(u) = \|u\|^2$, the functional Φ is sequentially weakly upper semicontinuous on X, therefore together with (3.6), (3.7) and the reflexivity of X, we obtain the existence of a function $\bar{u} \in X \setminus \{0\}$ such that

$$\Phi(\bar{u}) = \sup_X \Phi(u) > 0. \tag{3.8}$$

Let us check that (2.1) holds with $\mu = 1/(2\Phi(\bar{u})) = \Psi(\bar{u})/(I(\bar{u}) - I(0))$. Indeed, by the definition of μ and (3.8), one has at once

$$\Psi(u) - \mu(I(u) - I(0)) \geq 0 \quad \text{for all } u \in X,$$

and also

$$\Psi(\bar{u}) - \mu(e^{I(\bar{u})-I(0)} - 1) = \Psi(\bar{u})\left[1 - \frac{e^{I(\bar{u})-I(0)} - 1}{I(\bar{u}) - I(0)}\right] < 0$$

(notice that $I(\bar{u}) > I(0)$), which is our claim.

Moreover, the boundedness of I together with the coercivity of Ψ implies at once (2.2) for every $\lambda \in \mathbb{R}$. Thus Theorem 2.1 yields an open interval A and a positive number ρ such that for each $\lambda \in A$ the functional $u \mapsto \Psi(u) - \lambda e^{I(u)}$ has at least three critical points whose norms are less than ρ. As critical points are solutions of problem (Q_λ) and the zero function is a solution of (Q_λ) for any λ, we thus get two nontrivial solutions for (Q_λ). Moreover, standard arguments show that any critical point of the above functional is nonnegative and our theorem is proved. ∎

Proof of Theorem 1.2. The proof is similar to the one of Theorem 1.1 and we sketch only the differences. Notice that here we do not require the boundedness of F but a suitable growth assumption which ensures the coercivity of the energy functional.

We are going to apply Corollary 2.2 with

$$J = I^{-p+1}.$$

Clearly, J is of class C^1 with compact derivative,

$$J(u) > 0 \quad \forall u \in X,$$

and from (3.2),

$$\sup J(u) > J(0).$$

Concerning inequality (2.3), we proceed as in Theorem 1.1.

Let $\Phi : X \to \mathbb{R}$ be defined by

$$\Phi(u) = \begin{cases} \dfrac{\max\{\ln(J(u)/J(0)), 0\}}{\|u\|^2} & \text{if } u \neq 0, \\ 0 & \text{if } u = 0, \end{cases}$$

Then Φ is well defined, and satisfies $\sup_X \Phi > 0$.

As $f \in \mathcal{A}$, for $N \geq 2$ there exists $q > 1$ such that

$$|F(s)| \leq c(1 + |s|^q) \qquad \text{for every } s \in \mathbb{R}.$$

Then, for $\|u\|$ large enough, $J(u) \leq c_1 \|u\|^{q(-p+1)}$, and hence

$$0 \leq \Phi(u) \leq \frac{c_2 \ln \|u\| + c_3}{\|u\|^2}$$

for some positive constants c_2 and c_3. When $N = 1$, due to the growth assumption (jj), a similar estimate holds. Then

$$\lim_{\|u\| \to \infty} \Phi(u) = 0. \tag{3.9}$$

From (3.5), one has

$$\limsup_{\|u\|_q\to 0}\frac{\ln(J(u)/J(0))}{\|u\|^2}=(-p+1)\limsup_{\|u\|_q\to 0}\frac{\ln(I(u)/I(0))}{I(u)-I(0)}\,\frac{I(u)-I(0)}{\|u\|^2}$$
$$=(-p+1)\limsup_{\|u\|_q\to 0}\frac{\ln\big(\frac{I(u)-I(0)}{I(0)}+1\big)}{I(u)-I(0)}\,\frac{I(u)-I(0)}{\|u\|^2}\le 0.$$

An application of Lemma 3.1 with $f(u)=\ln(J(u)/J(0))$ and $g(u)=\|u\|^2$ shows that Φ is sequentially weakly upper semicontinuous on X.

Hence, there exists $\bar{u}\in X\setminus\{0\}$ such that $\Phi(\bar{u})=\max_X\Phi>0$ and condition (2.3) of Corollary 2.2 holds with $\mu=1/(2\Phi(\bar{u}))$. Condition (2.4) is again a consequence of assumption (jj) as

$$\Psi(u)-\lambda J(u)=\frac{1}{2}\|u\|^2-\lambda\Big[\int_\Omega F(u)\,dx\Big]^{-p+1}\ge\frac{1}{2}\|u\|^2-\lambda(c_4\|u\|^{r(-p+1)}+c_5)$$

for some positive constants c_4, c_5.

The conclusion is analogous to the one of Theorem 1.1 and gives the existence of an open interval A and of a positive number ρ such that for each $\lambda\in A$ the functional $u\mapsto\Psi(u)-\lambda J(u)$ has at least two nontrivial critical points with norms less than ρ. Such critical points are solutions of the problem

$$\begin{cases}-\Delta u=\lambda(-p+1)\dfrac{f(u)}{(\int_\Omega F(u)\,dx)^p} & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

and the conclusion follows at once by rescaling the parameter. ∎

Proof of Theorem 1.3. This is analogous to the proofs of Theorems 1.1 and 1.2. Notice that in this case we do not need any growth assumption on F when $N\ge 2$. Indeed the latter was required in Theorem 1.2 in order to guarantee the coercivity of the energy. In the present case, it is enough, for our purposes, to assume $f\in\mathcal{A}$ and assumption (ll) when $N=1$. ∎

REMARK 3.2. Assumption (1.1) implies that, for $N\ge 2$,

$$|F(s)|\le c(1+s^q)\quad\text{for } s\ge 0 \tag{3.10}$$

where $q>1$ if $N=2$, and $1<q<2^\star$ if $N>2$. In Theorem 1.2 we require (even when $N=1$) assumption (jj), which together with (j) gives

$$|F(s)|\le c(1+s^r)\quad\text{for } s\ge 0 \tag{3.11}$$

with $r<2/(1-p)$. When $N>2$ the two exponents appearing in (3.10) and (3.11), depending on different parameters, are not comparable while it is clear that (3.10) is a consequence of (3.11) for $N=2$. In Theorem 1.3 growth assumptions are not required when $N\ge 2$.

REMARK 3.3. We notice that the case $p > 1$ in Theorem 1.2 cannot be treated in the present setting. Indeed with the choice of $J = -I^{-p+1}$, it is not possible to apply Corollary 2.2.

REMARK 3.4. We point out that in the previous theorems, the energy functional associated to problem (P_λ) is sequentially weakly lower semicontinuous and coercive. Thus, the existence of one solution (the global minimum of the energy) is trivial.

REMARK 3.5. The same results, with obvious modifications in the hypotheses, can be obtained for quasilinear elliptic problems, i.e. replacing the Laplacian with the p-Laplacian.

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