Decompositions and asymptotic limit for bicontractions

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Abstract. The asymptotic limit of a bicontraction $T$ (that is, a pair of commuting contractions) on a Hilbert space $\mathcal{H}$ is used to describe a Nagy–Foiaş–Langer type decomposition of $T$. This decomposition is refined in the case when the asymptotic limit of $T$ is an orthogonal projection. The case of a bicontraction $T$ consisting of hyponormal (even quasinormal) contractions is also considered, where we have $S_{T^*} = S_{T^*}$.

1. Introduction. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$ with the identity element $I$. The range and the kernel of $T \in \mathcal{B}(\mathcal{H})$ are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. Recall that $T$ is hyponormal if $TT^* \leq T^*T$, and $T$ is quasinormal if $T^*T^2 = TT^*$. Obviously, every quasinormal operator is hyponormal.

A (closed) subspace $M \subset \mathcal{H}$ is invariant for $T$ if $TM \subset M$, and when $M$ is invariant for $T$ and $T^*$ one says that $M$ reduces (or $M$ is reducing for) $T$. Also, $P_M$ stands for the orthogonal projection in $\mathcal{B}(\mathcal{H})$ corresponding to $M$.

A bicontraction on $\mathcal{H}$ is a pair $T = (T_0, T_1)$ of commuting contractions on $\mathcal{H}$, that is, a pair of operators satisfying $\|T_i\| \leq 1$ ($i = 0, 1$) and $T_0T_1 = T_1T_0$. If $T_0$ and $T_1$ are isometries then $T$ is called a bi-isometry on $\mathcal{H}$.

Let $T = (T_0, T_1)$ be a bicontraction. It is known (see [D], [SNF], [K], [S1]) that the asymptotic limit of $T_i$ is defined by

$$S_{T_i}h = \lim_{n \to \infty} T_i^n T_i^* T_i^n h \quad (h \in \mathcal{H})$$

and clearly, $0 \leq S_{T_i} \leq T_i^* T_i$, $T_i^* S_{T_i} T_i = S_{T_i}$, $i = 0, 1$ (the last condition means that $T_i$ is an $S_{T_i}$-isometry [S1], [S2]). It follows that

$$T_0^m S_{T_1} T_0^m \leq T_0^m T_1^* T_1^0 T_0^m = T_1^* T_0^m T_0^m T_1^0$$

for any $m, n \in \mathbb{N}$, and letting $m \to \infty$ one obtains

$$0 \leq \lim_{m \to \infty} T_0^m S_{T_1} T_0^m \leq T_1^* S_{T_0} T_1^* \quad (n \in \mathbb{N}).$$

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Letting $n \to \infty$ we infer that

$$s\lim_{m \to \infty} T_0^m S_{T_1} T_0^m \leq s\lim_{n \to \infty} T_1^* S_{T_0} T_1^n,$$

and by symmetry equality holds in this relation. Thus, the asymptotic limit of $T$ can be defined by

$$S_T h = \lim_{m \to \infty} T_0^m S_{T_1} T_0^m h = \lim_{n \to \infty} T_1^* S_{T_0} T_1^n h$$

$$= \lim_{n \to \infty} \lim_{m \to \infty} T_0^m T_1^* T_1 T_0^m h = \lim_{n \to \infty} \lim_{m \to \infty} T_0^m T_1^* T_1 T_0^m h$$

for any $h \in \mathcal{H}$. Note that $0 \leq S_T \leq S_{T_1}$ and $T_i^* S_{T_1} = S_T$ for $i = 0, 1$. In fact,

$$S_T = \max \{ A \in \mathcal{B}(\mathcal{H}) : 0 \leq A \leq I, T_i^* A T_i = A, i = 0, 1 \}.$$

We say that $T$ is strongly stable if $\mathcal{N}(S_T) = \{0\}$, that is, $T_0^m T_1^n h \to 0$ $(m, n \to \infty)$ for $h \in \mathcal{H}$.

Our goal in this paper is to find some orthogonal decompositions of $\mathcal{H}$ induced by bicontractions $T$ for which $S_T$ is an orthogonal projection. So, in Section 2 we get some conditions on $T$ under which $S_T = S_{T_1}^2$. We describe in the language of asymptotic limits the Nagy–Foiaş–Langer type decomposition of $T$ relative to a bicontraction $T$. The case when $T$ consists of hyponormal or quasinormal contractions is considered here, where we show that $S_{T^*} = S_{T_1}^2$.

In Section 3 we use the operators $S_T$ and $S_{T_i}$ $(i = 0, 1)$ to refine the Nagy–Foiaş–Langer type decomposition for the bicontractions $T$ with $S_T = S_{T_1}^2$ (and $S_{T^*} = S_{T_1}^2$). This decomposition is related to the general Wold type decomposition of a bi-isometry, obtained by D. Popovici [P] and recently, in a different way, by Bercovici–Douglas–Foiaş [BDF].

2. Invariant subspaces induced by the asymptotic limit. As in the case of a single contraction (see [K]), many interesting facts for bicontractions arise in the case when $S_T$ is an orthogonal projection, that is, $S_T = S_{T_1}^2$, or equivalently $\mathcal{N}(S_T - S_{T_1}^2) = \mathcal{H}$. The following proposition, which extends Lemmas 1 and 2 of [KVP], gives interesting information for this case of bicontractions.

**Proposition 2.1.** For any bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ we have:

(i) $\mathcal{N}(S_T - S_{T_1}^2) = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$ is the maximum subspace of $\mathcal{H}$ which is invariant for $T_0$ and $T_1$ and on which $S_T$ commutes with $T_0$ and $T_1$.

(ii) $\mathcal{N}(I - S_T)$ and $\mathcal{N}(S_T)$ are the maximum invariant subspaces for $T_0$ and $T_1$ in $\mathcal{H}$ such that $T_0$ and $T_1$ are isometries on $\mathcal{N}(I - S_T)$, and $T$ is strongly stable on $\mathcal{N}(S_T)$. In addition,

$$\mathcal{N}(I - S_T) = \{ h \in \mathcal{H} : \| T_0^m T_1^n h \| = \| h \|, \forall m, n \in \mathbb{N} \}.$$
Moreover, if $N(I - S_{T_i})$ is invariant for $T_{1-i}$ ($i = 0, 1$) then

(2.2) \[ N(I - S_T) = N(I - S_{T_0}) \cap N(I - S_{T_1}). \]

Proof. Observe that $N(I - S_T)$ and $N(S_T)$ are contained in $N(S_T - S_T^2)$, and are orthogonal. So, $N(I - S_T) \oplus N(S_T) \subset N(S_T - S_T^2)$. Conversely, let $h \in N(S_T - S_T^2)$ be such that $h$ is orthogonal to $N(I - S_T) \oplus N(S_T)$. Then $S_T h \in N(I - S_T)$ and therefore $\langle h, S_T h \rangle = 0$, which means that $S_T h = 0$ or $h \in N(S_T)$. Hence $h = 0$, since $h$ is orthogonal to $N(S_T)$. Consequently,

\[ N(S_T - S_T^2) = N(I - S_T) \oplus N(S_T). \]

Now recall that $T_i^* S_T T_i = S_T$, whence $N(S_T)$ is invariant for $T_i$ ($i = 0, 1$). As we also have $(T_i$ is a contraction)

\[ T_i^* (I - S_T) T_i \leq I - S_T, \]

it follows that $N(I - S_T)$ is invariant for $T_i$ ($i = 0, 1$).

Furthermore, for $m, n, p, q \in \mathbb{N}$ one has

\[ T_0^{*(m+p)} T_1^{*(n+q)} T_1^{m+q} T_0^{m+p} \leq T_0^{*m} T_1^{*n} T_1^{m} T_0^{m}, \]

and setting $p, q \to \infty$ we get $S_T \leq T_0^{*m} T_1^{*n} T_1^{m} T_0^{m}$, whence

\[ I - T_0^{*m} T_1^{*n} T_1^{m} T_0^{m} \leq I - S_T. \]

This gives on one hand,

\[ N(I - S_T) \subset \{ h \in \mathcal{H} : \| T_0^{m} T_1^{n} h \| = \| h \|, \forall m, n \in \mathbb{N} \}. \]

On the other hand, if $\| T_0^{m} T_1^{n} h \| = \| h \| \text{ for } m, n \in \mathbb{N}$ then letting $m, n \to \infty$ one obtains $\| S_T h \| = \| h \|$, and since $0 \leq S_T \leq I$ one infers $h = S_T h$, that is, $h \in N(I - S_T)$. Hence the relation (2.1) holds.

Next, if $h \in N(S_T - S_T^2)$ and $h = h_1 \oplus h_0$ with $h_1 \in N(I - S_T)$, $h_0 \in N(S_T)$ then

\[ (S_T T_i - T_i S_T) h = T_i h_1 - T_i h_1 = 0, \quad i = 0, 1, \]

therefore $S_T$ commutes with $T_0$ and $T_1$ on $N(S_T - S_T^2)$.

Let now $\mathcal{M} \subset \mathcal{H}$ be another subspace invariant for $T_0$ and $T_1$ such that $S_T T_i k = T_i S_T k$ for $k \in \mathcal{M}$, $i = 0, 1$. Then $S_T T_0^{m} T_1^{n} k = T_0^{m} T_1^{n} S_T k$ for any $m, n \in \mathbb{N}$, and this implies ($T_i$ being an $S_T$-isometry)

\[ S_T k = T_0^{*m} T_1^{*n} S_T T_0^{m} T_1^{n} k = T_0^{*m} T_1^{*n} T_0^{m} T_1^{n} S_T k. \]

Letting $m, n \to \infty$ we get $S_T k = S_T^2 k$, that is, $k \in N(S_T - S_T^2)$. So $\mathcal{M} \subset N(S_T - S_T^2)$ and we conclude that $N(S_T - S_T^2)$ is the maximum invariant subspace for $T_i$ on which $S_T$ commutes with $T_i$, $i = 0, 1$, which proves (i).

It is clear (by (2.1)) that $T_i$ is an isometry on $N(I - S_T)$, $i = 0, 1$, and (by the definition of $S_T$) we have $T_0^{m} T_1^{n} h \to 0 (m, n \to \infty)$ for $h \in N(S_T)$, that is, $T$ is strongly stable on $N(S_T)$. In addition, it is obvious that $N(I - S_T)$ and $N(S_T)$ are the maximum subspaces with the above mentioned properties. This proves (ii).
Finally, if \( N(I - S_{T_i}) \) is invariant for \( T_{1-i} \) then \( N(I - S_{T_0}) \cap N(I - S_{T_1}) \) is invariant for \( T_0 \) and \( T_1 \), and clearly \( T_i \) is an isometry on this subspace for \( i = 0, 1 \). Since \( N(I - S_T) \subset N(I - S_{T_0}) \cap N(I - S_{T_1}) \) it follows that the two subspaces coincide (by the maximality of \( N(I - S_T) \) cited in (ii)).

**Corollary 2.2.** For a bicontraction \( T = (T_0, T_1) \) on \( \mathcal{H} \) we have \( S_T = S_T^2 \) if and only if \( S_T \) is completely nonunitary, because in this case \( T_i \) is an isometry on this subspace for \( i = 0, 1 \). Since \( S_T \) reduces \( T_i \) and \( T_i \) is an isometry on this subspace for \( i = 0, 1 \), it follows that the two subspaces coincide (by the maximality of \( N(I - S_T) \) cited in (ii)).

**Proof.** If \( S_T = S_T^2 \), then \( N(S_T - S_T^2) = \mathcal{H} \), so \( S_T \) commutes with \( T_0 \) and \( T_1 \) on \( \mathcal{H} \) (by Proposition 2.1). Conversely, if \( S_T T_i = T_i S_T \) \((i = 0, 1)\) then necessarily \( N(S_T - S_T^2) = \mathcal{H} \) (by the maximality of \( N(S_T - S_T^2) \) in Proposition 2.1(i)), that is, \( S_T = S_T^2 \).

Assume now that \( S_T = S_T^2 \). For \( m, n \in \mathbb{N} \) and \( h \in \mathcal{H} \) one has

\[
S_T h = T_0^{m} T_1^{m} S_T T_0^{m} T_1^{m} h = T_0^{m} T_1^{m} S_T^2 T_0^{m} T_1^{m} h = T_0^{m} T_1^{m} T_0^{m} S_T T_0^{m} h \to S_T^3 h \quad (m, n \to \infty),
\]

hence \( S_T = S_T^3 \). It follows that \( S_T = S_T^2 \).

This corollary extends the corresponding assertions for contractions in Lemma 1 and Proposition 1 of \([KVP]\).

A special case of bicontractions for which their asymptotic limits are orthogonal projections is mentioned in the following theorem.

As usual, a bicontraction \( T = (T_0, T_1) \) on \( \mathcal{H} \) is called completely nonunitary if there is no nonzero subspaces of \( \mathcal{H} \) which reduce \( T_0 \) and \( T_1 \) to unitary operators. Clearly, every strongly stable bicontraction \( T \) is completely nonunitary, because in this case \( \mathcal{H} = N(S_T) \), therefore \( N(I - S_T) = \{0\} \) (by Proposition 2.1(i)).

**Theorem 2.3.** Let \( T = (T_0, T_1) \) be a bicontraction on \( \mathcal{H} \) with \( T_0 \) and \( T_1 \) hyponormal. Then \( S_T^* = S_T^{2*} \) and the maximum subspace of \( \mathcal{H} \) which reduces \( T_0 \) and \( T_1 \) to unitary operators is

\[
N(I - S_T^*) = \bigcap_{m, n \geq 0} T_0^{m} T_1^{n} [N(I - S_{T_0^*}) \cap N(I - S_{T_1^*})].
\]

Moreover, \( T^* \) is strongly stable if and only if \( T \) is completely nonunitary.

**Proof.** Since \( T_i \) is hyponormal we know (see the proof of \([K] \text{ Theorem 5.3}\)) that \( S_{T_i^*} = S_{T_i^{2*}} \) and \( \mathcal{R}(S_{T_{i}^*}) = N(I - S_{T_{i}^*}) \) reduces \( T_i \) to a unitary operator, for \( i = 0, 1 \). As \( N(S_{T_{i}^*}) \) is invariant for \( T_{0}^* \) and \( T_{1}^* \), \( \mathcal{R}(S_{T_{i}^*}) \) will be invariant for \( T_0 \) and \( T_1 \). In addition, because

\[
\mathcal{R}(S_{T_{i}^*}) \subset \mathcal{R}(S_{T_{0}^*}) \cap \mathcal{R}(S_{T_{1}^*}) = N(I - S_{T_{0}^*}) \cap N(I - S_{T_{1}^*})
\]

it follows that \( T_0 \) and \( T_1 \) are isometries on \( \mathcal{R}(S_{T_{i}^*}) \). So, we infer from Propo-
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Proposition 2.1 that
\[ \mathcal{R}(S_{T^*}) \subset \mathcal{N}(I - S_T). \]

Take an arbitrary \( h = h_1 \oplus h_0 \in \mathcal{H} \) with \( h_1 \in \mathcal{R}(S_{T^*}), h_0 \in \mathcal{N}(S_{T^*}) \). We have (by the above inclusion)
\[ T_0 S_{T^*} h = T_0 S_{T^*} T_0^* T_0 h_1 = S_{T^*} T_0 h_1. \]

But \( T_0^* S_{T^*} T_0 h_0 = S_{T^*} h_0 = 0 \), that is, \( S_{T^*} T_0 h_0 \in \mathcal{N}(T_0^*) \subset \mathcal{N}(S_{T^*}) \), hence \( S_{T^*} T_0 h_0 = 0 \). Thus, we obtain \( T_0 S_{T^*} h = S_{T^*} T_0 h \), and by symmetry one has \( T_1 S_{T^*} h = S_{T^*} T_1 h \). This means that \( S_{T^*} \) commute with \( T_0 \) and \( T_1 \), and by Corollary 2.2 we have \( S_{T^*} = S_{T^*}. \)

Now it follows that \( \mathcal{N}(I - S_{T^*}) \) is the maximum subspace of \( \mathcal{H} \) which reduces \( T_0^* \) and \( T_1^* \) to isometries. In fact, by the above remark, \( \mathcal{N}(I - S_{T^*}) = \mathcal{R}(S_{T^*}) \) is the maximum subspace which reduces \( T_0 \) and \( T_1 \) to unitary operators. Obviously, this subspace is contained in the right side of (2.3), briefly denoted by \( \mathcal{N}_T. \)

Let \( h \in \mathcal{N}_T \) be orthogonal to \( \mathcal{N}(I - S_{T^*}) \). So \( h \in \mathcal{N}(S_{T^*}) \), that is, \( T_0^* T_1^* h = 0 \) \((m, n \to \infty)\). Since \( h \in \mathcal{N}_T \), for any \( m, n \in \mathbb{N} \) there exist \( h_{m,n} \in \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*}) \) such that \( h = T_0^* T_1^* h_{m,n} \). As \( \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*}) \) is invariant for \( T_0 \) and \( T_1 \), while \( T_0 \), \( T_1 \) are isometries on this subspace, we get
\[ h_{m,n} = T_0^* T_1^* h_{m,n} = T_0^* T_1^* h \to 0, \quad m, n \to \infty. \]
This yields \( \|h\| = \|h_{m,n}\| \to 0 \) \((m, n \to \infty)\), hence \( h = 0 \). Thus, (2.3) holds.

Finally, it is clear that \( \mathcal{N}(I - S_{T^*}) = \{0\} \) implies \( \mathcal{H} = \mathcal{N}(S_{T^*}) \), therefore \( T^* \) is strongly stable if (and only if, by the above remark) \( T \) is completely nonunitary. ■

Remark 2.4. W. Mlak proved in [M] that the “unitary part” in \( \mathcal{H} \) of a hyponormal contraction \( T_0 \) is \( \bigcap_{n \geq 0} T_0^n \mathcal{N}(I - T_0 T_0^*) \), by using the minimal unitary dilation of \( T_0 \). This fact was recovered in [S2] without using dilation, by an argument as above involving the asymptotic limit. In the present context we cannot use \( \mathcal{N}(I - T_0 T_0^*) \cap \mathcal{N}(I - T_1 T_1^*) \) in (2.3) instead of \( \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*}) \), because the former subspace is not invariant for \( T_0 \) and \( T_1 \), in general.

We say that a bicontraction \( T = (T_0, T_1) \) on \( \mathcal{H} \) is unitary if \( T_0 \) and \( T_1 \) are unitary operators. We now give the “asymptotic” version of the Nagy–Foiaș–Langer decomposition for bicontractions.

Theorem 2.5. For every bicontraction \( T \) on \( \mathcal{H} \) there exists a unique decomposition of \( \mathcal{H} \) of the form
\[ \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_u^\perp \]
(2.4)
such that $\mathcal{H}_u$ reduces $T$ to a unitary bicontraction and $\mathcal{H}^u_+$ reduces $T$ to a completely nonunitary bicontraction. In addition,

$$
(2.5) \quad \mathcal{H}_u = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_T S_{T^*}) = \mathcal{N}(I - S_{T^*} S_T) = \mathcal{N}(I - S_{T^*}^1 S_T^1) = \mathcal{N}(I - S_T^1 S_{T^*}^1).
$$

Proof. If $h \in \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*})$ then $h = S_T h = S_{T^*} h = S_T S_{T^*} h = S_{T^*} S_T h$, so $\mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) \subset \mathcal{N}(I - S_{T^*}) \cap \mathcal{N}(I - S_{T^*} S_T)$. Conversely, let $h \in \mathcal{N}(I - S_{T^*} S_T)$, that is, $h = S_{T^*} S_T h$. We have

$$
\|h\|^2 = \langle S_{T^*} h, S_T h \rangle \leq \|S_{T^*}^1 h\| \|S_T^1 h\| \leq \|S_{T^*}^1 h\| \|h\|,
$$
whence $\|h\| = \|S_{T^*}^1 h\|$, or equivalently $(I - S_T) h = 0$ (as $0 \leq S_T \leq I$).

Similarly, one has $\|h\| = \|S_T^1 h\|$, that is, $(I - S_{T^*}) h = 0$, and so

$$
\mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_T S_{T^*}) = \mathcal{N}(I - S_{T^*} S_T).
$$

Now, if $h = S_{T^*} S_T h$ then as above $\|h\| = \|S_{T^*}^1 h\| = \|S_{T^*}^1 h\|$, therefore $h = S_{T^*}^1 h = S_{T^*} h = S_{T^*} h = S_{T^*}^1 S_T S_{T^*}^1 h = S_{T^*}^1 S_T S_{T^*}^1 h$. This shows that $\mathcal{N}(I - S_{T^*} S_T) \subset \mathcal{N}(I - S_{T^*}^1 S_T S_{T^*}^1) \cap \mathcal{N}(I - S_{T^*} S_T S_{T^*}^1 S_T^1)$. Conversely, $h = S_{T^*}^1 S_T S_{T^*}^1 h$ gives

$$
\|h\|^2 = \|S_{T^*}^1 S_T S_{T^*}^1 h\|^2 \leq \|S_{T^*}^1 S_T h\|^2 \leq \|S_{T^*}^1 h\|^2 \leq \|h\|^2,
$$
whence $\|h\|^2 = \|S_{T^*}^1 S_T S_{T^*}^1 h\|^2 = \|S_{T^*}^1 h\|^2$. Hence $h = S_{T^*} h = S_{T^*}^1 h$ and there fore $\|S_{T^*}^1 h\| = \|S_{T^*} S_{T^*}^1 h\| = \|h\|$ (the last equality follows from our assumption), which yields $h = S_{T^*} h$. So, $\mathcal{N}(I - S_{T^*} S_T S_{T^*}^1 S_T^1)$ and (by symmetry) $\mathcal{N}(I - S_{T^*}^1 S_T S_{T^*}^1)$ are contained in $\mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*})$. Thus, the above equalities between subspaces are completed with the last two from (2.5).

Next, by (2.1) for $T$ and $T^*$ we see immediately that the subspace $\mathcal{H}_u := \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*})$ reduces $T_0$ and $T_1$ to unitary operators. In addition, if $\mathcal{M} \subset \mathcal{H}$ is another such subs pace, then $\mathcal{M} \subset \mathcal{H}_u$ by Proposition 2.1(ii). Hence $\mathcal{H}_u$ is the maximum subspace with the property above, and finally, the reducing decomposition (2.4) for $T$ is unique with $T$ is unitary on $\mathcal{H}_u$, and completely nonunitary on $\mathcal{H}^u_+$.

**Corollary 2.6.** For every bi-isometry $T = (T_0, T_1)$ on $\mathcal{H}$ we have $S_{T^*} = S_{T_0^* T_1^*}$, hence $\mathcal{H}_u = \mathcal{N}(I - S_{T_0^* T_1^*})$ and $\mathcal{H}^u_+ = \mathcal{N}(S_{T_0^* T_1^*})$ in (2.4). Moreover, $T$ is completely nonunitary if and only if $T_0 T_1$ is a (unilateral) shift on $\mathcal{H}$.

**Proof.** Since $T_0 T_1$ is an isometry, by Theorem 2.3 the maximum subspace of $\mathcal{H}$ which reduces $T_0 T_1$ to a unitary operator is $\mathcal{N}(I - S_{T_0^* T_1^*})$. So, by Theorem 2.5 one obtains $\mathcal{N}(I - S_{T^*}) \subset \mathcal{N}(I - S_{T_0^* T_1^*})$. On the other hand,
\[ \mathcal{N}(I - S_{T_0^*T_1^*}) = \mathcal{N}(I - S_{(T_0, T_1)}) = \bigcap_{n \geq 0} T_0^n T_1^n \mathcal{H}, \]

it follows immediately that \( \mathcal{N}(I - S_{T_0^*T_1^*}) \) reduces \( T_0 \) and \( T_1 \) to unitary operators, hence \( \mathcal{N}(I - S_{T_0^*T_1^*}) \subset \mathcal{N}(I - T_{i}^*) \) by Theorem 2.5. Thus \( \mathcal{N}(I - T_{i}^*) = \mathcal{N}(I - S_{T_{i}^*}) \), and since \( S_{T_i^*} \), \( S_{T_0^*T_1^*} = S_{(T_0, T_1)}^* \) are orthogonal projections, also \( \mathcal{N}(S_{T_i^*}) = \mathcal{N}(S_{T_0^*T_1^*}) \). We conclude that \( S_{T_i^*} = S_{T_0^*T_1^*} \), and the remaining assertions of the corollary follow from Theorems 2.3 and 2.5. \( \Box \)

Another interesting particular case of Theorem 2.3 is considered below. Notice that the case of a single quasinormal contraction was considered in [KVP1 Example 3].

**Proposition 2.7.** For every bicontraction \( T = (T_0, T_1) \) on \( \mathcal{H} \) with \( T_0 \) and \( T_1 \) quasinormal one has \( S_{T_i^*} = S_{T_i}^2 \). Moreover, \( S_T = S_T^2 \) if and only if either \( T_0^* \mid_{\mathcal{R}(S_T)} \) or \( T_1^* \mid_{\mathcal{R}(S_T)} \) is a coisometry.

In addition, \( S_T = S_{T_i^*} \) if and only if \( T_i^* \mid_{\mathcal{R}(S_T)} \) is normal and \( \mathcal{R}(S_T) \) is invariant for \( T_i T_i^* \) (\( i = 0, 1 \)). In this case \( \mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T_i^*}) \cap \mathcal{N}(I - S_{T_i^*}) \).

**Proof.** Clearly, \( S_{T_i^*} = S_{T_i}^2 \) by Theorem 2.3. Furthermore, because \( T_i \) is quasinormal, we have (see [31], or Lemma 2.8 below) \( S_{T_i} = S_{T_i}^2 \), so \( \mathcal{R}(S_{T_i}) = \mathcal{N}(I - S_{T_i}) \) and \( \mathcal{R}(S_T) \subset \mathcal{N}(I - S_{T_i}) \), \( i = 0, 1 \). So, if \( S_T = S_T^2 \) then \( \mathcal{R}(S_T) \) reduces \( T_0^* \) and \( T_1^* \) to coisometries.

Conversely, assume that, say, \( T_0^* \mid_{\mathcal{R}(S_T)} \) is a coisometry (\( \mathcal{R}(S_T) \) being invariant for \( T_0^* \) and \( T_1^* \)). Put \( T_0 = T_0^* \mid_{\mathcal{R}(S_T)} \). Then \( T_0 = P_{\mathcal{R}(S_T)} T_0 \mid_{\mathcal{R}(S_T)} \) is an isometry on \( \mathcal{R}(S_T) \). Hence for \( h \in \mathcal{H} \) we obtain

\[ \| S_{T} h \| = \| P_{\mathcal{R}(S_T)} T_0 S_{T} h \| \leq \| T_0 S_{T} h \| \leq \| S_{T} h \|, \]

whence \( T_0 S_{T} h = P_{\mathcal{R}(S_T)} T_0 S_{T} h \). We infer that \( \mathcal{R}(S_{T_i}) \) reduces \( T_i \), and since \( \mathcal{R}(S_{T_i}) \subset \mathcal{N}(I - S_{T_i}) \) we have for \( m, n \in \mathbb{N} \) and \( h \in \mathcal{H} \),

\[ S_{T} h = T_0^m T_1^m S_{T} h = T_0^m T_1^m T_1^m T_1^m S_{T} h. \]

Letting \( m, n \to \infty \) we infer that \( S_{T} = S_{T_i}^2 \).

Obviously, if \( S_T = S_{T_i^*} \) then \( \mathcal{R}(S_{T_i}) \) reduces \( T_i \) to unitary operators, \( i = 0, 1 \). Conversely, suppose that \( T_i^* \mid_{\mathcal{R}(S_T)} \) are normal operators for \( i = 0, 1 \). Then for \( h \in \mathcal{H} \) we have

\[ T_i^* P_{\mathcal{R}(S_T)} T_0 S_{T} h = P_{\mathcal{R}(S_T)} T_0 T_i^* S_{T} h = T_0 T_i^* S_{T} h, \]

since \( P_{\mathcal{R}(S_T)} T_0 T_i^* S_{T} h = 0 \) by the assumption that \( \mathcal{R}(S_T) \) is invariant for \( T_0 T_i^* \). It follows that \( T_i^* P_{\mathcal{N}(S_{T_i})} T_0 S_{T} h = 0 \), which gives \( P_{\mathcal{N}(S_{T_i})} T_0 S_{T} h = 0 \), that is, \( T_0 S_{T} h = P_{\mathcal{R}(S_T)} T_0 S_{T} h \). Hence \( \mathcal{R}(S_T) \) reduces \( T_0 \), and so \( T_0 T_i^* S_{T} h = \)
\(T_0^* T_0 S_T h = S_T h\) which means that \(T_0\) is unitary on \(R(S_T)\). By symmetry, \(R(S_T)\) also reduces \(T_1\) to a unitary operator, and by Theorem 2.3 we get

\[
R(S_T) = \mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_T).
\]

Finally, this leads to \(S_T = S_{T^*}\). In this case

\[
\mathcal{N}(I - S_T) \subset \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*}) \subset \mathcal{N}(I - S_{T_0}) \cap \mathcal{N}(I - S_{T_1}),
\]

and since \(\mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*})\) is invariant for \(T_0\) and \(T_1\) it follows (from the second inclusion) that \(T_0\) and \(T_1\) are isometries on this subspace. Thus \(\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*})\), by the maximality of \(\mathcal{N}(I - S_T)\) given in Proposition 2.1(ii).}

Let us remark that if \(T = (T_0, T_1)\) consists of quasinormal commuting contractions and either \(T_0 S_{T_1} = S_{T_1} T_0\) or \(T_1 S_{T_0} = S_{T_0} T_1\) then \(S_T = S_{T_0} S_{T_1} = S_{T_1} S_{T_0}\), hence \(S_T = S_T^2\). We see in the example below that the condition \(S_T = S_T^2\) does not ensure the commutativity of \(T_{1-i}\) with \(S_{T_i}\), \(i = 0, 1\). We first give

**Lemma 2.8.** For every quasinormal contraction \(T_0\) on \(\mathcal{H}\) one has \(S_{T_0} = S_{T_0^*} T_0 = S_{T_0}^2\).

**Proof.** Since \(T_0\) is quasinormal we have (by induction) \((T_0^* T_0)^n = T_0^* T_0^n\) for any \(n \in \mathbb{N}\). Then

\[
S_{T_0} h = \lim_{n \to \infty} T_0^{2n} T_0^2 h = \lim_{n \to \infty} (T_0^* T_0)^{2n} h = S_{T_0^*} T_0 h = S_{T_0}^2 h
\]

for \(h \in \mathcal{H}\). Moreover, the above operator is an orthogonal projection because \(T_0^* T_0\) is positive. ■

**Example 2.9.** Let \(S\) be the canonical shift on \(l^2_+\) and \(\mathcal{K} = R(S) \oplus l^2_+\). Put \(S_0 = S|_{R(S)}\) and let \(S_1 : l^2_+ \to R(S)\) be given by \(S_1 = S P_{N(S^*)}\). Consider \(T_0, T_1 \in \mathcal{B}(\mathcal{K})\) defined by the operator matrices

\[
T_0 = \begin{pmatrix} S_0 & S_1 \\ 0 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}
\]

relative to the above decomposition of \(\mathcal{K}\). We have

\[
T_0^* T_0 = I_{R(S)} \oplus P_{N(S^*)}, \quad T_0^* T_0^2 = T_0 = T_0 T_0^* T_0,
\]

hence \(T_0\), and also \(T_1\), are quasinormal contractions on \(\mathcal{K}\). In addition \(T_0 T_1 = T_1 T_0 = 0\), so \(T = (T_0, T_1)\) is a bicontraction on \(\mathcal{K}\), and clearly, by the above commutativity condition for \(T_0\) and \(T_1\) we have \(S_T = 0\).

On the other hand, (by Lemma 2.8) \(S_{T_0} = S_{T_0^*} T_0 = T_0^* T_0\) and

\[
T_1 S_{T_0} = 0 \oplus S P_{N(S^*)} = 0 \oplus S_1 \neq 0 = 0 \oplus P_{N(S^*)} S = S_{T_0 T_1}.
\]
Similarly, since $S_{T_1} = 0 \oplus I^2_{T_1}$ we get

$$T_0S_{T_1} = \begin{pmatrix} 0 & S_1 \\ 0 & 0 \end{pmatrix} \neq 0 = S_{T_1}T_0.$$  

We conclude that $S_T = S^2_T$ but $T_{1-i}S_{T_1} \neq S_{T_1}T_{1-i}$, or equivalently $T_{1-i}|T_i| \neq |T_i|T_{1-i}$ because $|T_i| = S^*_T$ in this case, for $i = 0, 1$. This also shows that the conditions $T_{1-i}|T_i| = |T_i|T_{1-i}$ ($i = 0, 1$) are not necessary to ensure $S_T = S^2_T$, when $T_0$ and $T_1$ are quasinormal.

3. **Decompositions in the case $S_T = S^2_T$.** The asymptotic limits can be used to refine the Nagy–Foiaş–Langer decomposition for bicontractions when $S_T$ is an orthogonal projection. This decomposition (to be given below) generalizes the Wold type decompositions for bi-isometries which appear in [P] and [BDF]. Recall that a similar result for contractions can be found in [K].

We say (briefly) that a subspace $\mathcal{M} \subset \mathcal{H}$ is in\textit{variant} (resp. reducing) for a bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ if $\mathcal{M}$ is invariant (resp. reducing) for $T_0$ and $T_1$. Also, we say that $T$ is coisometric on $\mathcal{H}$ if both $T_i$ are coisometries.

The statements of Theorem 3.1 and Corollary 3.2 below extend Theorem 1 and Corollary 1 of [KVP] obtained for a single contraction.

**Theorem 3.1.** Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ with $S_T = S^2_T$. Then $\mathcal{H}$ admits the decomposition

$$\mathcal{H} = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S^*_T) \oplus \mathcal{N}(I - S_T) \cap \mathcal{N}(S^*_T) \oplus \mathcal{N}(S_T) \quad (3.1)$$

where all the three summands reduce $T$ in such a way that $T$ is unitary on $\mathcal{N}(I - S_T) \cap \mathcal{N}(I - S^*_T)$, $T^*$ is coisometric and strongly stable on $\mathcal{N}(I - S_T) \cap \mathcal{N}(S^*_T)$, and $T$ is strongly stable on $\mathcal{N}(S_T)$.

Moreover, if $\mathcal{N}(S_T) \neq \{0\}$ and $S^*_T = S^2_{T^*}$ then $\mathcal{N}(S_T)$ admits the decomposition

$$\mathcal{N}(S_T) = \mathcal{N}(I - S^*_T) \cap \mathcal{N}(S_T) \oplus \mathcal{N}(S^*_T) \cap \mathcal{N}(S^*_T), \quad (3.2)$$

where the two summands reduce $T$, and $T$ is coisometric and strongly stable on $\mathcal{N}(I - S^*_T) \cap \mathcal{N}(S_T)$, while $T$ and $T^*$ are strongly stable on $\mathcal{N}(S_T) \cap \mathcal{N}(S^*_T)$.

**Proof.** Since $S_T = S^2_T$ one has $\mathcal{H} = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$ where $\mathcal{N}(I - S_T)$ reduces $T$ to a bi-isometry and $T$ is strongly stable on $\mathcal{N}(S_T)$.

Let $W = (W_0, W_1)$ where $W_i = T_i|\mathcal{N}(I - S_T)$, $i = 0, 1$. By (2.5), the maximum subspace which reduces $T$ to a unitary bicontraction is

$$\mathcal{H}_u = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S^*_T).$$

Now since $W_i$ is an isometry on $\mathcal{N}(I - S_T)$ it follows that $S_{W^*_i} = S^2_{W^*_i}$ for $i = 0, 1$, and by Corollary 2.6 we obtain $S_{W^*_i} = S^2_{W^*_i}$. Therefore
\( \mathcal{N}(I - S_T) = \mathcal{N}(I - S_{W^*}) \oplus \mathcal{N}(S_{W^*}) \)

where the summands reduce \( W_i \), and so \( T_i, i = 0, 1 \). We also have

\[
\mathcal{N}(I - S_{W^*}) = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) = \mathcal{H}_u,
\]

\[
\mathcal{N}(S_{W^*}) = \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}),
\]

hence \( T_i^m T_1^n h \to 0 \ (m, n \to \infty) \) for \( h \in \mathcal{N}(S_{W^*}) \), that is, \( T^* \) is co-isometric and strongly stable on \( \mathcal{N}(S_{W^*}) \).

Next suppose \( \mathcal{N}(S_T) \neq \{0\} \) and let \( W' = (W'_0, W'_1) \) where \( W'_i = T_i |_{\mathcal{N}(S_T)} \), \( i = 0, 1 \). Then relative to the decomposition

\[
\mathcal{H} = \mathcal{H}_u \oplus \mathcal{N}(S_{W^*}) \oplus \mathcal{N}(S_T)
\]

we have \( S_{T^*} = I \oplus 0 \oplus S_{W^*} \), whence

\[
\mathcal{N}(S_{T^*}) = \mathcal{N}(S_{W^*}) \oplus \mathcal{N}(S_{W^*}) \subset \mathcal{N}(S_{W^*}) \oplus \mathcal{N}(S_T).
\]

Since \( \mathcal{N}(S_{W^*}) \subset \mathcal{N}(I - S_T) = \mathcal{H} \cap \mathcal{N}(S_T) \) we infer that

\[
\mathcal{N}(S_{W^*}) = \mathcal{N}(S_T) \cap \mathcal{N}(S_{T^*}).
\]

On the other hand, since \( I - S_{T^*} = 0 \oplus I \oplus (I - S_{W^*}) \) we have

\[
\mathcal{N}(I - S_{T^*}) = \mathcal{H}_u \oplus \mathcal{N}(I - S_{W^*}) \subset \mathcal{H}_u \oplus \mathcal{N}(S_T),
\]

whence

\[
\mathcal{N}(I - S_{W^*}) = \mathcal{N}(I - S_{T^*}) \cap \mathcal{N}(S_T).
\]

Assume \( S_T = S_T^2 \) and \( S_{T^*} = S_{T^*}^2 \). Clearly, the second condition is equivalent to \( S_{W^*} = S_{W^*}^2 \), which also means

\[
\mathcal{N}(S_T) = \mathcal{N}(I - S_{W^*}) \oplus \mathcal{N}(S_{W^*}).
\]

Thus, the summands, reducing for \( W' \), also reduce \( T \) in such a way that \( T^* \) is a bi-isometry and \( T \) is strongly stable on \( \mathcal{N}(I - S_{W^*}) \), and \( T, T^* \) are strongly stable bicontractions on \( \mathcal{N}(S_{W^*}) \). \( \blacksquare \)

Corollary 3.2. For a bicontraction \( T = (T_0, T_1) \) on \( \mathcal{H} \) one has \( S_T = S_{T^*} \) if and only if \( T_i = U_i \oplus S_i \ (i = 0, 1) \) relative to a decomposition \( \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp \), where \( \mathcal{M} \) reduces \( T \) so that \( U = (U_0, U_1) \) is unitary on \( \mathcal{M} \), while \( S = (S_0, S_1) \) and \( S^* \) are strongly stable on \( \mathcal{M}^\perp \).

Proof. Suppose \( S_T = S_{T^*} \). Then for \( m, n \geq 1 \) we have

\[
S_T = T_0^m T_1^n S_T T_1^n T_0^m = T_0^m T_1^n T_0^m T_1^n T_0^m T_1^n T_0^m S_T T_1^n T_0^m T_1^n T_0^m,
\]

and letting \( m, n \to \infty \) we get \( S_T = S_T S_T S_T S_T = S_T^3 \). It follows that \( S_T^2 = S_T^4 \) and so \( S_T = S_T^2 \). By our assumption, \( \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}) = \{0\} \) and \( \mathcal{N}(I - S_{T^*}) \cap \mathcal{N}(S_T) = \{0\} \), so we infer from (3.3) and (3.2) that

\[
\mathcal{H} = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) \oplus \mathcal{N}(S_T) \cap \mathcal{N}(S_{T^*}) = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T).
\]

Thus \( T \) is unitary on \( \mathcal{M} = \mathcal{N}(I - S_T) \), while \( T \) and \( T^* \) are strongly stable on \( \mathcal{M}^\perp = \mathcal{N}(S_T) \), and \( T_i = T_i |_{\mathcal{M}} \oplus T_i |_{\mathcal{M}^\perp} \), \( i = 0, 1 \).
Conversely, if $T_i = U_i \oplus S_i$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and $\mathcal{M}$ reduces $T$ and $U_i$ is unitary on $\mathcal{M}$ for $i = 0, 1$, while $S = (S_0, S_1)$ and $S^*$ are strongly stable on $\mathcal{M}^\perp$, then $S_T = I \oplus 0 = S_T^*$. ■

The decomposition (3.1) can be refined by the general Wold type decomposition of a bi-isometry which was obtained in [P] and recently in [BDF]. So, the following result holds.

**Theorem 3.3.** Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ with $S_T = S_T^2$. Then $\mathcal{H}$ admits a unique decomposition of the form

\[(3.3) \quad \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_s \oplus \mathcal{H}_1 \oplus \mathcal{H}_0,\]

where all the summands reduce $T$, and where $T_0|_{\mathcal{H}_u \oplus \mathcal{H}_{us}}$ and $T_1|_{\mathcal{H}_u \oplus \mathcal{H}_{su}}$ are unitary, $T_0|_{\mathcal{H}_{su}}$ and $T_1|_{\mathcal{H}_{us}}$ are shift operators, $T$ is a bi-shift on $\mathcal{H}_s$, $T$ is strongly stable on $\mathcal{H}_0$, while $T$ is a bi-isometry on $\mathcal{H}_1$ and there is no nonzero reducing subspace for $T$ of $\mathcal{H}_1$ on which either $T$ is a bi-shift, or $T_0$ is unitary or $T_1$ is unitary. Moreover, $T_0T_1$ is a shift on $\mathcal{H}_1$.

**Proof.** Clearly, $\mathcal{H}_u = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_T^*)$ and $\mathcal{H}_0 = \mathcal{N}(S_T)$ by Theorem 3.1. Denote $W = (W_0, W_1)$, $W_i = T_i|_{\mathcal{N}(I - S_T)}$, $i = 0, 1$. Since $W$ is an isometry we have (by Corollary 2.6)

\[\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{W^*}) \oplus \mathcal{N}(S_{W^*}) = \mathcal{H}_u \oplus \mathcal{N}(S_{W_0^*}W_1^*).\]

So, we infer from (3.1) that

\[\mathcal{N}(I - S_T) \cap \mathcal{N}(S_T^*) = \mathcal{N}(S_{W_0^*}W_1^*) = \bigoplus_{n \geq 0} W_0^n W_1^n \mathcal{N}(W_0^*W_1^*)\]

\[\supset \bigoplus_{n \geq 0} W_1^n \bigcap_{m \geq 0} W_0^m \mathcal{N}(W_1^*) \supset \bigoplus_{n \geq 0} W_1^n \bigcap_{m \geq 0} W_0^m \bigoplus_{j \geq 0} \mathcal{N}(W_1^*W_0^j) =: \mathcal{H}_{us}.\]

Observe that the subspace

\[\mathcal{H}_{0*} := \bigcap_{j \geq 0} \mathcal{N}(W_1^*W_0^j) \subset \mathcal{N}(W_1^*)\]

is invariant for $W_0$, so for $T_0$, and the subspace

\[\bigcap_{m \geq 0} W_0^m \mathcal{H}_{0*} = \mathcal{N}(I - S_{(T_0|_{\mathcal{H}_{0*}})^*}) \subset \mathcal{N}(W_1^*)\]

is wandering for $W_1$ and it reduces $T_0|_{\mathcal{H}_{0*}}$ to a unitary operator. Hence the subspace

\[\mathcal{H}_{us} = \bigoplus_{n \geq 0} W_1^n \mathcal{N}(I - S_{(T_0|_{\mathcal{H}_{0*}})^*}) = W_0 \bigoplus_{n \geq 0} W_1^n (W_0|_{\mathcal{H}_{0*}})^* \mathcal{N}(I - S_{(T_0|_{\mathcal{H}_{0*}})^*})\]

reduces $W_1$ to a shift, and from the second equality we get $\mathcal{H}_{us} = W_0 \mathcal{H}_{us}$, so $\mathcal{H}_{us}$ also reduces $W_0$. This implies that $\mathcal{H}_{us}$ reduces $T_1$ to a shift and $T_0$ to a unitary operator.
Similarly, if \( \mathcal{H}_{1*} := \bigcap_{j \geq 0} \mathcal{N}(W_0^*W_1^j) \) then
\[
\mathcal{H}_{su} := \bigoplus_{m \geq 0} W_0^m \mathcal{N}(I - S_{(T_1|_{\mathcal{H}_{1*}})^*}) \subset \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*})
\]
reduces \( T_0 \) to a shift and \( T_1 \) to a unitary operator. Since \( S_{W_i^*} = S_{W_i^*}^2 \), \( i = 0, 1 \), and we have
\[
\mathcal{H}_{us} \subset \mathcal{N}(I - S_{W_0^*}) \cap \mathcal{N}(S_{W_1^*}),
\]
\[
\mathcal{H}_{su} \subset \mathcal{N}(I - S_{W_1^*}) \cap \mathcal{N}(S_{W_0^*}),
\]
it follows that the subspaces \( \mathcal{H}_u, \mathcal{H}_{us} \) and \( \mathcal{H}_{su} \) are pairwise orthogonal in \( \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}) \).

Now, the subspace \( \mathcal{H}_{0*} \cap \mathcal{H}_{1*} \subset \mathcal{N}(W_0^*) \cap \mathcal{N}(W_1^*) \) is wandering for the bi-isometry \( W = (W_0, W_1) \), and the subspace
\[
\mathcal{H}_s := \bigoplus_{m, n \geq 0} W_0^m W_1^n (\mathcal{H}_0^* \cap \mathcal{H}_1^*)
\]
is invariant for \( W \), and also for \( T \). In fact,
\[
W_0 \mathcal{H}_s = \bigoplus_{m \geq 1, n \geq 0} W_0^m W_1^n (\mathcal{H}_0^* \cap \mathcal{H}_1^*) = \mathcal{H}_s \bigoplus \bigoplus_{n \geq 0} W_1^n (\mathcal{H}_0^* \cap \mathcal{H}_1^*)
\]
whence (as \( W_0^* W_1^n \mathcal{H}_{1*} = \{0\} \), \( n \geq 0 \))
\[
W_0^* \mathcal{H}_s = \mathcal{H}_s + W_0^* \left( \bigoplus_{n \geq 0} W_1^n (\mathcal{H}_0^* \cap \mathcal{H}_1^*) \right) = \mathcal{H}_s.
\]

Similarly, \( W_1^* \mathcal{H}_s = \mathcal{H}_s \), and therefore \( \mathcal{H}_s \) reduces \( W \), and so \( T \), to a bi-shift. Since \( \mathcal{H}_s \subset \mathcal{N}(S_{W_0^*}) \cap \mathcal{N}(S_{W_1^*}) \), we have
\[
\mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}) \ominus \mathcal{H}_s \supset \mathcal{N}(I - S_{W_0^*}) \cup \mathcal{N}(I - S_{W_1^*}) \supset \mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su},
\]
whence the subspace
\[
\mathcal{H}_1 := \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}) \ominus (\mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su})
\]
is also reducing for \( T \). In addition it is easy to see (as in [P]) that the subspaces \( \mathcal{H}_{us}, \mathcal{H}_{su} \) and \( \mathcal{H}_s \) are maximal with the properties quoted above. This implies that \( \mathcal{H}_1 \) contains no nonzero reducing subspace for \( T \) on which either \( T \) is a bi-shift, or \( T_0 \) is unitary, or \( T_1 \) is unitary.

Finally, since \( \mathcal{H}_1 \subset \mathcal{N}(S_{T^*}) \), \( T^*|_{\mathcal{H}_1} \) is strongly stable, that is, \( T_0 T_1|_{\mathcal{H}_1} \) is a shift, by Corollary 2.6.

Remark 3.4. The structure of the subspaces \( \mathcal{H}_{us}, \mathcal{H}_{su} \) and \( \mathcal{H}_s \) for a bi-isometry \( V \) was obtained by D. Popovici [P]. Here we describe these subspaces as well as the other from decomposition (3.3) using the context of asymptotic limits of a bicontraction \( T = (T_0, T_1) \).
COROLLARY 3.5. Let \( T = (T_0, T_1) \) be a bicontraction on \( \mathcal{H} \) with \( S_T = S_T^2 \), \( S_{T^*} = S_{T^*}^2 \) and \( \mathcal{N}(S_T) \neq \{0\} \). Then \( \mathcal{H} \) admits a unique decomposition of the form

\[
\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_s \oplus \mathcal{H}_{uc} \oplus \mathcal{H}_{cu} \oplus \mathcal{H}_c \oplus \mathcal{H}_{11} \oplus \mathcal{H}_{00},
\]

where all summands reduce \( T \) and where \( T_0|_{\mathcal{H}_u} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{uc} \) and \( T_1|_{\mathcal{H}_u} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{cu} \) are unitary, \( T_0|_{\mathcal{H}_{su}} \) and \( T_1|_{\mathcal{H}_{us}} \) are shifts, \( T_0|_{\mathcal{H}_{uc}} \) and \( T_1|_{\mathcal{H}_{cu}} \) are coisometries, \( T \) and \( T^* \) are strongly stable on \( \mathcal{H}_{00} \), and there is no nonzero reducing subspace for \( \mathcal{H}_{11} \) on which either \( T_0 \) or \( T_1 \) is unitary, or \( T \) or \( T^* \) is a bi-shift.

In addition, \( T_i|_{\mathcal{H}_{11}} = Z_i \oplus Z_i' \) on \( \mathcal{H}_{11} = \mathcal{H}_1 \oplus \mathcal{H}_1' \) where \( Z_i \) are isometries and \( Z_0 Z_1 \) is a shift on \( \mathcal{H}_1 \), while \( Z_i' \) are coisometries, and \( Z_0' Z_1' \) is a co-shift on \( \mathcal{H}_1' \) for \( i = 0, 1 \).

Proof. By Theorem 3.3 for the bi-isometry \( W \) and the bicontraction \( W' \) (\( W, W' \) as in the proof of Theorem 3.1) we have

\[
\mathcal{N}(I - S_T) = \mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_s \oplus \mathcal{H}_1,
\]

and respectively

\[
\mathcal{N}(S_T) = \mathcal{H}_0 = \mathcal{H}_{uc} \oplus \mathcal{H}_{cu} \oplus \mathcal{H}_c \oplus \mathcal{H}_1' \oplus \mathcal{H}_{00}.
\]

Here \( \mathcal{H}_{00} = \mathcal{N}(S_T) \cap \mathcal{N}(S_{T^*}) \), \( \mathcal{H}_1' \) contains no nonzero reducing subspaces for \( T \) on which either \( T^* \) is a bi-shift, or the coisometries \( T_0 \) or \( T_1 \) are unitary, and in addition, \( T \) is strongly stable, that is, \( T_0 T_1 \) is a co-shift on \( \mathcal{H}_1' \). Clearly, the other subspaces of \( \mathcal{N}(S_T) \) have the meaning from (3.4) for the bi-isometry \( T^* \). So, putting \( \mathcal{H}_{11} = \mathcal{H}_1 \oplus \mathcal{H}_1' \) we get the decomposition (3.4) of \( \mathcal{H} = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T) \), in view of (3.1) and (3.2).

Since \( \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_s \subset \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}) \) we have necessarily

\[
\mathcal{H}_{us} \subset \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T_1^*})
= \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_T) \cap \mathcal{N}(S_T),
\]

(3.5)

\[
\mathcal{H}_{su} \subset \mathcal{N}(I - S_{T_1^*}) \cap \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T_0^*})
= \mathcal{N}(I - S_{T_1^*}) \cap \mathcal{N}(I - S_T) \cap \mathcal{N}(S_T),
\]

(3.6)

\[
\mathcal{H}_s \subset \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*}),
\]

(3.7)

but the inclusions may be strict, as in Remark 3.9 below.

By Theorem 3.1 of [KO] we also get the following

COROLLARY 3.6. Let \( T = (T_0, T_1) \) be a bicontraction on \( \mathcal{H} \). Then there exist a unique minimal Hilbert space \( \mathcal{K} \supset \mathcal{H} \) and a bicontraction \( \bar{T} = (\bar{T}_0, \bar{T}_1) \) on \( \mathcal{K} \) extending \( T \) (i.e. such that \( T|_{\mathcal{H}} = T \)) and admitting a unique decomposition of the form given in Theorem 3.3.
We find now when these inclusions become equalities. Clearly, we can reduce this problem to the case of a bi-isometry (by \((3.3)\)).

**Proposition 3.7.** Let \(T = (T_0, T_1)\) be a bi-isometry on \(H\). Then

(i) \(H_{us} = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})\) and \(H_{su} = \mathcal{N}(I - S_{T_1^*}) \cap \mathcal{N}(S_{T_0^*})\) if and only if \(H_s \oplus H_1 = \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})\), where \(H_1\) is the subspace appearing in decomposition \((3.3)\). In this case, \(H_s \oplus H_1\) is the maximum subspace which reduces \(T_i\) \((i = 0, 1)\) to a shift.

(ii) \(H_s = \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})\) and \(H_1 = \{0\}\) if and only if \(T_0\) and \(T_1\) doubly commute.

**Proof.** Suppose we have equalities in \((3.5)\) and \((3.6)\), where \(\mathcal{N}(I - S_T) = \mathcal{H}\). Since \(T_0\) is a shift on \(H_{su}\), that is, \(T_0^* h \to 0\) \((n \to \infty)\) for \(h \in H_{su}\), we have \(H_{su} \subset \mathcal{N}(S_{T_0^*})\). Thus, since \(S_{T_0^*} = S_{T_0^*}^2\) and \(S_{T_1^*} = S_{T_1^*}^2\) \((T\) is a bi-isometry), we get the decompositions

\[
\mathcal{H} = \mathcal{N}(I - S_{T_0^*}) \oplus \mathcal{N}(S_{T_0^*}) = \mathcal{N}(I - S_{T_1^*}) \oplus \mathcal{N}(I - S_{T_1^*}) \oplus \mathcal{H}_{su} \oplus [\mathcal{N}(S_{T_0^*}) \oplus \mathcal{H}_{su}]
\]

Then from \((3.3)\) we infer (as \(H_0 = \mathcal{N}(S_T) = \{0\}\) in this case) that \(H_s \oplus H_1 = \mathcal{N}(S_{T_0^*}) \oplus H_{su}\), or \(\mathcal{N}(S_{T_1^*}) = H_{su} \oplus H_s \oplus H_1\). By symmetry we also have \(\mathcal{N}(S_{T_0^*}) = H_{us} \oplus H_s \oplus H_1\), and so

\[
H_s \oplus H_1 \subset \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*}) =: \mathcal{H}_{ss}.
\]

Now if \(h \in \mathcal{H}_{ss}\) and we write \(h = h_1 \oplus h_0 = h_2 \oplus h_0'\) with \(h_1 \in H_{us}\), \(h_2 \in H_{su}\) and \(h_0, h_0' \in H_s \oplus H_1\), then \(h_1 \oplus (-h_2) \oplus (h_0 - h_0') = 0\), hence \(h_1 = h_2 = 0\) and \(h_0 = h_0'\). This implies \(h = h_0 \in H_s \oplus H_1\), and we conclude that \(H_s \oplus H_1 = H_{ss}\). Clearly, in this case the subspace \(H_{ss}\) reduces \(T_i\) \((i = 0, 1)\) to a shift, and it contains any other subspace of \(H\) with this property.

Conversely, assume that \(H_s \oplus H_1 = \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})\). Then as above we get the decomposition

\[
\mathcal{H} = H_u \oplus [\mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*})] \oplus \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*}) + H_s \oplus H_1,
\]

and from \((3.3)\) we infer that

\[
H_{us} \oplus H_{su} = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*}) \oplus \mathcal{N}(S_{T_0^*}) \cap [\mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})] \perp.
\]

Since \(H_{us} \subset \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})\) and \(H_{su} \subset \mathcal{N}(I - S_{T_1^*}) \cap \mathcal{N}(S_{T_0^*})\) (by \((3.3)\)), the preceding equality leads to \(H_{us} = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})\) and also (because \(S_{T_1^*} = S_{T_1^*}^2\))

\[
H_{su} = \mathcal{N}(S_{T_0^*}) \cap [\mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(S_{T_1^*})] \perp \cap \mathcal{N}(S_{T_0^*}) \cap \mathcal{N}(I - S_{T_1^*}),
\]

hence \(H_{su} = \mathcal{N}(I - S_{T_0^*}) \cap \mathcal{N}(S_{T_0^*})\). This completes the proof of (i).
For (ii) it is clear that if $\mathcal{H}_s = \mathcal{N}(S_{T_0}^1) \cap \mathcal{N}(S_{T_1}^1)$ and $\mathcal{H}_1 = \{0\}$ then $T_0$ and $T_1$ doubly commute on $\mathcal{H}_s$, and finally, they doubly commute on $\mathcal{H} = \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_s$ ($T$ being a bi-isometry).

Conversely, if $T_0 T_1^* = T_1 T_0$ then $\mathcal{N}(I - S_{T_0}^1)$ and $\mathcal{N}(S_{T_1}^1)$ reduce $T_{1-i}$, and so $\mathcal{N}(I - S_{T_1}^1) \cap \mathcal{N}(S_{T_0}^1)$ reduces $T_i$ (resp. $T_{i-1}$) to a unitary (resp. shift) operator, for $i = 0, 1$. Thus, it is needed that $\mathcal{H}_{us} = \mathcal{N}(I - S_{T_0}^1) \cap \mathcal{N}(S_{T_1}^1)$ and $\mathcal{H}_{su} = \mathcal{N}(I - S_{T_1}^1) \cap \mathcal{N}(S_{T_0}^1)$, which gives $\mathcal{H}_s \oplus \mathcal{H}_1 = \mathcal{N}(S_{T_0}^1) \cap \mathcal{N}(S_{T_1}^1)$.

But, in this case we have $\mathcal{H}_s = \mathcal{N}(S_{T_0}^1) \cap \mathcal{N}(S_{T_1}^1)$ because $T_0$ and $T_1$ doubly commute on $\mathcal{H}_s \oplus \mathcal{H}_1$, hence $\mathcal{H}_1 = \{0\}$. This ends the proof. □

**Remark 3.8.** In fact, this proposition shows that a bi-isometry $T = (T_0, T_1)$ on $\mathcal{H}$ induces an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{ss},$$

where the subspaces have the above meaning, if and only if $\mathcal{H}_{us} = \mathcal{N}(I - S_{T_0}^1) \cap \mathcal{N}(S_{T_1}^1)$ and $\mathcal{H}_{su} = \mathcal{N}(I - S_{T_1}^1) \cap \mathcal{N}(S_{T_0}^1)$, while in this case $\mathcal{H}_{ss} = \mathcal{N}(S_{T_0}^1) \cap \mathcal{N}(S_{T_1}^1)$. Hence $\mathcal{H}_{ss}$ reduces $T_0$ and $T_1$ to shift operators and it is the maximum subspace with this property.

Recall that the decomposition (3.8) is known as the Słociński decomposition (see [Sl]). Moreover in (3.8) we have $\mathcal{H}_{ss} = \mathcal{H}_s$ if and only if $T_0$ and $T_1$ doubly commute.

**Remark 3.9.** In Example 1 of [GS] a bi-isometry $T$ was given for which $\mathcal{H}_{us} = \mathcal{N}(I - S_{T_0}^1) \subset \mathcal{N}(S_{T_1}^1)$ and $\mathcal{H}_{su} \oplus \mathcal{H}_1 = \mathcal{N}(S_{T_0}^1)$ with $\mathcal{N}(S_{T_0}^1) \cap \mathcal{N}(S_{T_1}^1) = \{0\} = \mathcal{H}_u$. In view of the above strict inclusion, $\mathcal{N}(I - S_{T_1}^1) \subset \mathcal{H}_{su} \oplus \mathcal{H}_1$ and also $\mathcal{H}_1 \neq \{0\}$ because otherwise we get $\mathcal{H}_{su} = \mathcal{N}(I - S_{T_1}^1)$, a contradiction. So $\mathcal{H}_{us} \subset \mathcal{N}(I - S_{T_1}^1) = \mathcal{N}(I - S_{T_0}^1) \cap \mathcal{N}(S_{T_1}^1)$, even if $\mathcal{H}_{us} = \mathcal{N}(I - S_{T_0}^1) \cap \mathcal{N}(S_{T_1}^1)$, hence $T$ does not have a Słociński decomposition (3.8).

**Remark 3.10.** Consider the bicontraction $T = (T_0, T_1)$ on $\mathcal{K}$ from Example 2.9. Since $S_T = 0$, $T$ is strongly stable on $\mathcal{K}$. On the other hand, as $T_0, T_1$ are quasinormal, by Theorem 2.3 we have $S_{T^*} = S_{T^*_0}^2$ and $\mathcal{R}(S_{T^*_0}) \subset \mathcal{R}(S_T) = \{0\}$, that is, $S_{T^*_0} = \{0\}$. Hence $T^*$ is strongly stable on $\mathcal{K}$ and we have $\mathcal{K} = \mathcal{N}(S_T) = \mathcal{N}(S_{T^*_0}) = \mathcal{K}_00$ in the corresponding decomposition (3.4).

### 4. Remarks on invariant subspaces for bicontractions

To every bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ one can associate a bi-isometry $V = (V_0, V_1)$ on $\overline{\mathcal{R}(S_T)}$ such that

$$V_i S_{T_i}^{1/2} h = S_{T_i}^{1/2} T_i h \quad (h \in \mathcal{H}, \ i = 0, 1).$$

Clearly, $V_i$ is an isometry ($T_i$ being an $S_T$-isometry), and $V_0 V_1 = V_1 V_0$ because $T_0 T_1 = T_1 T_0$. Since $\mathcal{N}(S_T)$ is invariant for $S_{T_i}^{1/2} T_i$, $\overline{\mathcal{R}(S_T)}$ is invariant.
for $T_i^*S^{1/2}_T$, and the above definition of $V_i$ implies
\begin{equation}
S^{1/2}_T V_i^* k = T_i^* S^{1/2}_T k \quad (k \in \overline{R(S_T)}, \ i = 0, 1).
\end{equation}

This relation gives $V_i S_T V_i^* \leq S_T$ on $\overline{R(S_T)}$, hence $V_i^*$ is an $\widehat{S}_T$-contraction $(i = 0, 1)$, where $\widehat{S}_T = S_T |_{\overline{R(S_T)}}$. Other properties of $V$ are summarized in

**Proposition 4.1.** Let $T = (T_0, T_1)$ be a bicontraction on $H$ and $V = (V_0, V_1)$ be the bi-isometry on $\overline{R(S_T)}$ associated to $T$ as in (4.1). Then
\begin{equation}
\lim_{m,n \to \infty} V_0^m V_1^n \widehat{S}_T V_1^* V_0^m k = \lim_{m,n \to \infty} V_0^m V_1^n \widehat{S}_T^{1/2} V_1^* V_0^m k = k
\end{equation}
and
\begin{equation}
\lim_{m,n \to \infty} V_0^m V_1^n \widehat{S}_T V_1^* V_0^m k = \lim_{n \to \infty} V_1^n S_T^{1/2} S_T^* S_T^{1/2} V_1^* k = S_T^{1/2} S_T^* S_T^{1/2} k
\end{equation}
for every $k \in \overline{R(S_T)}$ and $i = 0, 1$, where the operator limit in (4.4) is considered as acting on $\overline{R(S_T)}$. Moreover, the operator $S_T^{1/2} S_T^* S_T^{1/2}$ commutes with $V_0$ and $V_1$ and $\mathcal{R}(S_T^{1/2} S_T^* S_T^{1/2})$, as a subspace of $\overline{R(S_T)}$, reduces $V_0$ and $V_1$ to unitary operators.

**Proof.** For every $k \in S_T^{1/2} h$ with $h \in H$ and any integers $m, n \geq 1$,
\begin{align*}
\| I - V_0^m V_1^n \widehat{S}_T V_1^* V_0^m k \| &= \| V_0^m V_1^n S_T^{1/2} (I - S_T) T_0^m T_1^n h \|^2 \\
&\leq \| (I - S_T)^{1/2} T_0^m T_1^n h \|^2 = \| T_0^m T_1^n h \|^2 - \| S_T^{1/2} T_0^m T_1^n h \|^2 \to 0
\end{align*}
as $m, n \to \infty$. Since $0 \leq I - S_T^{1/2} \leq I - S_T$ we get as above
\begin{align*}
\| I - V_0^m V_1^n \widehat{S}_T^{1/2} V_1^* V_0^m k \|^2 &\leq \| (I - S_T^{1/2})^{1/2} T_0^m T_1^n h \|^2 \\
&\leq \| (I - S_T)^{1/2} T_0^m T_1^n h \|^2 \to 0
\end{align*}
as $m, n \to \infty$. So, the first equality of (4.3) holds for every $k \in \overline{R(S_T)}$ (the corresponding sequences are bounded).

Now from (4.1) and (4.2) we obtain
\begin{equation*}
V_0^m V_1^n \widehat{S}_T V_1^* V_0^m k = S_T^{1/2} T_0^m T_1^n T_0^* T_1^* S_T^{1/2} k \to S_T^{1/2} S_T^* S_T^{1/2} k
\end{equation*}
as $m, n \to \infty$, for any $k \in \overline{R(S_T)}$, which proves the second equality of (4.4). Obviously, $\overline{R(S_T)}$ reduces the operator $S_T^{1/2} S_T^* S_T^{1/2}$ (which is self-adjoint), so this operator can be considered in $\mathcal{B}(\overline{R(S_T)})$. On the other hand, since
\begin{equation*}
V_i^m \widehat{S}_T V_i^* V_i^m k = S_T^{1/2} T_i^m T_i^* S_T^{1/2} k \to S_T^{1/2} S_T^* S_T^{1/2} k
\end{equation*}
as $m \to \infty$, we have (by the previous remark)
\begin{equation*}
S_T^{1/2} S_T^* S_T^{1/2} k = \lim_{n \to \infty} V_1^n S_T^{1/2} S_T^* S_T^{1/2} V_1^* k
\end{equation*}
for $k \in \overline{R(S_T)}$ and $i = 0, 1$. So, the first equality of (4.4) holds true.
For the last assertion notice that by (4.1) and (4.2), $V_i^*$ is a $S_T^{1/2}S_T^*S_T^{1/2}$-isometry, that is, $V_i^*S_T^{1/2}S_T^*S_T^{1/2}V_i^* = S_T^{1/2}S_T^*S_T^{1/2}$, because $T_i^*$ is an $S_T^*$-isometry, $i = 0, 1$. This also implies

$$S_T^{1/2}S_T^*S_T^{1/2}V_i^* = V_i^*S_T^{1/2}T_iS_T^*S_T^{1/2} = V_i^*S_T^{1/2}S_T^*S_T^{1/2},$$

which means that $S_T^{1/2}S_T^*S_T^{1/2}$ commutes with $V_i$ for $i = 0, 1$. This ensures that the range

$$\mathcal{R}(S_T^{1/2}S_T^*S_T^{1/2}) = \frac{\mathcal{R}(S_T^{1/2}S_T^*S_T^{1/2})}{\mathcal{R}(S_T^{1/2}S_T^*S_T)}$$

as a subspace of $\overline{\mathcal{R}(S_T)}$ reduces $V_0$ and $V_1$. Since from the second equality of (4.4) it follows that

$$\mathcal{R}(S_T^{1/2}S_T^*S_T^{1/2}) \subset \bigcap_{m \geq 0} \mathcal{R}(V_0^m) \cap \bigcap_{n \geq 0} \mathcal{R}(V_1^n) = \mathcal{N}(I - S_{V_0^*}) \cap \mathcal{N}(I - S_{V_1^*}),$$

we infer that $V_0$ and $V_1$ are unitary on $\mathcal{R}(S_T^{1/2}S_T^*S_T^{1/2})$. □

**Remark 4.2.** From (4.1) one can get the polar decomposition of $S_T^{1/2}T_i$ ($i = 0, 1$). Note $|S_T^{1/2}T_i| = S_T^{1/2}$, and put $\tilde{V}_i = J V_i P$ where $P$ is the projection of $\mathcal{H}$ onto $\mathcal{R}(S_T)$ and $J = P^*$ is the canonical embedding of $\mathcal{R}(S_T)$ into $\mathcal{H}$. Clearly, $\tilde{V}_i$ isometrically maps $\mathcal{R}(S_T) = \mathcal{N}(S_T) = \mathcal{N}(S^{1/2}_T T_i)$ onto $\mathcal{R}(\tilde{V}_i) \subset \mathcal{R}(S_T^{1/2} T_i) \subset \mathcal{R}(S_T)$, and

$$\mathcal{N}(\tilde{V}_i) = \mathcal{N}(P) = \mathcal{N}(S_T) = \mathcal{N}(S_T^{1/2} T_i).$$

Hence $\tilde{V}_i$ is a partial isometry in $\mathcal{B}(\mathcal{H})$, and the polar decomposition of $S_T^{1/2} T_i$ is $S_T^{1/2} T_i = \tilde{V}_i S_T^{1/2}$, while $\tilde{V}_i$ is even an extension of $V_i$, for $i = 0, 1$.

Observe also that for a bicontraction $T^* = (T_0^*, T_1^*)$ there are isometries $V_{*0}, V_{*1} \in \mathcal{B}(\overline{\mathcal{R}(S_T^*)})$ which satisfy

$$V_{*i} S_T^{1/2} k = S_T^{1/2} T_{*i}^* k \quad (k \in \overline{\mathcal{R}(S_T^*)}, \ i = 0, 1).$$

Recall that two bicontractions $T = (T_0, T_1)$ on $\mathcal{H}$ and $S = (S_0, S_1)$ on $\mathcal{K}$ are *similar* if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ satisfying $A T_i = S_i A$, $i = 0, 1$. If $A$ belongs to $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is only densely defined, i.e. $\mathcal{R}(A) = \mathcal{K}$ with $\mathcal{N}(A) = \{0\}$ and $A$ intertwines $T_i$ with $S_i$ ($i = 0, 1$), one says that $T$ is a *quasiaffine transform* of $S$. Finally, $T$ is *quasisimilar* to $S$ if $T$ and $S$ are quasiaffine transforms of each other.

As in the case of a single contraction (see [K]), we can characterize these concepts using the asymptotic limit operators $S_T$ and $S_{T^*}$.

We first give the following
Lemma 4.3. Let \( T = (T_0, T_1) \) be a bicontraction on \( \mathcal{H} \) such that \( \mathcal{N}(S_T) = (S_T^*) = \{0\} \). Then for \( i = 0, 1 \) we have

\[
\begin{align*}
V_i & S_{T_i}^{1/2} S_{T_i}^{1/2} = S_{T_i}^{1/2} S_{T_i}^{1/2} V_i^*; \\
S_{T_i}^{1/2} S_{T_i}^{1/2} V_i & = V_i^* S_{T_i}^{1/2} S_{T_i}^{1/2}, \\
S_{T_i}^{1/2} V_i & = T_i S_{T_i}^{1/2} S_{T_i}^{1/2}, \\
S_{T_i} S_{T_i} V_i & = T_i S_{T_i} S_{T_i}^{1/2};
\end{align*}
\]

Proof. The hypothesis implies \( \mathcal{H} = \mathcal{R}(S_T) = \mathcal{R}(S_{T_i}) \), so \( V_i \) and \( V_{si} \) are isometries on \( \mathcal{H} \). Then by (4.1) and (4.5) we get

\[
V_i S_{T_i}^{1/2} S_{T_i}^{1/2} = S_{T_i}^{1/2} T_i S_{T_i}^{1/2} = S_{T_i}^{1/2} S_{T_i}^{1/2} V_i^*;
\]

that is, (4.6). By duality we have \( V_{si} S_{T_i}^{1/2} S_{T_i}^{1/2} = S_{T_i}^{1/2} T_i S_{T_i}^{1/2} V_i^* \), whence one obtains (4.7). Now from (4.7) it follows that

\[
S_{T_i} S_{T_i}^{1/2} V_i = S_{T_i}^{1/2} V_i^* S_{T_i}^{1/2} S_{T_i}^{1/2} = (V_{si} S_{T_i}^{1/2})^* S_{T_i}^{1/2} S_{T_i}^{1/2} = T_i S_{T_i} S_{T_i}^{1/2};
\]

that is, (4.8), while (4.9) is immediate from (4.8). \( \blacksquare \)

Theorem 4.4. If \( T \) is a bicontraction on \( \mathcal{H} \) then:

(i) \( T \) is similar to a bi-isometry if and only if \( S_T \) is invertible.

(ii) \( T \) is similar to a unitary bicontraction if and only if \( S_T \) and \( S_{T_i} \) are invertible.

(iii) \( T \) is quasisimilar to a unitary bicontraction if and only if

\[
\mathcal{N}(S_T) = \mathcal{N}(S_{T_i}) = \{0\}.
\]

Proof. (i) If \( S_T \) is invertible then \( T \) is similar via \( S_T \) to the bi-isometry \( V = (V_0, V_1) \) given in (4.1). Conversely, suppose that \( T \) is similar to a bi-isometry \( S = (S_0, S_1) \) on \( \mathcal{K} \) via an invertible operator \( A \) from \( \mathcal{H} \) onto \( \mathcal{K} \). Let \( A = QA \) be the polar decomposition of \( A \), with \( Q \) unitary and \( |A| \) invertible. Since \( AT_i = S_i A \) we get \( S_i = Q|A|T_i|A|^{-1}Q^* \), whence \( |A|T_i = Q^* S_i Q |A| = W_i |A| \) where \( W_i = Q^* S_i Q \) is an isometry, \( i = 0, 1 \). It follows that \( |A| = W_i^* |A|T_i \), and also \( W_i = |A|T_i |A|^{-1} \), and both give \( A^* A = |A|^2 = T_i^* A^* A T_i \), for \( i = 0, 1 \). This forces that \( A^* A \leq S_T \), hence \( S_T \) is invertible.

(ii) The previous remark implies that if \( T \) is similar to a unitary bicontraction then \( S_T \) and \( S_{T_i} \) are invertible.

Conversely, assume that \( S_T \) and \( S_{T_i} \) are invertible, so \( AT_i = S_i A \) as above, and \( BT_i^* = S_{si} B \) where \( S_{si} \) are isometries on \( \mathcal{G} \) and \( B \in \mathcal{B}(\mathcal{H}, \mathcal{G}) \) is invertible. Since \( T_i = B^* S_{si}(B^*)^{-1} \) we get \( S_i A = AB^* S_{si}^* (B^*)^{-1} \) where \( S_{si}^* \) is a coisometry, therefore it is surjective. This yields \( \mathcal{R}(S_i) = \mathcal{K} \), that is, \( S_i \) is unitary, \( i = 0, 1 \). Hence \( T \) is similar to the unitary bicontraction \( S \).

(iii) Suppose that \( T \) is quasisimilar to \( U = (U_0, U_1) \) where \( U_i \) are unitary operators on \( \mathcal{K} \), \( i = 0, 1 \). If \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) is such that \( \mathcal{R}(A) = \mathcal{H}, \mathcal{N}(A) = \{0\} \)
and $AT_i = U_i A$ $(i = 0, 1)$ then $AT_i^m T^n_i = U_0^m U_1^n A$ for $m, n \in \mathbb{N}$. So, for $h \in \mathcal{N}(S_T)$ we have $T_0^m T_1^n h \to 0$ $(m, n \to \infty)$, hence $U_0^m U_1^n A h \to 0$ $(m, n \to \infty)$, which gives $Ah = 0$ and $h = 0$, too. Thus $\mathcal{N}(S_T) = \{0\}$, and similarly, since $U$ is a quasiaffine transform of $T$, $\mathcal{N}(S_{T^*}) = \{0\}$.

Conversely, assume that $\mathcal{N}(S_T) = \mathcal{N}(S_{T^*}) = \{0\}$, therefore $\mathcal{R}(S_T) = \mathcal{R}(S_{T^*}) = \mathcal{H}$. We infer that $\mathcal{N}(S_{T^*} S_T^{1/2}) = \{0\}$ and also $\mathcal{R}(S_{T^*} S_T^{1/2}) = \mathcal{H}$. By (4.1) and (4.8) and the previous remarks we conclude that $T$ is quasisimilar to $(V_0, V_1)$, and it remains to see that $V_0$ and $V_1$ are unitary. Indeed, since $\mathcal{N}(T_i^*) \subset \mathcal{N}(S_{T^*}) = \{0\}$ one has $\mathcal{N}(T_i^*) = \{0\}$. But by (4.2) we have $S_T^{1/2} \mathcal{N}(V_i^*) \subset \mathcal{N}(T_i^*)$, hence $\mathcal{N}(V_i^*) = \{0\}$, which means that $V_i$ is unitary, $i = 0, 1$. ■

As in the case of a single contraction, the above results can be used to make some remarks on the invariant subspaces of a bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$. Obviously, an invariant subspace of $T$ means a jointly invariant subspace of $T_0$ and $T_1$.

**Theorem 4.5.** The following statements hold for every bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$:

(i) If $\mathcal{N}(S_T) = \mathcal{N}(S_{T^*}) = \{0\}$ then either $T_0$ and $T_1$ are unitary scalar, or $T$ has nontrivial invariant subspaces which are hyperinvariant for $T_0$ or $T_1$.

(ii) If $S_T \neq 0$ and $S_{T^*} \neq 0$ then either $T_0$ and $T_1$ are unitary scalar, or $T$ has nontrivial invariant subspaces which are invariant for any operator which commutes with $T_0$ and $T_1$.

**Proof.** (i) The assumption of (i) ensures, by Theorem 4.4, that $T$ is quasisimilar to a bicontraction $U = (U_0, U_1)$ with $U_1$ unitary. If $U_0$ (or $U_1$) is nonscalar then $U_0$ (resp. $U_1$) has nontrivial hyperinvariant subspaces, and by [K, Corollary 4.8] it follows that $T_0$ (resp. $T_1$) has nontrivial hyperinvariant subspaces. Hence $T$ has nontrivial invariant subspaces, as in the case considered before. In the other case, one has $U_i = \lambda_i I$ with $|\lambda_i| = 1$, and since $T_i$ is a quasiaffine transform of $U_i$ by an injective operator, we infer $T_i = \lambda_i I$, $i = 0, 1$. Clearly, when $\dim \mathcal{H} > 1$, any nontrivial subspace of $\mathcal{H}$ is invariant for $T$.

Note also that $\mathcal{N}(S_{T_i}) = \mathcal{N}(S_{T_i^*}) = \{0\}$ for $i = 0, 1$ by the hypothesis of (i). Thus, one can directly apply [K, Corollary 4.11] for $T_i$ $(i = 0, 1)$ to obtain the conclusion of (i).

(ii) The assumption of (ii) gives $\mathcal{H} \neq \mathcal{N}(S_T)$ and $\mathcal{H} \neq \mathcal{N}(S_{T^*})$. So, if $\mathcal{N}(S_T) \neq \{0\}$ then $\mathcal{N}(S_T)$ is a nontrivial invariant subspace for $T$. Since $h \in \mathcal{N}(S_T)$ iff $T_0^m T_1^n h \to 0$ $(m, n \to \infty)$, it follows that $\mathcal{N}(S_T)$ is also invariant for any operator which commutes with $T_0$ and $T_1$. 


If \( \mathcal{N}(S_{T^*}) \neq \{0\} \) then, as above, \( \mathcal{N}(S_{T^*}) \) is a nontrivial invariant subspace for \( T^* \) and, also, for any operator that commutes with \( T^*_0 \) and \( T^*_1 \). In this case, \( \overline{\mathcal{R}(S_{T^*})} \) is a nontrivial invariant subspace for \( T \), which remains invariant for any commutant of \( T_0 \) and \( T_1 \).

The other case, namely \( \mathcal{N}(S_T) = \mathcal{N}(S_{T^*}) = \{0\} \), was discussed in (i).

**Corollary 4.6.** Let \( T \) be a bicontraction on \( \mathcal{H} \) which has no nontrivial invariant subspace. Then either \( T \) or \( T^* \) is strongly stable on \( \mathcal{H} \). More precisely, either \( T \) and \( T^* \) are strongly stable, or \( T \) is strongly stable and \( 0 < \|S_T h\| < \|h\| \) for all nonzero \( h \in \mathcal{H} \), or \( T^* \) is strongly stable and \( 0 < \|S_T h\| < \|h\| \) for all nonzero \( h \in \mathcal{H} \).

**Proof.** By the previous theorem, \( T \) has no nontrivial invariant subspaces iff \( S_T = 0 \) or \( S_{T^*} = 0 \), equivalently \( \mathcal{N}(S_T) = \mathcal{H} \) or \( \mathcal{N}(S_{T^*}) = \mathcal{H} \). When this happens, we also have \( \mathcal{H} = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T) \) or \( \mathcal{H} = \mathcal{N}(I - S_{T^*}) \oplus \mathcal{N}(S_{T^*}) \), that is, \( \mathcal{N}(I - S_T) = \{0\} \) or \( \mathcal{N}(I - S_{T^*}) = \{0\} \). Hence only the following cases are admissible:

(a) \( \mathcal{H} = \mathcal{N}(S_T) = \mathcal{N}(S_{T^*}) \) which means that \( T \) and \( T^* \) are strongly stable,

(b) \( \mathcal{H} = \mathcal{N}(S_T) \) and \( \mathcal{N}(S_{T^*}) = \mathcal{N}(I - S_{T^*}) = \{0\} \), so \( T \) is strongly stable and \( 0 < \|S_T h\| < \|h\| \) for \( 0 \neq h \in \mathcal{H} \),

(c) \( \mathcal{H} = \mathcal{N}(S_{T^*}) \) and \( \mathcal{N}(S_T) = \mathcal{N}(I - S_T) = \{0\} \), meaning that \( T^* \) is strongly stable and \( 0 < \|S_T h\| < \|h\| \) for \( 0 \neq h \in \mathcal{H} \).

In the usual terminology (which also appears in [KO]), a bicontraction \( T \) belongs to the class \( C_0 \) (resp. \( C_1 \)) if \( \mathcal{N}(S_T) = \mathcal{H} \) (resp. \( \mathcal{N}(S_T) = \{0\} \)). Also, \( T \) belongs to \( C_0 \) (resp. \( C_1 \)) if \( T^* \) belongs to \( C_0 \) (resp. \( C_1 \)). For \( \alpha, \beta \in \{0, 1\} \), the class \( C_{\alpha\beta} \) is defined as \( C_\alpha \cap C_\beta \). Thus, Theorem 4.5 shows that any bicontraction of class \( C_{11} \) has nontrivial invariant subspaces, while Corollary 4.6 implies that every bicontraction without nontrivial invariant subspaces belongs to \( C_{01} \) or \( C_{10} \). Concerning these latter classes, the following fact can also be proved.

**Theorem 4.7.** Every bicontraction that does not belong to the class \( C_{00} \) has nontrivial invariant subspaces if and only if every bicontraction which is a quasiaffine transform of a unitary bicontraction has nontrivial invariant subspaces.

**Proof.** Let \( T = (T_0, T_1) \) be a bicontraction such that either \( T \) or \( T^* \) is not strongly stable, that is, \( S_T \neq 0 \) or \( S_{T^*} \neq 0 \). Suppose that \( T \) has no nontrivial invariant subspace, and firstly that \( S_T \neq 0 \). This forces \( \mathcal{N}(S_T) = \{0\} \) and hence \( \mathcal{N}(T_i) = \{0\} \), so \( T_i \neq 0 \) for \( i = 0, 1 \). Since \( (I - V_i V_i^*) S_T^{1/2} T_i = 0 \), \( V_i \) being given by (4.1), the assumption on \( T \) implies \( (I - V_i V_i^*) S_T^{1/2} = 0 \), \( i = 0, 1 \) (otherwise, \( \overline{\mathcal{R}(T_i)} \) is a nontrivial invariant subspace of \( T \)). As \( \overline{\mathcal{R}(S_T)} = \mathcal{H} \) it
follows that $V_i$ is unitary for $i = 0, 1$, hence $T$ is a quasi-affine transform by (4.1) of the unitary bicontraction $V = (V_0, V_1)$. By duality, in the case $S_{T^*} \neq 0$ it follows that $T^*$ is a quasi-affine transform of the unitary bicontraction $V_* = (V_{*0}, V_{*1})$ given in (4.5). We proved that, under the cited assumption on $T$, there exist bicontractions (either $T$ or $T^*$) without nontrivial invariant subspaces, that are quasi-affine transforms of unitary bicontractions.

Conversely, let $T$ be a bicontraction on $\mathcal{H}$ which is a quasi-affine transform of a unitary bicontraction $U = (U_0, U_1)$ on $\mathcal{K}$ by an operator $A \in B(\mathcal{H}, \mathcal{K})$, such that $T$ has no nontrivial invariant subspaces. Assuming that $T$ is strongly stable, that is, $\mathcal{N}(S_T) = \mathcal{H}$, we get, for $0 \neq h \in \mathcal{H}$,

$$\|Ah\| = \|U_0^m U_1^n Ah\| = \|AT_0^m T_1^n h\| \to 0 \quad (m, n \to \infty),$$

which yields $h = 0$ ($A$ being injective), a contradiction. Hence $T$ is not strongly stable, in particular, $T$ is not in the class $C_{00}$.

Note that Corollary 4.6 and Theorem 4.7 are direct extensions of [K] Corollary 5.9 and Theorem 4.14.

Finally, notice that some of the above facts concerning invariant subspaces for bicontractions are known (even for multicontractions) and obtained by a different method (see e.g. [KO] Theorems 2.2 and 2.3). Here we pointed out the role of asymptotic limit operators in the above problems, which is similar to the case of a single contraction (see [K]).

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