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Decompositions and asymptotic limit for bicontractions

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Abstract. The asymptotic limit of a bicontraction T (that is, a pair of commuting contractions) on a Hilbert space $\mathcal{H}$ is used to describe a Nagy–Foiaş–Langer type decomposition of T. This decomposition is refined in the case when the asymptotic limit of T is an orthogonal projection. The case of a bicontraction T consisting of hyponormal (even quasinormal) contractions is also considered, where we have $S_{T^*} = S^2_{T^*}$.

1. Introduction. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$ with the identity element I. The range and the kernel of $T \in \mathcal{B}(\mathcal{H})$ are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. Recall that T is *hyponormal* if $TT^* \le T^*T$, and T is *quasinormal* if $T^*T^2 = TT^*T$. Obviously, every quasinormal operator is hyponormal.

A (closed) subspace $\mathcal{M} \subset \mathcal{H}$ is *invariant* for T if $T\mathcal{M} \subset \mathcal{M}$, and when $\mathcal{M}$ is invariant for T and T^* one says that $\mathcal{M}$ *reduces* (or $\mathcal{M}$ *is reducing for*) T. Also, $P_{\mathcal{M}}$ stands for the orthogonal projection in $\mathcal{B}(\mathcal{H})$ corresponding to $\mathcal{M}$.

A *bicontraction* on $\mathcal{H}$ is a pair $T = (T_0, T_1)$ of commuting contractions on $\mathcal{H}$, that is, a pair of operators satisfying $\|T_i\| \le 1$ $(i = 0, 1)$ and $T_0T_1 = T_1T_0$. If T_0 and T_1 are isometries then T is called a *bi-isometry* on $\mathcal{H}$.

Let $T = (T_0, T_1)$ be a bicontraction. It is known (see [D], [SNF], [K], [S1]) that the asymptotic limit of T_i is defined by

$$S_{T_i}h = \lim_{n\to\infty} T_i^{*n}T_i^n h \quad (h \in \mathcal{H})$$

and clearly, $0 \le S_{T_i} \le T_i^*T_i$, $T_i^*S_{T_i}T_i = S_{T_i}$, $i = 0, 1$ (the last condition means that T_i is an S_{T_i}-isometry [S1], [S2]). It follows that

$$T_0^{*m}S_{T_1}T_0^m \le T_0^{*m}T_1^{*n}T_1^nT_0^m = T_1^{*n}T_0^{*m}T_0^mT_1^n$$

for any $m, n \in \mathbb{N}$, and letting $m \to \infty$ one obtains

$$0 \le \operatorname*{s-lim}_{m\to\infty} T_0^{*m}S_{T_1}T_0^m \le T_1^{*n}S_{T_0}T_1^n \quad (n \in \mathbb{N}).$$

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Letting $n \to \infty$ we infer that

$$\operatorname*{s\text{-}lim}_{m\to\infty} T_0^{*m} S_{T_1} T_0^m \le \operatorname*{s\text{-}lim}_{n\to\infty} T_1^{*n} S_{T_0} T_1^n,$$

and by symmetry equality holds in this relation. Thus, the *asymptotic limit* of T can be defined by

$$\begin{aligned} S_T h &= \lim_{m\to\infty} T_0^{*m} S_{T_1} T_0^m h = \lim_{n\to\infty} T_1^{*n} S_{T_0} T_1^n h \\ &= \lim_{m\to\infty}\lim_{n\to\infty} T_0^{*m} T_1^{*n} T_1^n T_0^m h = \lim_{n\to\infty}\lim_{m\to\infty} T_0^{*m} T_1^{*n} T_1^n T_0^m h \end{aligned}$$

for any $h \in \mathcal{H}$. Note that $0 \le S_T \le S_{T_i}$ and $T_i^* S_T T_i = S_T$ for $i = 0, 1$. In fact,

$$S_T = \max\{A \in \mathcal{B}(\mathcal{H}) : 0 \le A \le I,\ T_i^* A T_i = A,\ i = 0, 1\}.$$

We say that T is *strongly stable* if $\mathcal{N}(S_T) = \{0\}$, that is, $T_0^m T_1^n h \to 0$ $(m, n \to \infty)$ for $h \in \mathcal{H}$.

Our goal in this paper is to find some orthogonal decompositions of $\mathcal{H}$ induced by bicontractions T for which S_T is an orthogonal projection. So, in Section 2 we get some conditions on T under which $S_T = S_T^2$. We describe in the language of asymptotic limits the Nagy–Foiaş–Langer type decomposition of T relative to a bicontraction T. The case when T consists of hyponormal or quasinormal contractions is considered here, where we show that $S_{T^*} = S_{T^*}^2$.

In Section 3 we use the operators S_T and S_{T_i} $(i = 0, 1)$ to refine the Nagy–Foiaş–Langer type decomposition for the bicontractions T with $S_T = S_T^2$ (and $S_{T^*} = S_{T^*}^2$). This decomposition is related to the general Wold type decomposition of a bi-isometry, obtained by D. Popovici [P] and recently, in a different way, by Bercovici–Douglas–Foiaş [BDF].

2. Invariant subspaces induced by the asymptotic limit. As in the case of a single contraction (see [K]), many interesting facts for bicontractions arise in the case when S_T is an orthogonal projection, that is, $S_T = S_T^2$, or equivalently $\mathcal{N}(S_T - S_T^2) = \mathcal{H}$. The following proposition, which extends Lemmas 1 and 2 of [KVP], gives interesting information for this case of bicontractions.

PROPOSITION 2.1. *For any bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ we have:*

(i) *$\mathcal{N}(S_T - S_T^2) = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$ is the maximum subspace of $\mathcal{H}$ which is invariant for T_0 and T_1 and on which S_T commutes with T_0 and T_1.*

(ii) *$\mathcal{N}(I - S_T)$ and $\mathcal{N}(S_T)$ are the maximum invariant subspaces for T_0 and T_1 in $\mathcal{H}$ such that T_0 and T_1 are isometries on $\mathcal{N}(I - S_T)$, and T is strongly stable on $\mathcal{N}(S_T)$. In addition,*

$$\mathcal{N}(I - S_T) = \{h \in \mathcal{H} : \|T_0^m T_1^n h\| = \|h\|,\ \forall m, n \in \mathbb{N}\}. \tag{2.1}$$

Moreover, if $\mathcal{N}(I - S_{T_i})$ *is invariant for* T_{1-i} $(i = 0, 1)$ *then*

$$\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T_0}) \cap \mathcal{N}(I - S_{T_1}). \tag{2.2}$$

Proof. Observe that $\mathcal{N}(I-S_T)$ and $\mathcal{N}(S_T)$ are contained in $\mathcal{N}(S_T - S_T^2)$, and are orthogonal. So, $\mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T) \subset \mathcal{N}(S_T - S_T^2)$. Conversely, let $h \in \mathcal{N}(S_T - S_T^2)$ be such that h is orthogonal to $\mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$. Then $S_T h \in \mathcal{N}(I - S_T)$ and therefore $\langle h, S_T h\rangle = 0$, which means that $S_T h = 0$ or $h \in \mathcal{N}(S_T)$. Hence $h = 0$, since h is orthogonal to $\mathcal{N}(S_T)$. Consequently,

$$\mathcal{N}(S_T - S_T^2) = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T).$$

Now recall that $T_i^* S_T T_i = S_T$, whence $\mathcal{N}(S_T)$ is invariant for T_i $(i = 0, 1)$. As we also have (T_i is a contraction)

$$T_i^*(I - S_T)T_i \le I - S_T,$$

it follows that $\mathcal{N}(I - S_T)$ is invariant for T_i $(i = 0, 1)$.

Furthermore, for $m, n, p, q \in \mathbb{N}$ one has

$$T_0^{*(m+p)} T_1^{*(n+q)} T_1^{n+q} T_0^{m+p} \le T_0^{*m} T_1^{*n} T_1^n T_0^m,$$

and setting $p, q \to \infty$ we get $S_T \le T_0^{*m} T_1^{*n} T_1^n T_0^m$, whence

$$I - T_0^{*m} T_1^{*n} T_1^n T_0^m \le I - S_T.$$

This gives on one hand,

$$\mathcal{N}(I - S_T) \subset \{h \in \mathcal{H} : \|T_0^m T_1^n h\| = \|h\|,\ \forall m, n \in \mathbb{N}\}.$$

On the other hand, if $\|T_0^m T_1^n h\| = \|h\|$ for $m, n \in \mathbb{N}$ then letting $m, n \to \infty$ one obtains $\|S_T h\| = \|h\|$, and since $0 \le S_T \le I$ one infers $h = S_T h$, that is, $h \in \mathcal{N}(I - S_T)$. Hence the relation (2.1) holds.

Next, if $h \in \mathcal{N}(S_T - S_T^2)$ and $h = h_1 \oplus h_0$ with $h_1 \in \mathcal{N}(I - S_T)$, $h_0 \in \mathcal{N}(S_T)$ then

$$(S_T T_i - T_i S_T)h = T_i h_1 - T_i h_1 = 0, \qquad i = 0, 1,$$

therefore S_T commutes with T_0 and T_1 on $\mathcal{N}(S_T - S_T^2)$.

Let now $\mathcal{M} \subset \mathcal{H}$ be another subspace invariant for T_0 and T_1 such that $S_T T_i k = T_i S_T k$ for $k \in \mathcal{M}$, $i = 0, 1$. Then $S_T T_0^m T_1^n k = T_0^m T_1^n S_T k$ for any $m, n \in \mathbb{N}$, and this implies (T_i being an S_T-isometry)

$$S_T k = T_0^{*m} T_1^{*n} S_T T_0^m T_1^n k = T_0^{*m} T_1^{*n} T_0^m T_1^n S_T k.$$

Letting $m, n \to \infty$ we get $S_T k = S_T^2 k$, that is, $k \in \mathcal{N}(S_T - S_T^2)$. So $\mathcal{M} \subset \mathcal{N}(S_T - S_T^2)$ and we conclude that $\mathcal{N}(S_T - S_T^2)$ is the maximum invariant subspace for T_i on which S_T commutes with T_i, $i = 0, 1$, which proves (i).

It is clear (by (2.1)) that T_i is an isometry on $\mathcal{N}(I - S_T)$, $i = 0, 1$, and (by the definition of S_T) we have $T_0^m T_1^n h \to 0$ $(m, n \to \infty)$ for $h \in \mathcal{N}(S_T)$, that is, T is strongly stable on $\mathcal{N}(S_T)$. In addition, it is obvious that $\mathcal{N}(I - S_T)$ and $\mathcal{N}(S_T)$ are the maximum subspaces with the above mentioned properties. This proves (ii).

Finally, if $\mathcal{N}(I - S_{T_i})$ is invariant for T_{1-i} then $\mathcal{N}(I - S_{T_0}) \cap \mathcal{N}(I - S_{T_1})$ is invariant for T_0 and T_1, and clearly T_i is an isometry on this subspace for $i = 0, 1$. Since $\mathcal{N}(I - S_T) \subset \mathcal{N}(I - S_{T_0}) \cap \mathcal{N}(I - S_{T_1})$ it follows that the two subspaces coincide (by the maximality of $\mathcal{N}(I - S_T)$ cited in (ii)). ∎

Corollary 2.2. *For a bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ we have $S_T = S_T^2$ if and only if $S_T T_i = T_i S_T$ for $i = 0, 1$. Furthermore, if $S_T = S_{T^*}$ then $S_T = S_T^2$.*

Proof. If $S_T = S_T^2$ then $\mathcal{N}(S_T - S_T^2) = \mathcal{H}$, so S_T commutes with T_0 and T_1 on $\mathcal{H}$ (by Proposition 2.1). Conversely, if $S_T T_i = T_i S_T$ $(i = 0, 1)$ then necessarily $\mathcal{N}(S_T - S_T^2) = \mathcal{H}$ (by the maximality of $\mathcal{N}(S_T - S_T^2)$ in Proposition 2.1(i)), that is, $S_T = S_T^2$.

Assume now that $S_T = S_{T^*}$. For $m, n \in \mathbb{N}$ and $h \in \mathcal{H}$ one has

$$\begin{aligned} S_T h &= T_0^{*m} T_1^{*n} S_T T_1^n T_0^m h = T_0^{*m} T_1^{*n} S_{T^*} T_1^n T_0^m h \\ &= T_0^{*m} T_1^{*n} T_1^n T_0^m S_{T^*} T_0^{*m} T_1^{*n} T_1^n T_0^m h \to S_T^3 h \quad (m, n \to \infty), \end{aligned}$$

hence $S_T = S_T^3$. It follows that $S_T = S_T^2$. ∎

This corollary extends the corresponding assertions for contractions in Lemma 1 and Proposition 1 of [KVP].

A special case of bicontractions for which their asymptotic limits are orthogonal projections is mentioned in the following theorem.

As usual, a bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ is called *completely nonunitary* if there is no nonzero subspaces of $\mathcal{H}$ which reduce T_0 and T_1 to unitary operators. Clearly, every strongly stable bicontraction T is completely nonunitary, because in this case $\mathcal{H} = \mathcal{N}(S_T)$, therefore $\mathcal{N}(I - S_T) = \{0\}$ (by Proposition 2.1(i)).

Theorem 2.3. *Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ with T_0 and T_1 hyponormal. Then $S_{T^*} = S_{T^*}^2$ and the maximum subspace of $\mathcal{H}$ which reduces T_0 and T_1 to unitary operators is*

$$\mathcal{N}(I - S_{T^*}) = \bigcap_{m,n \geq 0} T_0^m T_1^n [\mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*})]. \tag{2.3}$$

Moreover, T^* is strongly stable if and only if T is completely nonunitary.

Proof. Since T_i is hyponormal we know (see the proof of [K, Theorem 5.3]) that $S_{T_i^*} = S_{T_i^*}^2$ and $\mathcal{R}(S_{T_i^*}) = \mathcal{N}(I - S_{T_i^*})$ reduces T_i to a unitary operator, for $i = 0, 1$. As $\mathcal{N}(S_{T^*})$ is invariant for T_0^* and T_1^*, $\overline{\mathcal{R}(S_{T^*})}$ will be invariant for T_0 and T_1. In addition, because

$$\overline{\mathcal{R}(S_{T^*})} \subset \mathcal{R}(S_{T_0^*}) \cap \mathcal{R}(S_{T_1^*}) = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*})$$

it follows that T_0 and T_1 are isometries on $\overline{\mathcal{R}(S_{T^*})}$. So, we infer from Proposition 2.1 that

$$\overline{\mathcal{R}(S_{T^*})} \subset \mathcal{N}(I - S_T).$$

Take an arbitrary $h = h_1 \oplus h_0 \in \mathcal{H}$ with $h_1 \in \overline{\mathcal{R}(S_{T^*})}$, $h_0 \in \mathcal{N}(S_{T^*})$. We have (by the above inclusion)

$$T_0 S_{T^*} h = T_0 S_{T^*} h_1 = T_0 S_{T^*} T_0^* T_0 h_1 = S_{T^*} T_0 h_1.$$

But $T_0^* S_{T^*} T_0 h_0 = S_{T^*} h_0 = 0$, that is, $S_{T^*} T_0 h_0 \in \mathcal{N}(T_0^*) \subset \mathcal{N}(S_{T^*})$, hence $S_{T^*} T_0 h_0 = 0$. Thus, we obtain $T_0 S_{T^*} h = S_{T^*} T_0 h$, and by symmetry one has $T_1 S_{T^*} h = S_{T^*} T_1 h$. This means that S_{T^*} commute with T_0 and T_1, and by Corollary 2.2 we have $S_{T^*} = S_{T^*}^2$.

Now it follows that $\mathcal{N}(I - S_{T^*})$ is the maximum subspace of $\mathcal{H}$ which reduces T_0^* and T_1^* to isometries. In fact, by the above remark, $\mathcal{N}(I - S_{T^*}) = \mathcal{R}(S_{T^*})$ is the maximum subspace which reduces T_0 and T_1 to unitary operators. Obviously, this subspace is contained in the right side of (2.3), briefly denoted by $\mathcal{N}_T$.

Let $h \in \mathcal{N}_T$ be orthogonal to $\mathcal{N}(I - S_{T^*})$. So $h \in \mathcal{N}(S_{T^*})$, that is, $T_0^{*m} T_1^{*n} h \to 0$ $(m, n \to \infty)$. Since $h \in \mathcal{N}_T$, for any $m, n \in \mathbb{N}$ there exist $h_{m,n} \in \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*})$ such that $h = T_0^m T_1^n h_{m,n}$. As $\mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*})$ is invariant for T_0 and T_1, while T_0, T_1 are isometries on this subspace, we get

$$h_{m,n} = T_0^{*m} T_1^{*n} T_1^n T_0^m h_{m,n} = T_0^{*m} T_1^{*n} h \to 0, \qquad m, n \to \infty.$$

This yields $\|h\| = \|h_{m,n}\| \to 0$ $(m, n \to \infty)$, hence $h = 0$. Thus, (2.3) holds.

Finally, it is clear that $\mathcal{N}(I - S_{T^*}) = \{0\}$ implies $\mathcal{H} = \mathcal{N}(S_{T^*})$, therefore T^* is strongly stable if (and only if, by the above remark) T is completely nonunitary. ∎

Remark 2.4. W. Mlak proved in [M] that the "unitary part" in $\mathcal{H}$ of a hyponormal contraction T_0 is $\bigcap_{n \geq 0} T_0^n \mathcal{N}(I - T_0 T_0^*)$, by using the minimal unitary dilation of T_0. This fact was recovered in [S2] without using dilation, by an argument as above involving the asymptotic limit. In the present context we cannot use $\mathcal{N}(I - T_0 T_0^*) \cap \mathcal{N}(I - T_1 T_1^*)$ in (2.3) instead of $\mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*})$, because the former subspace is not invariant for T_0 and T_1, in general.

We say that a bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ is *unitary* if T_0 and T_1 are unitary operators. We now give the "asymptotic" version of the Nagy–Foiaş–Langer decomposition for bicontractions.

Theorem 2.5. *For every bicontraction T on $\mathcal{H}$ there exists a unique decomposition of $\mathcal{H}$ of the form*

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_u^{\perp} \tag{2.4}$$

such that $\mathcal{H}_u$ reduces T to a unitary bicontraction and $\mathcal{H}_u^{\perp}$ reduces T to a completely nonunitary bicontraction. In addition,

$$\begin{aligned}(2.5)\qquad \mathcal{H}_u &= \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_T S_{T^*}) = \mathcal{N}(I - S_{T^*} S_T)\\ &= \mathcal{N}(I - S_T^{1/2} S_{T^*} S_T^{1/2}) = \mathcal{N}(I - S_{T^*}^{1/2} S_T S_{T^*}^{1/2}).\end{aligned}$$

Proof. If $h \in \mathcal{N}(I-S_T)\cap\mathcal{N}(I-S_{T^*})$ then $h = S_T h = S_{T^*}h = S_T S_{T*}h = S_{T^*} S_T h$, so $\mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) \subset \mathcal{N}(I - S_T S_{T^*}) \cap \mathcal{N}(I - S_{T^*} S_T)$. Conversely, let $h \in \mathcal{N}(I - S_T S_{T^*})$, that is, $h = S_T S_{T^*} h$. We have

$$\|h\|^2 = \langle S_{T^*} h, S_T h\rangle \le \|S_{T^*}^{1/2} h\|\,\|S_T^{1/2} h\| \le \|S_T^{1/2} h\|\,\|h\|,$$

whence $\|h\| = \|S_T^{1/2}\|$, or equivalently $(I - S_T)h = 0$ (as $0 \le S_T \le I$). Similarly, one has $\|h\| = \|S_{T^*}^{1/2} h\|$, that is, $(I - S_{T^*})h = 0$, and so

$$\mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_T S_{T^*}) = \mathcal{N}(I - S_{T^*} S_T).$$

Now, if $h = S_T S_{T^*} h$ then as above $\|h\| = \|S_T^{1/4} h\| = \|S_{T^*}^{1/4} h\|$, therefore $h = S_T^{1/2} h = S_{T^*}^{1/2} h = S_T h = S_{T^*} h = S_T^{1/2} S_{T^*} S_T^{1/2} h = S_{T^*}^{1/2} S_T S_{T^*}^{1/2} h$. This shows that $\mathcal{N}(I - S_T S_{T^*}) \subset \mathcal{N}(I - S_T^{1/2} S_{T^*} S_T^{1/2}) \cap \mathcal{N}(I - S_{T^*}^{1/2} S_T S_{T^*}^{1/2})$. Conversely, $h = S_T^{1/2} S_{T^*} S_T^{1/2} h$ gives

$$\|h\|^2 = \|S_{T^*}^{1/2} S_T^{1/2} h\|^2 \le \|S_T^{1/2} h\|^2 \le \|S_T^{1/4} h\|^2 \le \|h\|^2,$$

whence $\|h\|^2 = \|S_T^{1/2} h\|^2 = \|S_T^{1/4} h\|^2$. Hence $h = S_T h = S_T^{1/2} h$ and therefore $\|S_{T^*}^{1/2} h\| = \|S_{T^*}^{1/2} S_T^{1/2} h\| = \|h\|$ (the last equality follows from our assumption), which yields $h = S_{T^*} h$. So, $\mathcal{N}(I - S_T^{1/2} S_{T^*} S_T^{1/2})$ and (by symmetry) $\mathcal{N}(I - S_{T^*}^{1/2} S_T S_{T^*}^{1/2})$ are contained in $\mathcal{N}(I-S_T)\cap\mathcal{N}(I-S_{T^*})$. Thus, the above equalities between subspaces are completed with the last two from (2.5).

Next, by (2.1) for T and T^* we see immediately that the subspace $\mathcal{H}_u := \mathcal{N}(I-S_T)\cap\mathcal{N}(I-S_{T^*})$ reduces T_0 and T_1 to unitary operators. In addition, if $\mathcal{M} \subset \mathcal{H}$ is another such subspace, then $\mathcal{M} \subset \mathcal{H}_u$ by Proposition 2.1(ii). Hence $\mathcal{H}_u$ is the maximum subspace with the property above, and finally, the reducing decomposition (2.4) for T is unique with T is unitary on $\mathcal{H}_u$, and completely nonunitary on $\mathcal{H}_u^{\perp}$. ∎

COROLLARY 2.6. *For every bi-isometry $T = (T_0, T_1)$ on $\mathcal{H}$ we have $S_{T^*} = S_{T_0^* T_1^*}$, hence $\mathcal{H}_u = \mathcal{N}(I - S_{T_0^* T_1^*})$ and $\mathcal{H}_u^{\perp} = \mathcal{N}(S_{T_0^* T_1^*})$ in* (2.4). *Moreover, T is completely nonunitary if and only if $T_0 T_1$ is a* (*unilateral*) *shift on $\mathcal{H}$.*

Proof. Since $T_0 T_1$ is an isometry, by Theorem 2.3 the maximum subspace of $\mathcal{H}$ which reduces $T_0 T_1$ to a unitary operator is $\mathcal{N}(I - S_{T_0^* T_1^*})$. So, by Theorem 2.5 one obtains $\mathcal{N}(I - S_{T^*}) \subset \mathcal{N}(I - S_{T_0^* T_1^*})$. On the other hand, since

$$\mathcal{N}(I - S_{T_0^*T_1^*}) = \mathcal{N}(I - S_{(T_0,T_1)^*}) = \bigcap_{n\geq 0} T_0^n T_1^n \mathcal{H},$$

it follows immediately that $\mathcal{N}(I - S_{T_0^*T_1^*})$ reduces T_0 and T_1 to unitary operators, hence $\mathcal{N}(I - S_{T_0^*T_1^*}) \subset \mathcal{N}(I - S_{T^*})$ by Theorem 2.5. Thus $\mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_{T_0^*T_1^*})$, and since S_{T^*}, $S_{T_0^*T_1^*} = S_{(T_0,T_1)^*}$ are orthogonal projections, also $\mathcal{N}(S_{T^*}) = \mathcal{N}(S_{T_0^*T_1^*})$. We conclude that $S_{T^*} = S_{T_0^*T_1^*}$, and the remaining assertions of the corollary follow from Theorems 2.3 and 2.5. ∎

Another interesting particular case of Theorem 2.3 is considered below. Notice that the case of a single quasinormal contraction was considered in [KVP, Example 3].

PROPOSITION 2.7. *For every bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ with T_0 and T_1 quasinormal one has $S_{T^*} = S_{T^*}^2$. Moreover, $S_T = S_T^2$ if and only if either $T_0^*|_{\overline{R(S_T)}}$ or $T_1^*|_{\overline{R(S_T)}}$ is a coisometry.*

In addition, $S_T = S_{T^*}$ if and only if $T_i^*|_{\overline{R(S_T)}}$ is normal and $\overline{R(S_T)}$ is invariant for $T_iT_i^*$ $(i=0,1)$. In this case $\mathcal{N}(I-S_T)=\mathcal{N}(I-S_{T_0^*})\cap\mathcal{N}(I-S_{T_1^*})$.

Proof. Clearly, $S_{T^*} = S_{T^*}^2$ by Theorem 2.3. Furthermore, because T_i is quasinormal, we have (see [S1], or Lemma 2.8 below) $S_{T_i} = S_{T_i}^2$ so $\mathcal{R}(S_{T_i}) = \mathcal{N}(I - S_{T_i})$ and $\overline{R(S_T)} \subset \mathcal{N}(I - S_{T_i})$, $i = 0, 1$. So, if $S_T = S_T^2$ then $\mathcal{R}(S_T)$ reduces T_0^* and T_1^* to coisometries.

Conversely, assume that, say, $T_0^*|_{\overline{R(S_T)}}$ is a coisometry ($\overline{R(S_T)}$ being invariant for T_0^* and T_1^*). Put $T_{0*} = T_0^*|_{\overline{R(S_T)}}$. Then $T_{0*}^* = P_{\overline{R(S_T)}}T_0|_{\overline{R(S_T)}}$ is an isometry on $\overline{R(S_T)}$. Hence for $h \in \mathcal{H}$ we obtain

$$\|S_Th\| = \|P_{\overline{R(S_T)}}T_0S_Th\| \leq \|T_0S_Th\| \leq \|S_Th\|,$$

whence $T_0S_Th = P_{\overline{R(S_T)}}T_0S_Th$. We infer that $\overline{R(S_T)}$ reduces T_0, and since $\mathcal{R}(S_T) \subset \mathcal{N}(I - S_{T_1})$ we have for $m, n \in \mathbb{N}$ and $h \in \mathcal{H}$,

$$S_Th = T_0^{*m}T_0^mS_Th = T_0^{*m}T_1^{*n}T_1^nT_0^mS_Th.$$

Letting $m, n \to \infty$ we infer that $S_T = S_T^2$.

Obviously, if $S_T = S_{T^*}$ then $R(S_T)$ reduces T_i to unitary operators, $i = 0, 1$. Conversely, suppose that $T_i^*|_{\overline{R(S_T)}}$ are normal operators for $i = 0, 1$. Then for $h \in \mathcal{H}$ we have

$$T_0^*P_{\overline{R(S_T)}}T_0S_Th = P_{\overline{R(S_T)}}T_0T_0^*S_Th = T_0T_0^*S_Th,$$

since $P_{\mathcal{N}(S_T)}T_0T_0^*S_Th = 0$ by the assumption that $\overline{R(S_T)}$ is invariant for $T_0T_0^*$. It follows that $T_0^*P_{\mathcal{N}(S_T)}T_0S_Th = 0$, which gives $P_{\mathcal{N}(S_T)}T_0S_Th = 0$, that is, $T_0S_Th = P_{\overline{R(S_T)}}T_0S_Th$. Hence $\overline{R(S_T)}$ reduces T_0, and so $T_0T_0^*S_Th =$

$T_0^*T_0S_Th = S_Th$ which means that T_0 is unitary on $\overline{R(S_T)}$. By symmetry, $\overline{R(S_T)}$ also reduces T_1 to a unitary operator, and by Theorem 2.3 we get

$$\overline{R(S_T)} = \mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_T).$$

Finally, this leads to $S_T = S_{T^*}$. In this case

$$\mathcal{N}(I - S_T) \subset \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*}) \subset \mathcal{N}(I - S_{T_0}) \cap \mathcal{N}(I - S_{T_1}),$$

and since $\mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*})$ is invariant for T_0 and T_1 it follows (from the second inclusion) that T_0 and T_1 are isometries on this subspace. Thus $\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*})$, by the maximality of $\mathcal{N}(I - S_T)$ given in Proposition 2.1(ii). ∎

Let us remark that if $T = (T_0, T_1)$ consists of quasinormal commuting contractions and either $T_0S_{T_1} = S_{T_1}T_0$ or $T_1S_{T_0} = S_{T_0}T_1$ then $S_T = S_{T_0}S_{T_1} = S_{T_1}S_{T_0}$, hence $S_T = S_T^2$. We see in the example below that the condition $S_T = S_T^2$ does not ensure the commutativity of T_{1-i} with S_{T_i}, $i = 0, 1$. We first give

Lemma 2.8. *For every quasinormal contraction T_0 on $\mathcal{H}$ one has $S_{T_0} = S_{T_0^*T_0} = S_{T_0}^2$.*

Proof. Since T_0 is quasinormal we have (by induction) $(T_0^*T_0)^n = T_0^{*n}T_0^n$ for any $n \in \mathbb{N}$. Then

$$S_{T_0}h = \lim_{n\to\infty} T_0^{*2n}T_0^{2n}h = \lim_{n\to\infty} (T_0^*T_0)^{2n}h = S_{T_0^*T_0}h = S_{T_0}^2 h$$

for $h \in \mathcal{H}$. Moreover, the above operator is an orthogonal projection because $T_0^*T_0$ is positive. ∎

Example 2.9. Let S be the canonical shift on l_+^2 and $\mathcal{K} = \mathcal{R}(S) \oplus l_+^2$. Put $S_0 = S|_{\mathcal{R}(S)}$ and let $S_1 : l_+^2 \to \mathcal{R}(S)$ be given by $S_1 = SP_{\mathcal{N}(S^*)}$. Consider $T_0, T_1 \in \mathcal{B}(\mathcal{K})$ defined by the operator matrices

$$T_0 = \begin{pmatrix} S_0 & S_1 \\ 0 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$$

relative to the above decomposition of $\mathcal{K}$. We have

$$T_0^*T_0 = I_{\mathcal{R}(S)} \oplus P_{\mathcal{N}(S^*)}, \quad T_0^*T_0^2 = T_0 = T_0T_0^*T_0,$$

hence T_0, and also T_1, are quasinormal contractions on $\mathcal{K}$. In addition $T_0T_1 = T_1T_0 = 0$, so $T = (T_0, T_1)$ is a bicontraction on $\mathcal{K}$, and clearly, by the above commutativity condition for T_0 and T_1 we have $S_T = 0$.

On the other hand, (by Lemma 2.8) $S_{T_0} = S_{T_0^*T_0} = T_0^*T_0$ and

$$T_1S_{T_0} = 0 \oplus SP_{\mathcal{N}(S^*)} = 0 \oplus S_1 \neq 0 = 0 \oplus P_{\mathcal{N}(S^*)}S = S_{T_0}T_1.$$

Similarly, since $S_{T_1} = 0 \oplus I_{l^2_+}$ we get

$$T_0 S_{T_1} = \begin{pmatrix} 0 & S_1 \\ 0 & 0 \end{pmatrix} \neq 0 = S_{T_1} T_0.$$

We conclude that $S_T = S_T^2$ but $T_{1-i} S_{T_i} \neq S_{T_i} T_{1-i}$, or equivalently $T_{1-i}|T_i| \neq |T_i| T_{1-i}$ because $|T_i| = S_{T_i}$ in this case, for $i = 0, 1$. This also shows that the conditions $T_{1-i}|T_i| = |T_i|T_{1-i}$ $(i = 0, 1)$ are not necessary to ensure $S_T = S_T^2$, when T_0 and T_1 are quasinormal.

3. Decompositions in the case $S_T = S_T^2$. The asymptotic limits can be used to refine the Nagy–Foiaş–Langer decomposition for bicontractions when S_T is an orthogonal projection. This decomposition (to be given below) generalizes the Wold type decompositions for bi-isometries which appear in [P] and [BDF]. Recall that a similar result for contractions can be found in [K].

We say (briefly) that a subspace $\mathcal{M} \subset \mathcal{H}$ is *invariant* (resp. *reducing*) for a bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ if $\mathcal{M}$ is invariant (resp. reducing) for T_0 and T_1. Also, we say that T is *coisometric* on $\mathcal{H}$ if both T_i are coisometries.

The statements of Theorem 3.1 and Corollary 3.2 below extend Theorem 1 and Corollary 1 of [KVP] obtained for a single contraction.

THEOREM 3.1. *Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ with $S_T = S_T^2$. Then $\mathcal{H}$ admits the decomposition*

$$\mathcal{H} = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) \oplus \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}) \oplus \mathcal{N}(S_T) \tag{3.1}$$

where all the three summands reduce T in such a way that T is unitary on $\mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*})$, T^* is coisometric and strongly stable on $\mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*})$, and T is strongly stable on $\mathcal{N}(S_T)$.

Moreover, if $\mathcal{N}(S_T) \neq \{0\}$ and $S_{T^*} = S_{T^*}^2$ then $\mathcal{N}(S_T)$ admits the decomposition

$$\mathcal{N}(S_T) = \mathcal{N}(I - S_{T^*}) \cap \mathcal{N}(S_T) \oplus \mathcal{N}(S_{T^*}) \cap \mathcal{N}(S_T), \tag{3.2}$$

where the two summands reduce T, and T is coisometric and strongly stable on $\mathcal{N}(I - S_{T^*}) \cap \mathcal{N}(S_T)$, while T and T^* are strongly stable on $\mathcal{N}(S_T) \cap \mathcal{N}(S_{T^*})$.

Proof. Since $S_T = S_T^2$ one has $\mathcal{H} = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$ where $\mathcal{N}(I - S_T)$ reduces T to a bi-isometry and T is strongly stable on $\mathcal{N}(S_T)$.

Let $W = (W_0, W_1)$ where $W_i = T_i|_{\mathcal{N}(I - S_T)}$, $i = 0, 1$. By (2.5), the maximum subspace which reduces T to a unitary bicontraction is

$$\mathcal{H}_u = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}).$$

Now since W_i is an isometry on $\mathcal{N}(I - S_T)$ it follows that $S_{W_i^*} = S_{W_i^*}^2$ for $i = 0, 1$, and by Corollary 2.6 we obtain $S_{W^*} = S_{W^*}^2$. Therefore

$$\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{W^*}) \oplus \mathcal{N}(S_{W^*})$$

where the summands reduce W_i, and so T_i, $i = 0, 1$. We also have

$$\mathcal{N}(I - S_{W^*}) = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) = \mathcal{H}_u,$$
$$\mathcal{N}(S_{W^*}) = \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}),$$

hence $T_0^{*m} T_1^{*n} h \to 0$ $(m, n \to \infty)$ for $h \in \mathcal{N}(S_{W^*})$,that is, T^* is co-isometric and strongly stable on $\mathcal{N}(S_{W^*})$.

Next suppose $\mathcal{N}(S_T) \neq \{0\}$ and let $W' = (W'_0, W'_1)$ where $W'_i = T_i|_{\mathcal{N}(S_T)}$, $i = 0, 1$. Then relative to the decomposition

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{N}(S_{W^*}) \oplus \mathcal{N}(S_T)$$

we have $S_{T^*} = I \oplus 0 \oplus S_{W'^*}$, whence

$$\mathcal{N}(S_{T^*}) = \mathcal{N}(S_{W^*}) \oplus \mathcal{N}(S_{W'^*}) \subset \mathcal{N}(S_{W^*}) \oplus \mathcal{N}(S_T).$$

Since $\mathcal{N}(S_{W^*}) \subset \mathcal{N}(I - S_T) = \mathcal{H} \ominus \mathcal{N}(S_T)$ we infer that

$$\mathcal{N}(S_{W'^*}) = \mathcal{N}(S_{T^*}) \cap \mathcal{N}(S_T).$$

On the other hand, since $I - S_{T^*} = 0 \oplus I \oplus (I - S_{W'^*})$ we have

$$\mathcal{N}(I - S_{T^*}) = \mathcal{H}_u \oplus \mathcal{N}(I - S_{W'^*}) \subset \mathcal{H}_u \oplus \mathcal{N}(S_T),$$

whence

$$\mathcal{N}(I - S_{W'^*}) = \mathcal{N}(I - S_{T^*}) \cap \mathcal{N}(S_T).$$

Assume $S_T = S_T^2$ and $S_{T^*} = S_{T^*}^2$. Clearly, the second condition is equivalent to $S_{W'^*} = S_{W'^*}^2$, which also means

$$\mathcal{N}(S_T) = \mathcal{N}(I - S_{W'^*}) \oplus \mathcal{N}(S_{W'^*}).$$

Thus, the summands, reducing for W', also reduce T in such a way that T^* is a bi-isometry and T is strongly stable on $\mathcal{N}(I - S_{W'^*})$, and T, T^* are strongly stable bicontractions on $\mathcal{N}(S_{W'^*})$. ■

COROLLARY 3.2. *For a bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ one has $S_T = S_{T^*}$ if and only if $T_i = U_i \oplus S_i$ $(i = 0, 1)$ relative to a decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where $\mathcal{M}$ reduces T so that $U = (U_0, U_1)$ is unitary on $\mathcal{M}$, while $S = (S_0, S_1)$ and S^* are strongly stable on $\mathcal{M}^\perp$.*

Proof. Suppose $S_T = S_{T^*}$. Then for $m, n \geq 1$ we have

$$S_T = T_0^{*m} T_1^{*n} S_{T^*} T_1^n T_0^m = T_0^{*m} T_1^{*n} T_1^n T_0^m S_{T^*} T_0^{*m} T_1^{*n} T_1^n T_0^m,$$

and letting $m, n \to \infty$ we get $S_T = S_T S_{T^*} S_T = S_T^3$. It follows that $S_T^2 = S_T^4$ and so $S_T = S_T^2$. By our assumption, $\mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}) = \{0\}$ and $\mathcal{N}(I - S_{T^*}) \cap \mathcal{N}(S_T) = \{0\}$, so we infer from (3.1) and (3.2) that

$$\mathcal{H} = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) \oplus \mathcal{N}(S_T) \cap \mathcal{N}(S_{T^*}) = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T).$$

Thus T is unitary on $\mathcal{M} = \mathcal{N}(I - S_T)$, while T and T^* are strongly stable on $\mathcal{M}^\perp = \mathcal{N}(S_T)$, and $T_i = T_i|_{\mathcal{M}} \oplus T_i|_{\mathcal{M}^\perp}$, $i = 0, 1$.

Conversely, if $T_i = U_i \oplus S_i$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and $\mathcal{M}$ reduces T and U_i is unitary on $\mathcal{M}$ for $i = 0, 1$, while $S = (S_0, S_1)$ and S^* are strongly stable on $\mathcal{M}^\perp$, then $S_T = I \oplus 0 = S_{T^*}$. ■

The decomposition (3.1) can be refined by the general Wold type decomposition of a bi-isometry which was obtained in [P] and recently in [BDF]. So, the following result holds.

THEOREM 3.3. *Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ with $S_T = S_T^2$. Then $\mathcal{H}$ admits a unique decomposition of the form*

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_s \oplus \mathcal{H}_1 \oplus \mathcal{H}_0, \tag{3.3}$$

where all the summands reduce T, and where $T_0|_{\mathcal{H}_u \oplus \mathcal{H}_{us}}$ and $T_1|_{\mathcal{H}_u \oplus \mathcal{H}_{su}}$ are unitary, $T_0|_{\mathcal{H}_{su}}$ and $T_1|_{\mathcal{H}_{us}}$ are shift operators, T is a bi-shift on $\mathcal{H}_s$, T is strongly stable on $\mathcal{H}_0$, while T is a bi-isometry on $\mathcal{H}_1$ and there is no nonzero reducing subspace for T of $\mathcal{H}_1$ on which either T is a bi-shift, or T_0 is unitary or T_1 is unitary. Moreover, T_0T_1 is a shift on $\mathcal{H}_1$.

Proof. Clearly, $\mathcal{H}_u = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*})$ and $\mathcal{H}_0 = \mathcal{N}(S_T)$ by Theorem 3.1. Denote $W = (W_0, W_1)$, $W_i = T_i|_{\mathcal{N}(I - S_T)}$, $i = 0, 1$. Since W is an isometry we have (by Corollary 2.6)

$$\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{W^*}) \oplus \mathcal{N}(S_{W^*}) = \mathcal{H}_u \oplus \mathcal{N}(S_{W_0^* W_1^*}).$$

So, we infer from (3.1) that

$$\mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}) = \mathcal{N}(S_{W_0^* W_1^*}) = \bigoplus_{n \geq 0} W_0^n W_1^n \mathcal{N}(W_0^* W_1^*)$$
$$\supset \bigoplus_{n \geq 0} W_1^n \bigcap_{m \geq 0} W_0^m \mathcal{N}(W_1^*) \supset \bigoplus_{n \geq 0} W_1^n \bigcap_{m \geq 0} W_0^m \bigoplus_{j \geq 0} \mathcal{N}(W_1^* W_0^j) =: \mathcal{H}_{us}.$$

Observe that the subspace

$$\mathcal{H}_{0*} := \bigcap_{j \geq 0} \mathcal{N}(W_1^* W_0^j) \subset \mathcal{N}(W_1^*)$$

is invariant for W_0, so for T_0, and the subspace

$$\bigcap_{m \geq 0} W_0^m \mathcal{H}_{0*} = \mathcal{N}(I - S_{(T_0|_{\mathcal{H}_{0*}})^*}) \subset \mathcal{N}(W_1^*)$$

is wandering for W_1 and it reduces $T_0|_{\mathcal{H}_{0*}}$ to a unitary operator. Hence the subspace

$$\mathcal{H}_{us} = \bigoplus_{n \geq 0} W_1^n \mathcal{N}(I - S_{(T_0|_{\mathcal{H}_{0*}})^*}) = W_0 \bigoplus_{n \geq 0} W_1^n (W_0|_{\mathcal{H}_{0*}})^* \mathcal{N}(I - S_{(T_0|_{\mathcal{H}_{0*}})^*})$$

reduces W_1 to a shift, and from the second equality we get $\mathcal{H}_{us} = W_0 \mathcal{H}_{us}$, so $\mathcal{H}_{us}$ also reduces W_0. This implies that $\mathcal{H}_{us}$ reduces T_1 to a shift and T_0 to a unitary operator.

Similarly, if $\mathcal{H}_{1*} := \bigcap_{j\geq 0} \mathcal{N}(W_0^* W_1^j)$ then

$$\mathcal{H}_{su} := \bigoplus_{m\geq 0} W_0^m \mathcal{N}(I - S_{(T_1|_{\mathcal{H}_{1*}})^*}) \subset \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*})$$

reduces T_0 to a shift and T_1 to a unitary operator. Since $S_{W_i^*} = S_{W_i^*}^2$, $i = 0, 1$, and we have

$$\mathcal{H}_{us} \subset \mathcal{N}(I - S_{W_0^*}) \cap \mathcal{N}(S_{W_1^*}),$$
$$\mathcal{H}_{su} \subset \mathcal{N}(I - S_{W_1^*}) \cap \mathcal{N}(S_{W_0^*}),$$

it follows that the subspaces $\mathcal{H}_u$, $\mathcal{H}_{us}$ and $\mathcal{H}_{su}$ are pairwise orthogonal in $\mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*})$.

Now, the subspace $\mathcal{H}_{0*} \cap \mathcal{H}_{1*} \subset \mathcal{N}(W_0^*) \cap \mathcal{N}(W_1^*)$ is wandering for the bi-isometry $W = (W_0, W_1)$, and the subspace

$$\mathcal{H}_s := \bigoplus_{m,n\geq 0} W_0^m W_1^n (\mathcal{H}_0^* \cap \mathcal{H}_1^*)$$

is invariant for W, and also for T. In fact,

$$W_0\mathcal{H}_s = \bigoplus_{m\geq 1, n\geq 0} W_0^m W_1^n (\mathcal{H}_0^* \cap \mathcal{H}_1^*) = \mathcal{H}_s \ominus \bigoplus_{n\geq 0} W_1^n (\mathcal{H}_0^* \cap \mathcal{H}_1^*)$$

whence (as $W_0^* W_1^n \mathcal{H}_{1*} = \{0\}$, $n \geq 0$)

$$W_0^*\mathcal{H}_s = \mathcal{H}_s + W_0^*\Big(\bigoplus_{n\geq 0} W_1^n (\mathcal{H}_0^* \cap \mathcal{H}_1^*)\Big) = \mathcal{H}_s.$$

Similarly, $W_1^*\mathcal{H}_s = \mathcal{H}_s$, and therefore $\mathcal{H}_s$ reduces W, and so T, to a bi-shift. Since $\mathcal{H}_s \subset \mathcal{N}(S_{W_0^*}) \cap \mathcal{N}(S_{W_1^*})$, we have

$$\mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}) \ominus \mathcal{H}_s \supset \mathcal{N}(I - S_{W_0^*}) \vee \mathcal{N}(I - S_{W_1^*}) \supset \mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su},$$

whence the subspace

$$\mathcal{H}_1 := \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}) \ominus (\mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su})$$

is also reducing for T. In addition it is easy to see (as in [P]) that the subspaces $\mathcal{H}_{us}$, $\mathcal{H}_{su}$ and $\mathcal{H}_s$ are maximal with the properties quoted above. This implies that $\mathcal{H}_1$ contains no nonzero reducing subspace for T on which either T is a bi-shift, or T_0 is unitary, or T_1 is unitary.

Finally, since $\mathcal{H}_1 \subset \mathcal{N}(S_{T^*})$, $T^*|_{\mathcal{H}_1}$ is strongly stable, that is, $T_0T_1|_{\mathcal{H}_1}$ is a shift, by Corollary 2.6. ∎

Remark 3.4. The structure of the subspaces $\mathcal{H}_{us}$, $\mathcal{H}_{su}$ and $\mathcal{H}_s$ for a bi-isometry V was obtained by D. Popovici [P]. Here we describe these subspaces as well as the other from decomposition (3.3) using the context of asymptotic limits of a bicontraction $T = (T_0, T_1)$.

Corollary 3.5. *Let $T=(T_0,T_1)$ be a bicontraction on $\mathcal{H}$ with $S_T=S_T^2$, $S_{T^*}=S_{T^*}^2$ and $\mathcal{N}(S_T)\neq\{0\}$. Then $\mathcal{H}$ admits a unique decomposition of the form*

$$\mathcal{H}=\mathcal{H}_u\oplus\mathcal{H}_{us}\oplus\mathcal{H}_{su}\oplus\mathcal{H}_s\oplus\mathcal{H}_{uc}\oplus\mathcal{H}_{cu}\oplus\mathcal{H}_c\oplus\mathcal{H}_{11}\oplus\mathcal{H}_{00}, \tag{3.4}$$

where all summands reduce T and where $T_0|_{\mathcal{H}_u\oplus\mathcal{H}_{us}\oplus\mathcal{H}_{uc}}$ and $T_1|_{\mathcal{H}_u\oplus\mathcal{H}_{su}\oplus\mathcal{H}_{cu}}$ are unitary, $T_0|_{\mathcal{H}_{su}}$ and $T_1|_{\mathcal{H}_{us}}$ are shifts, $T_0|_{\mathcal{H}_{cu}}$ and $T_1|_{\mathcal{H}_{uc}}$ are coshifts, T and T^* are strongly stable on $\mathcal{H}_{00}$, and there is no nonzero reducing subspace for T of $\mathcal{H}_{11}$ on which either T_0 or T_1 is unitary, or T or T^* is a bi-shift.

In addition, $T_i|_{\mathcal{H}_{11}}=Z_i\oplus Z_i'$ on $\mathcal{H}_{11}=\mathcal{H}_1\oplus\mathcal{H}_1'$ where Z_i are isometries and Z_0Z_1 is a shift on $\mathcal{H}_1$, while Z_i' are coisometries, and $Z_0'Z_1'$ is a co-shift on $\mathcal{H}_1'$, for $i=0,1$.

Proof. By Theorem 3.3 for the bi-isometry W and the bicontraction W' (W,W' as in the proof of Theorem 3.1) we have

$$\mathcal{N}(I-S_T)=\mathcal{H}_u\oplus\mathcal{H}_{us}\oplus\mathcal{H}_{su}\oplus\mathcal{H}_s\oplus\mathcal{H}_1,$$

and respectively

$$\mathcal{N}(S_T)=\mathcal{H}_0=\mathcal{H}_{uc}\oplus\mathcal{H}_{cu}\oplus\mathcal{H}_c\oplus\mathcal{H}_1'\oplus\mathcal{H}_{00}.$$

Here $\mathcal{H}_{00}=\mathcal{N}(S_T)\cap\mathcal{N}(S_{T^*})$, $\mathcal{H}_1'$ contains no nonzero reducing subspaces for T on which either T^* is a bi-shift, or the coisometries T_0 or T_1 are unitary, and in addition, T is strongly stable, that is, T_0T_1 is a co-shift on $\mathcal{H}_1'$. Clearly, the other subspaces of $\mathcal{N}(S_T)$ have the meaning from (3.4) for the bi-isometry T^*. So, putting $\mathcal{H}_{11}=\mathcal{H}_1\oplus\mathcal{H}_1'$ we get the decomposition (3.4) of $\mathcal{H}=\mathcal{N}(I-S_T)\oplus\mathcal{N}(S_T)$, in view of (3.1) and (3.2). ∎

Since $\mathcal{H}_{us}\oplus\mathcal{H}_{su}\oplus\mathcal{H}_s\subset\mathcal{N}(I-S_T)\cap\mathcal{N}(S_{T^*})$ we have necessarily

$$\begin{aligned}\mathcal{H}_{us}&\subset\mathcal{N}(I-S_{T_0^*})\cap\mathcal{N}(I-S_T)\cap\mathcal{N}(S_{T_1^*})\\&=\mathcal{N}(I-S_{T_0^*})\cap\mathcal{N}(I-S_T)\cap\mathcal{N}(S_T),\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{H}_{su}&\subset\mathcal{N}(I-S_{T_1^*})\cap\mathcal{N}(I-S_T)\cap\mathcal{N}(S_{T_0^*})\\&=\mathcal{N}(I-S_{T_1^*})\cap\mathcal{N}(I-S_T)\cap\mathcal{N}(S_T),\end{aligned} \tag{3.6}$$

and

$$\mathcal{H}_s\subset\mathcal{N}(I-S_T)\cap\mathcal{N}(S_{T_0^*})\cap\mathcal{N}(S_{T_1^*}), \tag{3.7}$$

but the inclusions may be strict, as in Remark 3.9 below.

By Theorem 3.1 of [KO] we also get the following

Corollary 3.6. *Let $T=(T_0,T_1)$ be a bicontraction on $\mathcal{H}$. Then there exist a unique minimal Hilbert space $\mathcal{K}\supset\mathcal{H}$ and a bicontraction $\tilde{T}=(\tilde{T}_0,\tilde{T}_1)$ on $\mathcal{K}$ extending T (i.e. such that $\tilde{T}|_{\mathcal{H}}=T$) and admitting a unique decomposition of the form given in Theorem 3.3.*

We find now when these inclusions become equalities. Clearly, we can reduce this problem to the case of a bi-isometry (by (3.3)).

PROPOSITION 3.7. *Let* $T = (T_0, T_1)$ *be a bi-isometry on* $\mathcal{H}$. *Then*

(i) $\mathcal{H}_{us} = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$ *and* $\mathcal{H}_{su} = \mathcal{N}(I - S_{T_1^*}) \cap \mathcal{N}(S_{T_0^*})$ *if and only if* $\mathcal{H}_s \oplus \mathcal{H}_1 = \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$, *where* $\mathcal{H}_1$ *is the subspace appearing in decomposition* (3.3). *In this case,* $\mathcal{H}_s \oplus \mathcal{H}_1$ *is the maximum subspace which reduces* T_i $(i = 0, 1)$ *to a shift.*

(ii) $\mathcal{H}_s = \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$ *and* $\mathcal{H}_1 = \{0\}$ *if and only if* T_0 *and* T_1 *doubly commute.*

Proof. Suppose we have equalities in (3.5) and (3.6), where $\mathcal{N}(I - S_T) = \mathcal{H}$. Since T_0 is a shift on $\mathcal{H}_{su}$, that is, $T_0^{*n} h \to 0$ $(n \to \infty)$ for $h \in \mathcal{H}_{su}$, we have $\mathcal{H}_{su} \subset \mathcal{N}(S_{T_0^*})$. Thus, since $S_{T_0^*} = S^2_{T_0^*}$ and $S_{T^*} = S^2_{T^*}$ (T is a bi-isometry), we get the decompositions

$$\begin{aligned}\mathcal{H} &= \mathcal{N}(I - S_{T_0^*}) \oplus \mathcal{N}(S_{T_0^*}) \\ &= \mathcal{N}(I - S_{T^*}) \oplus [\mathcal{N}(I - S_{T_0^*}) \ominus \mathcal{N}(I - S_{T^*})] \oplus \mathcal{H}_{su} \oplus [\mathcal{N}(S_{T_0^*}) \ominus \mathcal{H}_{su}] \\ &= \mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus [\mathcal{N}(S_{T_0^*}) \ominus \mathcal{H}_{su}].\end{aligned}$$

Then from (3.3) we infer (as $\mathcal{H}_0 = \mathcal{N}(S_T) = \{0\}$ in this case) that $\mathcal{H}_s \oplus \mathcal{H}_1 = \mathcal{N}(S_{T_0^*}) \ominus \mathcal{H}_{su}$, or $\mathcal{N}(S_{T_0^*}) = \mathcal{H}_{su} \oplus \mathcal{H}_s \oplus \mathcal{H}_1$. By symmetry we also have $\mathcal{N}(S_{T_1^*}) = \mathcal{H}_{us} \oplus \mathcal{H}_s \oplus \mathcal{H}_1$, and so

$$\mathcal{H}_s \oplus \mathcal{H}_1 \subset \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*}) =: \mathcal{H}_{ss}.$$

Now if $h \in \mathcal{H}_{ss}$ and we write $h = h_1 \oplus h_0 = h_2 \oplus h_0'$ with $h_1 \in \mathcal{H}_{us}$, $h_2 \in \mathcal{H}_{su}$ and $h_0, h_0' \in \mathcal{H}_s \oplus \mathcal{H}_1$, then $h_1 \oplus (-h_2) \oplus (h_0 - h_0') = 0$, hence $h_1 = h_2 = 0$ and $h_0 = h_0'$. This implies $h = h_0 \in \mathcal{H}_s \oplus \mathcal{H}_1$, and we conclude that $\mathcal{H}_s \oplus \mathcal{H}_1 = \mathcal{H}_{ss}$. Clearly, in this case the subspace $\mathcal{H}_{ss}$ reduces T_i $(i = 0, 1)$ to a shift, and it contains any other subspace of $\mathcal{H}$ with this property.

Conversely, assume that $\mathcal{H}_s \oplus \mathcal{H}_1 = \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$. Then as above we get the decomposition

$$\mathcal{H} = \mathcal{H}_u \oplus [\mathcal{N}(I - S_{T_0^*}) \ominus \mathcal{N}(I - S_{T^*})] \oplus [\mathcal{N}(S_{T_0^*}) \ominus \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})] \oplus \mathcal{H}_s \oplus \mathcal{H}_1,$$

and from (3.3) we infer that

$$\mathcal{H}_{us} \oplus \mathcal{H}_{su} = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*}) \oplus \mathcal{N}(S_{T_0^*}) \cap [\mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})]^{\perp}.$$

Since $\mathcal{H}_{us} \subset \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(S_{T^*})$ and $\mathcal{H}_{su} \subset \mathcal{N}(I - S_{T_1^*}) \cap \mathcal{N}(S_{T_0^*})$ (by (3.3)), the preceding equality leads to $\mathcal{H}_{us} = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$ and also (because $S_{T_1^*} = S^2_{T_1^*}$)

$$\mathcal{H}_{su} = \mathcal{N}(S_{T_0^*}) \cap [\mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})]^{\perp} \supset \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*}),$$

hence $\mathcal{H}_{su} = \mathcal{N}(I - S_{T_1^*}) \cap \mathcal{N}(S_{T_0^*})$. This completes the proof of (i).

For (ii) it is clear that if $\mathcal{H}_s = \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$ and $\mathcal{H}_1 = \{0\}$ then T_0 and T_1 doubly commute on $\mathcal{H}_s$, and finally, they doubly commute on $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_s$ (T being a bi-isometry).

Conversely, if $T_0T_1^* = T_1^*T_0$ then $\mathcal{N}(I - S_{T_i^*})$ and $\mathcal{N}(S_{T_i^*})$ reduce T_{1-i}, and so $\mathcal{N}(I-S_{T_i^*}) \cap \mathcal{N}(S_{T_{1-i}^*})$ reduces T_i (resp. T_{1-i}) to a unitary (resp. shift) operator, for $i = 0, 1$. Thus, it is needed that $\mathcal{H}_{us} = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$ and $\mathcal{H}_{su} = \mathcal{N}(I - S_{T_1^*}) \cap \mathcal{N}(S_{T_0^*})$, which gives $\mathcal{H}_s \oplus \mathcal{H}_1 = \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$. But, in this case we have $\mathcal{H}_s = \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$ because T_0 and T_1 doubly commute on $\mathcal{H}_s \oplus \mathcal{H}_1$, hence $\mathcal{H}_1 = \{0\}$. This ends the proof. ∎

Remark 3.8. In fact, this proposition shows that a bi-isometry $T = (T_0, T_1)$ on $\mathcal{H}$ induces an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{ss}, \tag{3.8}$$

where the subspaces have the above meaning, if and only if $\mathcal{H}_{us} = \mathcal{N}(I-S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$ and $\mathcal{H}_{su} = \mathcal{N}(I - S_{T_1^*}) \cap \mathcal{N}(S_{T_0^*})$, while in this case $\mathcal{H}_{ss} = \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$. Hence $\mathcal{H}_{ss}$ reduces T_0 and T_1 to shift operators and it is the maximum subspace with this property.

Recall that the decomposition (3.8) is known as the *Słociński decomposition* (see [Sl]). Moreover in (3.8) we have $\mathcal{H}_{ss} = \mathcal{H}_s$ if and only if T_0 and T_1 doubly commute.

Remark 3.9. In Example 1 of [GS] a bi-isometry T was given for which $\mathcal{H}_{us} = \mathcal{N}(I - S_{T_0^*}) \subsetneq \mathcal{N}(S_{T_1^*})$ and $\mathcal{H}_{su} \oplus \mathcal{H}_1 = \mathcal{N}(S_{T_0^*})$ with $\mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*}) = \{0\} = \mathcal{H}_u$. In view of the above strict inclusion, $\mathcal{N}(I - S_{T_1^*}) \subset \mathcal{H}_{su} \oplus \mathcal{H}_1$ and also $\mathcal{H}_1 \neq \{0\}$ because otherwise we get $\mathcal{H}_{su} = \mathcal{N}(I - S_{T_1^*})$, a contradiction. So $\mathcal{H}_{su} \subsetneq \mathcal{N}(I - S_{T_1^*}) = \mathcal{N}(I - S_{T_1^*}) \cap \mathcal{N}(S_{T_0^*})$, even if $\mathcal{H}_{us} = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})$, hence T does not have a Słociński decomposition (3.8).

Remark 3.10. Consider the bicontraction $T = (T_0, T_1)$ on $\mathcal{K}$ from Example 2.9. Since $S_T = 0$, T is strongly stable on $\mathcal{K}$. On the other hand, as T_0, T_1 are quasinormal, by Theorem 2.3 we have $S_{T^*} = S_{T^*}^2$ and $\mathcal{R}(S_{T^*}) \subset \mathcal{R}(S_T) = \{0\}$, that is, $S_{T^*} = \{0\}$. Hence T^* is strongly stable on $\mathcal{K}$ and we have $\mathcal{K} = \mathcal{N}(S_T) = \mathcal{N}(S_{T^*}) = \mathcal{K}_{00}$ in the corresponding decomposition (3.4).

4. Remarks on invariant subspaces for bicontractions. To every bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ one can associate a bi-isometry $V = (V_0, V_1)$ on $\overline{\mathcal{R}(S_T)}$ such that

$$V_i S_T^{1/2} h = S_T^{1/2} T_i h \quad (h \in \mathcal{H},\ i = 0, 1). \tag{4.1}$$

Clearly, V_i is an isometry (T_i being an S_T-isometry), and $V_0V_1 = V_1V_0$ because $T_0T_1 = T_1T_0$. Since $\mathcal{N}(S_T)$ is invariant for $S_T^{1/2}T_i$, $\overline{\mathcal{R}(S_T)}$ is invariant for $T_i^* S_T^{1/2}$, and the above definition of V_i implies

$$S_T^{1/2} V_i^* k = T_i^* S_T^{1/2} k \quad (k \in \overline{\mathcal{R}(S_T)},\ i = 0, 1). \tag{4.2}$$

This relation gives $V_i S_T V_i^* \leq S_T$ on $\overline{\mathcal{R}(S_T)}$, hence V_i^* is an $\widehat{S_T}$-contraction ($i = 0, 1$), where $\widehat{S_T} = S_T|_{\overline{\mathcal{R}(S_T)}}$. Other properties of V are summarized in

Proposition 4.1. *Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ and $V = (V_0, V_1)$ be the bi-isometry on $\overline{\mathcal{R}(S_T)}$ associated to T as in* (4.1). *Then*

$$\lim_{m,n\to\infty} V_0^{*m} V_1^{*n} \widehat{S_T} V_1^n V_0^m k = \lim_{m,n\to\infty} V_0^{*m} V_1^{*n} \widehat{S_T}^{1/2} V_1^n V_0^m k = k \tag{4.3}$$

and

$$\lim_{m,n\to\infty} V_0^m V_1^n \widehat{S_T} V_1^{*n} V_0^{*m} k = \lim_{n\to\infty} V_{1-i}^n S_T^{1/2} S_{T_i^*} S_T^{1/2} V_{1-i}^{*n} k = S_T^{1/2} S_{T^*} S_T^{1/2} k \tag{4.4}$$

for every $k \in \overline{\mathcal{R}(S_T)}$ and $i = 0, 1$, where the operator limit in (4.4) *is considered as acting on $\overline{\mathcal{R}(S_T)}$. Moreover, the operator $S_T^{1/2} S_{T^*} S_T^{1/2}$ commutes with V_0 and V_1 and $\overline{\mathcal{R}(S_T^{1/2} S_{T^*} S_T^{1/2})}$, as a subspace of $\overline{\mathcal{R}(S_T)}$, reduces V_0 and V_1 to unitary operators.*

Proof. For every $k \in S_T^{1/2} h$ with $h \in \mathcal{H}$ and any integers $m, n \geq 1$,

$$\begin{aligned}\|I - V_0^{*m} V_1^{*n} \widehat{S_T} V_1^n V_0^m k\|^2 &= \|V_0^{*m} V_1^{*n} S_T^{1/2} (I - S_T) T_0^m T_1^n h\|^2 \\ &\leq \|(I - S_T)^{1/2} T_0^m T_1^n h\|^2 = \|T_0^m T_1^n h\|^2 - \|S_T^{1/2} T_0^m T_1^n h\|^2 \to 0\end{aligned}$$

as $m, n \to \infty$. Since $0 \leq I - S_T^{1/2} \leq I - S_T$ we get as above

$$\begin{aligned}\|I - V_0^{*m} V_1^{*n} \widehat{S_T}^{1/2} V_1^n V_0^m k\|^2 &\leq \|(I - S_T^{1/2})^{1/2} T_0^m T_1^n h\|^2 \\ &\leq \|(I - S_T)^{1/2} T_0^m T_1^n h\|^2 \to 0\end{aligned}$$

as $m, n \to \infty$. So, the first equality of (4.3) holds for every $k \in \overline{\mathcal{R}(S_T)}$ (the corresponding sequences are bounded).

Now from (4.1) and (4.2) we obtain

$$V_0^m V_1^n \widehat{S_T} V_1^{*n} V_0^{*m} k = S_T^{1/2} T_0^m T_1^n T_1^{*n} T_0^{*m} S_T^{1/2} k \to S_T^{1/2} S_{T^*} S_T^{1/2} k$$

as $m, n \to \infty$, for any $k \in \overline{\mathcal{R}(S_T)}$, which proves the second equality of (4.4). Obviously, $\overline{\mathcal{R}(S_T)}$ reduces the operator $S_T^{1/2} S_{T^*} S_T^{1/2}$ (which is self-adjoint), so this operator can be considered in $\mathcal{B}(\overline{\mathcal{R}(S_T)})$. On the other hand, since

$$V_i^m \widehat{S_T} V_i^{*m} k = S_T^{1/2} T_i^m T_i^{*m} S_T^{1/2} k \to S_T^{1/2} S_{T_i^*} S_T^{1/2} k$$

as $m \to \infty$, we have (by the previous remark)

$$S_T^{1/2} S_{T^*} S_T^{1/2} k = \lim_{n\to\infty} V_{1-i}^n S_T^{1/2} S_{T_i^*} S_T^{1/2} V_{1-i}^{*n} k$$

for $k \in \overline{\mathcal{R}(S_T)}$ and $i = 0, 1$. So, the first equality of (4.4) holds true.

For the last assertion notice that by (4.1) and (4.2), V_i^* is a $S_T^{1/2}S_{T^*}S_T^{1/2}$-isometry, that is, $V_iS_T^{1/2}S_{T^*}S_T^{1/2}V_i^* = S_T^{1/2}S_{T^*}S_T^{1/2}$, because T_i^* is an S_{T^*}-isometry, $i = 0, 1$. This also implies

$$S_T^{1/2}S_{T^*}S_T^{1/2}V_i^* = V_i^*S_T^{1/2}T_iS_{T^*}T_i^*S_T^{1/2} = V_i^*S_T^{1/2}S_{T^*}S_T^{1/2},$$

which means that $S_T^{1/2}S_{T^*}S_T^{1/2}$ commutes with V_i for $i = 0, 1$. This ensures that the range

$$\overline{\mathcal{R}(S_T^{1/2}S_{T^*}S_T^{1/2})} = \overline{S_T^{1/2}S_{T^*}S_T^{1/2}\mathcal{R}(S_T^{1/2})} = \overline{\mathcal{R}(S_T^{1/2}S_{T^*}S_T)}$$

as a subspace of $\overline{\mathcal{R}(S_T)}$ reduces V_0 and V_1. Since from the second equality of (4.4) it follows that

$$\overline{\mathcal{R}(S_T^{1/2}S_{T^*}S_T^{1/2})} \subset \bigcap_{m\geq 0} \mathcal{R}(V_0^m) \cap \bigcap_{n\geq 0} \mathcal{R}(V_1^n) = \mathcal{N}(I - S_{V_0^*}) \cap \mathcal{N}(I - S_{V_1^*}),$$

we infer that V_0 and V_1 are unitary on $\overline{\mathcal{R}(S_T^{1/2}S_{T^*}S_T^{1/2})}$. ∎

REMARK 4.2. From (4.1) one can get the polar decomposition of $S_T^{1/2}T_i$ ($i = 0, 1$). Note $|S_T^{1/2}T_i| = S_T^{1/2}$, and put $\widetilde{V}_i = JV_iP$ where P is the projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(S_T)}$ and $J = P^*$ is the canonical embedding of $\overline{\mathcal{R}(S_T)}$ into $\mathcal{H}$. Clearly, $\widetilde{V}_i$ isometrically maps $\overline{\mathcal{R}(S_T)} = \mathcal{N}(S_T)^\perp = \mathcal{N}(S_T^{1/2}T_i)^\perp$ onto $\mathcal{R}(\widetilde{V}_i) \subset \overline{\mathcal{R}(S_T^{1/2}T_i)} \subset \overline{\mathcal{R}(S_T)}$, and

$$\mathcal{N}(\widetilde{V}_i) = \mathcal{N}(P) = \mathcal{N}(S_T) = \mathcal{N}(S_T^{1/2}T_i).$$

Hence $\widetilde{V}_i$ is a partial isometry in $\mathcal{B}(\mathcal{H})$, and the polar decomposition of $S_T^{1/2}T_i$ is $S_T^{1/2}T_i = \widetilde{V}_iS_T^{1/2}$, while $\widetilde{V}_i$ is even an extension of V_i, for $i = 0, 1$.

Observe also that for a bicontraction $T^* = (T_0^*, T_1^*)$ there are isometries $V_{*0}, V_{*1} \in \mathcal{B}(\overline{\mathcal{R}(S_{T^*})})$ which satisfy

$$V_{*i}S_{T^*}^{1/2}k = S_{T^*}^{1/2}T_i^*k \quad (k \in \overline{\mathcal{R}(S_{T^*})},\ i = 0, 1). \tag{4.5}$$

Recall that two bicontractions $T = (T_0, T_1)$ on $\mathcal{H}$ and $S = (S_0, S_1)$ on $\mathcal{K}$ are *similar* if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ satisfying $AT_i = S_iA$, $i = 0, 1$. If A belonging to $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is only densely defined, i.e. $\overline{\mathcal{R}(A)} = \mathcal{K}$ with $\mathcal{N}(A) = \{0\}$ and A intertwines T_i with S_i ($i = 0, 1$), one says that T is a *quasiaffine transform* of S. Finally, T is *quasisimilar* to S if T and S are quasiaffine transforms of each other.

As in the case of a single contraction (see [K]), we can characterize these concepts using the asymptotic limit operators S_T and S_{T^*}.

We first give the following

LEMMA 4.3. *Let $T=(T_0,T_1)$ be a bicontraction on $\mathcal{H}$ such that $\mathcal{N}(S_T)=\mathcal{N}(S_{T^*})=\{0\}$. Then for $i=0,1$ we have*

$$V_iS_T^{1/2}S_{T^*}^{1/2}=S_T^{1/2}S_{T^*}^{1/2}V_{*i}^*, \tag{4.6}$$

$$S_{T^*}^{1/2}S_T^{1/2}V_i=V_{*i}^*S_{T^*}^{1/2}S_T^{1/2}, \tag{4.7}$$

$$S_{T^*}S_T^{1/2}V_i=T_iS_{T^*}S_T^{1/2}, \tag{4.8}$$

$$S_{T^*}S_TT_i=T_iS_{T^*}S_T. \tag{4.9}$$

Proof. The hypothesis implies $\mathcal{H}=\overline{\mathcal{R}(S_T)}=\overline{\mathcal{R}(S_{T^*})}$, so V_i and V_{*i} are isometries on $\mathcal{H}$. Then by (4.1) and (4.5) we get

$$V_iS_T^{1/2}S_{T^*}^{1/2}=S_T^{1/2}T_iS_{T^*}^{1/2}=S_T^{1/2}S_{T^*}^{1/2}V_{*i}^*,$$

that is, (4.6). By duality we have $V_{*i}S_{T^*}^{1/2}S_T^{1/2}=S_{T^*}^{1/2}S_T^{1/2}V_i^*$, whence one obtains (4.7). Now from (4.7) it follows that

$$S_{T^*}S_T^{1/2}V_i=S_{T^*}^{1/2}V_{*i}^*S_{T^*}^{1/2}S_T^{1/2}=(V_{*i}S_{T^*}^{1/2})^*S_{T^*}^{1/2}S_T^{1/2}=T_iS_{T^*}S_T^{1/2},$$

that is, (4.8), while (4.9) is immediate from (4.8). ∎

THEOREM 4.4. *If T is a bicontraction on $\mathcal{H}$ then:*

(i) *T is similar to a bi-isometry if and only if S_T is invertible.*
(ii) *T is similar to a unitary bicontraction if and only if S_T and S_{T^*} are invertible.*
(iii) *T is quasisimilar to a unitary bicontraction if and only if*

$$\mathcal{N}(S_T)=\mathcal{N}(S_{T^*})=\{0\}.$$

Proof. (i) If S_T is invertible then T is similar via S_T to the bi-isometry $V=(V_0,V_1)$ given in (4.1). Conversely, suppose that T is similar to a bi-isometry $S=(S_0,S_1)$ on $\mathcal{K}$ via an invertible operator A from $\mathcal{H}$ onto $\mathcal{K}$. Let $A=Q|A|$ be the polar decomposition of A, with Q unitary and $|A|$ invertible. Since $AT_i=S_iA$ we get $S_i=Q|A|T_i|A|^{-1}Q^*$, whence $|A|T_i=Q^*S_iQ|A|=W_i|A|$ where $W_i=Q^*S_iQ$ is an isometry, $i=0,1$. It follows that $|A|=W_i^*|A|T_i$, and also $W_i=|A|T_i|A|^{-1}$, and both give $A^*A=|A|^2=T_i^*A^*AT_i$, for $i=0,1$. This forces that $A^*A\le S_T$, hence S_T is invertible.

(ii) The previous remark implies that if T is similar to a unitary bicontraction then S_T and S_{T^*} are invertible.

Conversely, assume that S_T and S_{T^*} are invertible, so $AT_i=S_iA$ as above, and $BT_i^*=S_{*i}B$ where S_{*i} are isometries on $\mathcal{G}$ and $B\in\mathcal{B}(\mathcal{H},\mathcal{G})$ is invertible. Since $T_i=B^*S_{*i}^*(B^*)^{-1}$ we get $S_iA=AB^*S_{*i}^*(B^*)^{-1}$ where S_{*i}^* is a coisometry, therefore it is surjective. This yields $\mathcal{R}(S_i)=\mathcal{K}$, that is, S_i is unitary, $i=0,1$. Hence T is similar to the unitary bicontraction S.

(iii) Suppose that T is quasisimilar to $U=(U_0,U_1)$ where U_i are unitary operators on $\mathcal{K}$, $i=0,1$. If $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ is such that $\overline{\mathcal{R}(A)}=\mathcal{H}$, $\mathcal{N}(A)=\{0\}$

and $AT_i = U_iA$ $(i = 0,1)$ then $AT_0^mT_1^n = U_0^mU_1^nA$ for $m,n \in \mathbb{N}$. So, for $h \in \mathcal{N}(S_T)$ we have $T_0^mT_1^nh \to 0$ $(m,n \to \infty)$, hence $U_0^mU_1^nAh \to 0$ $(m,n \to \infty)$, which gives $Ah = 0$ and $h = 0$, too. Thus $\mathcal{N}(S_T) = \{0\}$, and similarly, since U is a quasiaffine transform of T, $\mathcal{N}(S_{T^*}) = \{0\}$.

Conversely, assume that $\mathcal{N}(S_T) = \mathcal{N}(S_{T^*}) = \{0\}$, therefore $\overline{\mathcal{R}(S_T)} = \overline{\mathcal{R}(S_{T^*})} = \mathcal{H}$. We infer that $\mathcal{N}(S_{T^*}S_T^{1/2}) = \{0\}$ and also $\overline{\mathcal{R}(S_{T^*}S_T^{1/2})} = \mathcal{H}$. By (4.1) and (4.8) and the previous remarks we conclude that T is quasisimilar to (V_0,V_1), and it remains to see that V_0 and V_1 are unitary. Indeed, since $\mathcal{N}(T_i^*) \subset \mathcal{N}(S_{T^*}) = \{0\}$ one has $\mathcal{N}(T_i^*) = \{0\}$. But by (4.2) we have $S_T^{1/2}\mathcal{N}(V_i^*) \subset \mathcal{N}(T_i^*)$, hence $\mathcal{N}(V_i^*) = \{0\}$, which means that V_i is unitary, $i = 0,1$. ∎

As in the case of a single contraction, the above results can be used to make some remarks on the invariant subspaces of a bicontraction $T = (T_0,T_1)$ on $\mathcal{H}$. Obviously, an invariant subspace of T means a jointly invariant subspace of T_0 and T_1.

THEOREM 4.5. *The following statements hold for every bicontraction $T = (T_0,T_1)$ on $\mathcal{H}$:*

(i) *If $\mathcal{N}(S_T) = \mathcal{N}(S_{T^*}) = \{0\}$ then either T_0 and T_1 are unitary scalar, or T has nontrivial invariant subspaces which are hyperinvariant for T_0 or T_1.*
(ii) *If $S_T \neq 0$ and $S_{T^*} \neq 0$ then either T_0 and T_1 are unitary scalar, or T has nontrivial invariant subspaces which are invariant for any operator which commutes with T_0 and T_1.*

Proof. (i) The assumption of (i) ensures, by Theorem 4.4, that T is quasisimilar to a bicontraction $U = (U_0,U_1)$ with U_i unitary. If U_0 (or U_1) is nonscalar then U_0 (resp. U_1) has nontrivial hyperinvariant subspaces, and by [K, Corollary 4.8] it follows that T_0 (resp. T_1) has nontrivial hyperinvariant subspaces. Hence T has nontrivial invariant subspaces, as in the case considered before. In the other case, one has $U_i = \lambda_iI$ with $|\lambda_i| = 1$, and since T_i is a quasiaffine transform of U_i by an injective operator, we infer $T_i = \lambda_iI$, $i = 0,1$. Clearly, when $\dim\mathcal{H} > 1$, any nontrivial subspace of $\mathcal{H}$ is invariant for T.

Note also that $\mathcal{N}(S_{T_i}) = \mathcal{N}(S_{T_i^*}) = \{0\}$ for $i = 0,1$ by the hypothesis of (i). Thus, one can directly apply [K, Corollary 4.11] for T_i $(i = 0,1)$ to obtain the conclusion of (i).

(ii) The assumption of (ii) gives $\mathcal{H} \neq \mathcal{N}(S_T)$ and $\mathcal{H} \neq \mathcal{N}(S_{T^*})$. So, if $\mathcal{N}(S_T) \neq \{0\}$ then $\mathcal{N}(S_T)$ is a nontrivial invariant subspace for T. Since $h \in \mathcal{N}(S_T)$ iff $T_0^mT_1^nh \to 0$ $(m,n \to \infty)$, it follows that $\mathcal{N}(S_T)$ is also invariant for any operator which commutes with T_0 and T_1.

If $\mathcal{N}(S_{T^*}) \neq \{0\}$ then, as above, $\mathcal{N}(S_{T^*})$ is a nontrivial invariant subspace for T^* and, also, for any operator that commutes with T_0^* and T_1^*. In this case, $\overline{\mathcal{R}(S_{T^*})}$ is a nontrivial invariant subspace for T, which remains invariant for any commutant of T_0 and T_1.

The other case, namely $\mathcal{N}(S_T) = \mathcal{N}(S_{T^*}) = \{0\}$, was discussed in (i). ∎

COROLLARY 4.6. *Let T be a bicontraction on $\mathcal{H}$ which has no nontrivial invariant subspace. Then either T or T^* is strongly stable on $\mathcal{H}$. More precisely, either T and T^* are strongly stable, or T is strongly stable and $0 < \|S_{T^*}h\| < \|h\|$ for all nonzero $h \in \mathcal{H}$, or T^* is strongly stable and $0 < \|S_T h\| < \|h\|$ for all nonzero $h \in \mathcal{H}$.*

Proof. By the previous theorem, T has no nontrivial invariant subspaces iff $S_T = 0$ or $S_{T^*} = 0$, equivalently $\mathcal{N}(S_T) = \mathcal{H}$ or $\mathcal{N}(S_{T^*}) = \mathcal{H}$. When this happens, we also have $\mathcal{H} = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$ or $\mathcal{H} = \mathcal{N}(I - S_{T^*}) \oplus \mathcal{N}(S_{T^*})$, that is, $\mathcal{N}(I - S_T) = \{0\}$ or $\mathcal{N}(I - S_{T^*}) = \{0\}$. Hence only the following cases are admissible:

(a) $\mathcal{H} = \mathcal{N}(S_T) = \mathcal{N}(S_{T^*})$ which means that T and T^* are strongly stable,
(b) $\mathcal{H} = \mathcal{N}(S_T)$ and $\mathcal{N}(S_{T^*}) = \mathcal{N}(I - S_{T^*}) = \{0\}$, so T is strongly stable and $0 < \|S_{T^*}h\| < \|h\|$ for $0 \neq h \in \mathcal{H}$,
(c) $\mathcal{H} = \mathcal{N}(S_{T^*})$ and $\mathcal{N}(S_T) = \mathcal{N}(I - S_T) = \{0\}$, meaning that T^* is strongly stable and $0 < \|S_T h\| < \|h\|$ for $0 \neq h \in \mathcal{H}$. ∎

In the usual terminology (which also appears in [KO]), a bicontraction T belongs to the class $C_{0\cdot}$ (resp. $C_{1\cdot}$) if $\mathcal{N}(S_T) = \mathcal{H}$ (resp. $\mathcal{N}(S_T) = \{0\}$). Also, T belongs to $C_{\cdot 0}$ (resp. $C_{\cdot 1}$) if T^* belongs to $C_{0\cdot}$ (resp. $C_{1\cdot}$). For $\alpha, \beta \in \{0, 1\}$, the class $C_{\alpha\beta}$ is defined as $C_{\alpha\cdot} \cap C_{\cdot\beta}$. Thus, Theorem 4.5 shows that any bicontraction of class C_{11} has nontrivial invariant subspaces, while Corollary 4.6 implies that every bicontraction without nontrivial invariant subspaces belongs to C_{01} or C_{10}. Concerning these latter classes, the following fact can also be proved.

THEOREM 4.7. *Every bicontraction that does not belong to the class C_{00} has nontrivial invariant subspaces if and only if every bicontraction which is a quasiaffine transform of a unitary bicontraction has nontrivial invariant subspaces.*

Proof. Let $T = (T_0, T_1)$ be a bicontraction such that either T or T^* is not strongly stable, that is, $S_T \neq 0$ or $S_{T^*} \neq 0$. Suppose that T has no nontrivial invariant subspace, and firstly that $S_T \neq 0$. This forces $\mathcal{N}(S_T) = \{0\}$ and hence $\mathcal{N}(T_i) = \{0\}$, so $T_i \neq 0$ for $i = 0, 1$. Since $(I - V_i V_i^*) S_T^{1/2} T_i = 0$, V_i being given by (4.1), the assumption on T implies $(I - V_i V_i^*) S_T^{1/2} = 0$, $i = 0, 1$ (otherwise, $\overline{\mathcal{R}(T_i)}$ is a nontrivial invariant subspace of T). As $\overline{\mathcal{R}(S_T)} = \mathcal{H}$ it follows that V_i is unitary for $i=0,1$, hence T is a quasiaffine transform by (4.1) of the unitary bicontraction $V=(V_0,V_1)$. By duality, in the case $S_{T^*}\neq 0$ it follows that T^* is a quasiaffine transform of the unitary bicontraction $V_*=(V_{*0},V_{*1})$ given in (4.5). We proved that, under the cited assumption on T, there exist bicontractions (either T or T^*) without nontrivial invariant subspaces, that are quasiaffine transforms of unitary bicontractions.

Conversely, let T be a bicontraction on $\mathcal{H}$ which is a quasiaffine transform of a unitary bicontraction $U=(U_0,U_1)$ on $\mathcal{K}$ by an operator $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$, such that T has no nontrivial invariant subspaces. Assuming that T is strongly stable, that is, $\mathcal{N}(S_T)=\mathcal{H}$, we get, for $0\neq h\in\mathcal{H}$,

$$\|Ah\|=\|U_0^mU_1^nAh\|=\|AT_0^mT_1^nh\|\to 0\quad (m,n\to\infty),$$

which yields $h=0$ (A being injective), a contradiction. Hence T is not strongly stable, in particular, T is not in the class C_{00}. ∎

Note that Corollary 4.6 and Theorem 4.7 are direct extensions of [K, Corollary 5.9 and Theorem 4.14].

Finally, notice that some of the above facts concerning invariant subspaces for bicontractions are known (even for multicontractions) and obtained by a different method (see e.g. [KO, Theorems 2.2 and 2.3]). Here we pointed out the role of asymptotic limit operators in the above problems, which is similar to the case of a single contraction (see [K]).

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