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Reduction of semialgebraic constructible functions

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In memory of Stanisław Łojasiewicz (1926–2002)

Abstract. Let R be a real closed field with a real valuation v. A $\mathbb{Z}$-valued semialgebraic function on R^n is called algebraic if it can be written as the sign of a symmetric bilinear form over $R[X_1, \dots, X_n]$. We show that the reduction of such a function with respect to v is again algebraic on the residue field. This implies a corresponding result for limits of algebraic functions in definable families.

Introduction. One of the most fundamental results in real algebraic geometry is the theorem of Tarski–Seidenberg, stating that if $S \subset \mathbb{R}^n$ is semialgebraic and $f : \mathbb{R}^n \to \mathbb{R}^k$ is a polynomial map, then $f(S)$ is also semialgebraic.

However, if S is algebraic, then in general $f(S)$ is no longer so. The simplest example is where $S = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$, the unit circle in $\mathbb{R}^2$, and f is the projection onto the first coordinate in $\mathbb{R}^2$. Then $f(S)$ is the unit ball $\{x \in \mathbb{R} \mid x^2 \leq 1\}$ in $\mathbb{R}$ which is obviously not algebraic. So the consideration of images is too rough in order to save information on the algebraicity of S.

Also, let $\varphi : \mathbb{R}^n \to \mathbb{Z}$ be a constructible function, that is, $\varphi(\mathbb{R}^n)$ is finite and $\varphi^{-1}(z)$ is semialgebraic for all $z \in \mathbb{Z}$. Let again $f : \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial map. It is not a priori clear how to define the image function $f_*\varphi : \mathbb{R}^k \to \mathbb{Z}$, but it turns out that a well behaved definition is

$$f_*\varphi(x) = \int_{f^{-1}(x)} \varphi \, d\chi,$$

where the right hand side is the so-called Euler integral, to be explained in Section 1. This extension of the classical Euler characteristic to constructible functions appeared in many contexts.

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Now let $\varphi = 1_S$ where S is algebraic. Then $f_*\varphi$ is algebraic in the following sense. There exist finitely many polynomials $f_1, \dots, f_m \in \mathbb{R}[X_1, \dots X_k]$ such that $f_*\varphi(x) = \operatorname{sign} f_1(x) + \cdots + \operatorname{sign} f_m(x)$ for all $x \in \mathbb{R}^k$. In the example where S is the unit circle in $\mathbb{R}^2$, we have $f_*\varphi(x) = 2$ for $x^2 < 1$, $f_*\varphi(x) = 1$ for $x^2 = 1$ and $f_*\varphi(x) = 0$ for $x^2 > 1$. Thus $f_*\varphi = \operatorname{sign} f_1 + \operatorname{sign} f_2$ for $f_1 = 1 - X^2$ and $f_2 \equiv 1$.

More generally, a beautiful theorem of Parusiński–Szafraniec [P-S] states that if $\varphi : \mathbb{R}^n \to \mathbb{Z}$ is algebraic, then so is $f_*\varphi$.

We will show a corresponding result for limits in semialgebraic families, where one has a similar situation:

Let $S \subset \mathbb{R}^n \times \mathbb{R}^k$ be semialgebraic. For $t \in \mathbb{R}^k$ let $S_t := S \cap \mathbb{R}^n \times \{t\}$. Let $(t_m) \in \mathbb{R}^k$ be a sequence such that (S_{t_n}) tends to T in the Hausdorff sense. It is known that T is again semialgebraic [Br1], [Br2], which holds correspondingly in the o-minimal context [L-S], [vdD1], [vdD2]. Again, if S is algebraic, this need not be the case for T.

For instance, consider the family of ellipses $E_t \subset \mathbb{R}^2$ where

$$E_t = \{x \in \mathbb{R}^2 \mid x_1^2 + t^2 x_2^2 = 1\},$$

and let $(t_n) \to \infty$. Then $(E_{t_n}) \to \{x \in \mathbb{R}^2 \mid x_1^2 \le 1,\ x_2 = 0\}$, which is not algebraic.

So again, Hausdorff limits are not appropriate. In Section 6 we will introduce a different kind of convergence using local Euler integrals, which extends to constructible functions. Then we will show that the limit of a sequence of constructible functions in an algebraic family is again algebraic (Theorem 6.4). The main ingredient is our Theorem 5.1 which expresses the situation in terms of valuations and reduction maps.

1. Euler integral. Let R be a real closed field and let ω be the class of all semialgebraic sets $S \subset R^n$ for some n. A function $\varphi : R^n \to \mathbb{Z}$ is called *constructible* if the range of φ is finite and $\varphi^{-1}(z)$ is semialgebraic for all $z \in \mathbb{Z}$.

For any semialgebraic set $S \subset R^n$ the Euler characteristic (with compact supports) $\chi(S)$ is defined. If S is semialgebraically isomorphic to an open d-cell, then $\chi(S) = (-1)^d$. For arbitrary S, this allows us to compute $\chi(S)$ from a cell decomposition, which always exists, and it turns out that the computation of $\chi(S)$ does not depend on the decomposition. One can extend the Euler characteristic to constructible functions as follows (compare [V]). Let $\varphi = \sum a_i 1_{S_i}$ where the sum is finite, $a_i \in \mathbb{Z}$ and $R^n \supset S_i$ is semialgebraic. Then $\chi(\varphi) := \sum a_i \chi(S_i)$. Again, this is independent of the representation of φ.

One also writes

$$\chi(\varphi) = \int \varphi \, d\chi = \int\limits_{R^n} \varphi(x) \, d\chi(x),$$

and if $T \subset R^n$ is semialgebraic, then

$$\int\limits_T \varphi \, d\chi := \int\limits_{R^n} 1_T \cdot \varphi \, d\chi;$$

this is called *Euler integration.* This name is justified by the following rules:

(i) If $\varphi, \psi : R^n \to \mathbb{Z}$ are constructible, then $\int(\varphi+\psi)d\chi = \int \varphi d\chi + \int \psi d\chi$.

(ii) If, moreover, $f : R^n \to R^m$ is a continuous semialgebraic map, then

$$\int\limits_{R^n} \varphi(x) \, d\chi(x) = \int\limits_{R^m} \Big(\int\limits_{f^{-1}(y)} \varphi(x) \, d\chi(x) \Big) \, d\chi(y)$$

(Fubini–Cavalieri).

In the situation of (ii) we define $f_*\varphi : R^m \to \mathbb{Z}$, $y \mapsto \int_{f^{-1}(y)} \varphi(x) \, d\chi(x)$. Thus

$$\int\limits_{R^n} \varphi(x) \, d\chi(x) = \int\limits_{R^m} f_*\varphi(y) \, d\chi(y).$$

2. Algebraic functions. Again, let R be a real closed field.

DEFINITION 2.1. A constructible function $\varphi : R^n \to \mathbb{Z}$ is called *algebraic* if there are finitely many polynomials $p_1, \ldots, p_k \in R[X]$, $X = (X_1, \ldots, X_n)$, such that $\varphi(x) = \operatorname{sign}(p_1(x)) + \cdots + \operatorname{sign}(p_n(x))$ for all $x \in R^n$.

EXAMPLE 2.2. Let $V \subset R^n$ be a real algebraic set. Then 1_V is algebraic. In fact, there is a positive polynomial p for V, that is, $p(x) > 0$ for $x \notin V$ and $p(x) = 0$ for $x \in V$. Now $1_V = \operatorname{sign}(1) + \operatorname{sign}(-p)$.

REMARK 2.3. Let $\varphi, \psi : R^n \to \mathbb{Z}$ be algebraic. Then

(a) $\varphi + \psi$ and $\varphi \cdot \psi$ are algebraic.

(b) Let $V \subset R^n$ be an algebraic subset. Then ζ is algebraic, where $\zeta(x) = \varphi(x)$ for $x \notin V$ and $\zeta(x) = \psi(x)$ for $x \in V$.

(c) Let S be a symmetric $n \times n$ matrix with coefficients in $R[X]$. Then $x \mapsto \operatorname{sign}_x S$ (signature of S at x) is algebraic.

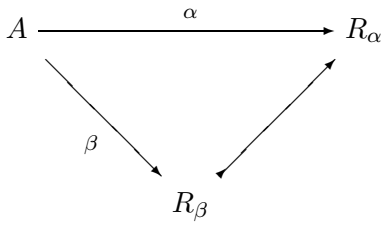
Here (a) is obvious, (b) follows easily from (a) and Example 2.2, and (c) follows from (b) by induction on the dimension, since we can diagonalize S over function fields, that is, up to algebraic sets of smaller dimensions.

The fundamental property of algebraic functions is (see [P-S]):

THEOREM 2.4 (Parusiński–Szafraniec). *Let $f : R^n \to R^m$ be a polynomial map. If $\varphi : R^n \to \mathbb{Z}$ is algebraic, then so is $f_*\varphi : R^m \to \mathbb{Z}$.*

We are going to show a corresponding result for reduction maps. This requires more preparations (for more details see [B-C-R, Chap. 7], [A-B-R, Chap. 2]):

Let $V = \mathrm{Spec}(A)$ be an algebraic variety over R. Then $\mathrm{Hom}(A, R) = V(R)$ is the variety of closed real points in V. We extend $V(R) = \mathrm{Hom}(A, R)$ to the space $\mathrm{Spec}_{\mathrm{r}}(A) = \{\alpha : A \to R_\alpha\}/\sim$ called the real spectrum of A, where R_α is real closed, α is a homomorphism and $\sim$ is generated by commutative triangles



From the model completeness of the theory of real closed fields it is clear that elementary objects and properties in $V(R)$ and $\mathrm{Spec}_{\mathrm{r}}(A)$ correspond to each other. For instance, a constructible (algebraic) function φ on $V(R)$ extends uniquely to a constructible (algebraic) function $\widetilde{\varphi}$ on $\mathrm{Spec}_{\mathrm{r}}(A)$.

Let $\alpha \in \mathrm{Spec}_{\mathrm{r}}(A)$. Then $\mathrm{Ker}(\alpha) =: \mathrm{supp}(\alpha)$ (support of α) is a prime ideal of A. We define $\dim(\alpha) := \dim(A/\mathrm{supp}(\alpha))$. One defines $\mathrm{Spec}_{\mathrm{r}}(A)$ correspondingly for any commutative ring with unit, and also constructible and algebraic functions. For instance, if $A = K$ is a field, then $\mathrm{Spec}_{\mathrm{r}}(K)$ is the space of all orderings of K. The representation theorem [Be-Br] provides a criterion for the algebraicity of a constructible function $\varphi : \mathrm{Spec}_{\mathrm{r}}(K) \to \mathbb{Z}$. For general A one has

$$\mathrm{Spec}_{\mathrm{r}}(A) = \bigcup_{p \in \mathrm{Spec}(A)} \mathrm{Spec}_{\mathrm{r}}(k(p))$$

where $k(p)$ is the residue field of $p \in \mathrm{Spec}(A)$. This allows us to reduce questions about $V(R)$, via $\mathrm{Spec}_{\mathrm{r}}(A)$, to features and tools of the real algebra of fields. We will follow this path in the next sections.

3. The reduction map. Throughout this section we fix a real closed field R together with a real valuation $v : R \to \Gamma \cup \infty$, where Γ is the value group. We denote by $B \subset R$ the valuation ring, by m the maximal ideal and by $\overline{R}$ the residue field. Note that v is henselian, Γ is divisible and $\overline{R}$ is again real closed. The residue map $\varrho : B \to \overline{R}$, $b \mapsto \varrho(b) =: \overline{b}$, admits a section $\sigma : \overline{R} \to B$ which we fix in the following. So we consider $\overline{R}$ as a subfield of R. We denote by ϱ also the map $B^n \to \overline{R}^n$, $x_i \mapsto \varrho(x_i)$, $i = 1, \ldots, n$. One has [Br1, Th. 1.7]

PROPOSITION 3.1. *Let $S \subset B^n$ be definable in the language of valued ordered fields. Then $\varrho(S) \subset \overline{R}^n$ is semialgebraic.*

Again, we want to extend ϱ to constructible functions. For this, let $\varphi : R^n \to \mathbb{Z}$ be constructible, as in Section 1, and let $a \in B^n$. For $\lambda \in R$, $\lambda > 0$ let

$$D(a,\lambda) := \{x \in R^n \mid (a-x)_i < \lambda \text{ for } i = 1,\dots,n\}$$

and

$$\chi(a,\lambda)(\varphi) := \int\limits_{D(a,\lambda)} \varphi \, d\chi.$$

The map $]0,\infty[\to \mathbb{Z}$, $\lambda \mapsto \chi(a,\lambda)(\varphi)$, is constructible. Hence there exist $0 < \lambda_1 < \lambda_2$, $\lambda_1 \in m$, $\lambda_2 \in B \setminus m$, such that $\chi(a,\lambda)(\varphi)$ is constant on $]\lambda_1,\lambda_2[$, so we define

$$\chi(a,m)(\varphi) := \int\limits_{\varrho^{-1}(\overline{a})} \varphi \, d\chi := \int\limits_{D(a,\lambda)} \varphi \, d\chi$$

for $\lambda \in]\lambda_1,\lambda_2[$ and $\overline{a} = \varrho(a)$. Then we get

PROPOSITION 3.2. *Let $\varphi : R^n \to \mathbb{Z}$ be constructible. Then so is $\varrho_*\varphi : \overline{R}^n \to \mathbb{Z}$, where for $\overline{a} \in \overline{R}^n$,*

$$\varrho_*\varphi(\overline{a}) := \int\limits_{\varrho^{-1}(\overline{a})} \varphi \, d\chi.$$

Proof. For all $z \in \mathbb{Z}$ the set $\{a \in B^n \mid \chi(a,m)(\varphi) = z\} =: e(z,\varphi)$ is definable in the language of valued fields. By Hardt's theorem [Ha] it is also clear that $\chi(a,m)(\varphi)$ is bounded for fixed φ and $a \in B^n$. So we can write

$$\chi(a,m)(\varphi) = \sum z_i 1_{e(z_i,\varphi)}$$

where the sum is finite. Clearly every function $1_{e(z_i,\varphi)}$ is constant on the fibers of ϱ. Hence

$$\varrho_*\varphi = \sum z_i 1_{\varrho(e(z_i,\varphi))},$$

which by Proposition 3.1 is constructible. ∎

4. The reduction map on the real spectrum. In Section 2 we saw that to every constructible function $\varphi : R^n \to \mathbb{Z}$ there corresponds a constructible function $\widetilde{\varphi} : \mathrm{Spec_r}(A) \to \mathbb{Z}$, where $A = R[X_1,\dots,X_n]$. If, in particular, S is semialgebraic and $\varphi = 1_S$ then $\widetilde{\varphi} = 1_{\widetilde{S}}$, the constructible set $\widetilde{S} \subset \mathrm{Spec_r}(A)$ being defined by the same equations as S. The set $B^n \subset R^n$ is not semialgebraic. Nevertheless, we define

$$\widetilde{B}^n = \{\alpha \in \mathrm{Spec_r}(A) \mid \forall f \in A \text{ with } \alpha(f) > 0 \; \exists x \in B^n \text{ with } f(x) > 0\}.$$

We are going to study a reduction

$$\widetilde{\varrho} : \widetilde{B}^n \to \mathrm{Spec}_{\mathrm{r}}(\overline{A}),$$

where $\overline{A} = \overline{R}[X_1, \ldots, X_n]$. The constructions and results below are taken from [Br1, §3]. Let $\alpha \in \widetilde{B}^n$. Recall that α corresponds to an ultrafilter, say $\varphi(\alpha)$, of semialgebraic sets in R^n. Then the sets $\varrho(S \cap B^n \mid S \in \varphi(\alpha))$ generate a unique ultrafilter φ' of semialgebraic sets in $\overline{R}^n$. So we define $\widetilde{\varrho}(\alpha)$ to be the corresponding element in $\mathrm{Spec}_{\mathrm{r}}(\overline{A})$, that is, $\varphi' = \varphi(\widetilde{\varrho}(\alpha))$.

Now let $V \subset R^n$ be a closed integral algebraic subvariety with $\dim(V) = k$, and let $\overline{V}_1, \ldots, \overline{V}_r$ be the irreducible components of $\mathrm{cl}_{\mathrm{z}}\, \varrho(V \cap B^n)$ (where cl_{z} denotes the Zariski closure), for which $\dim \overline{V}_i = k$. For each $i = 1, \ldots, r$ we shall define a set v_{ij} of valuations of $R(V)$ which extend the given valuation v on R in a natural way. For this let

$$B[V] := \{f + I(V) \in R[V] \mid f \in B(X)\}$$

and

$$p_i := \{f + I(V) \in R[V] \mid f \in B[X] \text{ and } \overline{f}(x) = 0 \ \forall x \in \overline{V}_i\}.$$

Here $X = (X_1, \ldots, X_n)$, $x = (x_1, \ldots, x_n)$ and $\overline{f}$ is the componentwise reduction of f. Then p_i is a prime ideal in $B[V]$ and $B[V]/p_i \simeq \overline{R}[\overline{V}_i]$. Let $A_i = B[V]_{p_i}$ and $\widetilde{A}$ the integral closure of A_i in $R(V)$. In general, $A_i \neq \widetilde{A}_i$ and also $\widetilde{A}_i$ is possibly not a valuation ring in $R(V)$. However, one has

Proposition 4.1. *Under the above notations, $\widetilde{A}_i$ is semilocal with maximal ideals m_{ij}, $j = 1, \ldots, s(i)$, where $m_{ij} \cap B[V] = p_i$. The localization $(\widetilde{A}_i)_{m_{ij}}$ is the valuation ring of a valuation v_{ij} of $R(V)$ extending v. The residue field of v_{ij} is a finite extension of $\overline{R}(\overline{V}_i)$ and the value group of v_{ij} is divisible for $j = 1, \ldots, s(i)$.*

Quite generally, an ordering α of a field K is called *compatible* with a valuation v of K if the valuation ring of v is convex with respect to α. In that case α induces an ordering $v(\alpha)$ on the residue field. With these notations we have moreover:

Proposition 4.2. *The ordering $\alpha \in \mathrm{Spec}_{\mathrm{r}}(R(V))$ is compatible with one of the valuations v_{ij} if and only if $\alpha \in \widetilde{B}^n$ and $\dim(\widetilde{\varrho}(\alpha)) = \dim(\alpha) = \dim(V)$. In that case $\widetilde{\varrho}(\alpha) \in \mathrm{Spec}_{\mathrm{r}}(\overline{R}(\overline{V}_i))$ and $v_{ij}(\alpha)$ extends $\widetilde{\varrho}(\alpha)$.*

Finally, in the above situation, we would like to have some information on the number of those $\alpha \in \mathrm{Spec}_{\mathrm{r}}(R(V))$ for which $\widetilde{\varrho}(\alpha) = \overline{\alpha} \in \mathrm{Spec}_{\mathrm{r}}(\overline{R}(\overline{V}_i))$. Here we have

Proposition 4.3. *Under the above notations, for $i = 1, \ldots, r$ there exists a nondegenerate quadratic form φ_i over $\overline{R}(\overline{V}_i)$ such that for all $\overline{\alpha} \in$*

$\mathrm{Spec_r}(\overline{R}(\overline{V}_i))$ *one has*

$$\#\{\alpha \in \mathrm{Spec_r}(R(V)) \mid \widetilde{\varrho}(\alpha) = \overline{\alpha}\} = \mathrm{sign}_{\overline{\alpha}}(\varphi_i).$$

In fact, φ_i *is the trace form of* $\widetilde{A}_i/p_i\widetilde{A}_i$ *over* $\overline{R}(\overline{V}_i)$.

More generally, let $g \in B[V]$ and for $x, y \in \widetilde{A}_i/p_i\widetilde{A}_i$ let $\varphi_i(g)(x,y) := \mathrm{tr}\, gxy$, where tr is again the trace of $\widetilde{A}_i/p_i\widetilde{A}_i$ over $\overline{R}(\overline{V}_i)$.

PROPOSITION 4.4. *For* $\overline{\alpha} \in \mathrm{Spec_r}(\overline{R}(\overline{V}_i))$ *one has*

$$\begin{aligned}\mathrm{sign}_{\overline{\alpha}}(\varphi_i(g)) = {} & \#\{\alpha \in \mathrm{Spec_r}(R(V)) \mid \widetilde{\varrho}(\alpha) = \overline{\alpha},\ \alpha(g) > 0\} \\ & - \#\{\alpha \in \mathrm{Spec_r}(R(V)) \mid \widetilde{\varrho}(\alpha) = \overline{\alpha},\ \alpha(g) < 0\}.\end{aligned}$$

REMARK 4.5. Propositions 4.3 and 4.4 also hold if V is not necessarily integral, but merely reduced. Consider the following situation: $\overline{W} \subset \overline{R}^n$ is an irreducible variety. If we fix a section $\overline{R} \hookrightarrow R$, then $\overline{W}$ corresponds to a subvariety W, again irreducible, of R^n. Consider $R \times W \subset R^n$ (possibly replace n by $n+1$), let $f \in R[R \times W]$ and let $V = Z(f)$ be the zero set of f. Assume that $\overline{W}$ coincides with one of the $\overline{V}_i$, say $\overline{W} = \overline{V}_1$. Assume further, for simplicity, that f, regarded as a polynomial over $R(W)$, is separable. Now let $\overline{\alpha} \in \mathrm{Spec_r}(\overline{R}(\overline{W}))$. There is a unique $\alpha \in \mathrm{Spec_r}(R(W))$ with $\widetilde{\varrho}(\alpha) = \overline{\alpha}$. Consider the corresponding real closed field $R_\alpha \supset R(W)$ and the valuation ring $B_\alpha \subset R_\alpha$ where

$$B_\alpha = \{x \in R_\alpha \mid \exists b \in B : x \leq b\}.$$

Also let m_α be the corresponding maximal ideal. Then the zeros of f in B_α correspond to those $\alpha \in \mathrm{Spec_r}(R(V))$ which are compatible with a valuation v_{ij} and for which $\widetilde{\varrho}(\alpha) = \overline{\alpha}$. Moreover, $\alpha \in m_\alpha$ if and only if α is compatible with a valuation v_{1j}.

5. Reduction of algebraic functions. In this section we show our main result:

THEOREM 5.1. *Let* $\varphi : R^n \to \mathbb{Z}$ *be algebraic. Then so is* $\varrho_*\varphi : \overline{R}^n \to \mathbb{Z}$.

Here we use the notations of Section 3. For the proof we will need several steps and the lemma below which is essentially [P-S, Lemma 6], but note that unlike that article we always consider the Euler characteristic with compact supports.

LEMMA 5.2. *Let* R *be a real closed field,* $a < b \in R$ *and* $f = a_d X^d + \cdots + a_0$, $a_d \neq 0$, *a polynomial such that* $f(a)$, $f(b) \neq 0$. *Moreover, for* $2 \leq k \leq d$ *let*

$$V_k := \{x \in R \mid f(x) = \cdots = f^{(k-1)}(x) = 0\}$$

and

$$Z_{k,a,b} := \sum_{x \in V_k,\, a<x<b} \mathrm{sign}(f^{(k)}(x)).$$

Then

$$\int_{]a,b[} \operatorname{sign}(f) d\chi = -\frac{1}{2}\left(\operatorname{sign}(f(a)) + \operatorname{sign}(f(b))\right) - \sum_{2\le 2k\le d} Z_{2k,a,b}.$$

Proof of Theorem 5.1. First of all, we may assume that $\varphi(x) = \operatorname{sign}(f(x))$ for all $x \in R^n$, where f is a polynomial $\in R[X_1, \dots, X_n]$.

1. Without changing $\varphi = \operatorname{sign}(f)$ we may assume that $f \in B[X_1, \dots, X_n]$, but $f \notin m[X_1, \dots, X_n]$. Let $\overline{f} \in \overline{R}[X_1, \dots, X_n]$ be the componentwise reduction of f, so $\overline{f} \neq 0$. Clearly, $\varrho_*\varphi(y) = \operatorname{sign}\overline{f}(y)$ for all those $y \in \overline{R}^n$ for which $\overline{f}(y) \neq 0$. In other words, $\varrho^*\varphi$ is algebraic up to a set of smaller dimension.

2. Now let $W' \subset \overline{R}^n$ be an algebraic variety such that the following hold:

- There is an algebraic function $\varphi' : \overline{R}^n \to \mathbb{Z}$ with $\varrho_*\varphi = \varphi'$ on $\overline{R}^n \setminus W'$.
- W' is of minimal dimension and has a minimal number of components subject to the above property.

We want to show that $W' = \emptyset$. Assume $W' \neq \emptyset$. So let $\overline{W}$ be a component of W' with $\dim \overline{W} = \dim W'$. We will show

3. $\varrho_*\varphi$ is generically algebraic on $\overline{W}$ (algebraic up to a set of smaller dimension). By Remark 2.3 this would contradict $W' \neq \emptyset$. As before, we fix a section $\overline{R} \hookrightarrow B$. Thus $\overline{W}$ corresponds to a subvariety W of R^n. Let N be the normal bundle of W in R^n:

$$N = \{(x, a) \mid x \in R^n,\ a \in W,\ x \perp T_a W\}.$$

Here $T_a W$ is the tangent space of W at a. (If a is a singular point, then $T_a W = R^n$.) Similarly, we have

$$\overline{N} = \{(y, b) \mid y \in \overline{R}^n,\ b \in \overline{W},\ y \perp T_b \overline{W}\}.$$

Let $\overline{r} : \overline{W} \to \overline{R}$ be a polynomial map such that $\overline{r}(b) > 0$ if b is nonsingular and $\overline{r}(b) = 0$ if b is singular. Let

$$\overline{U}_{\overline{r}} := \{(y, b) \in \overline{N} \mid \langle y, y\rangle < \overline{r}(b)\},$$
$$U_r := \{(x, a) \in N \mid \langle x, x\rangle < r(a)\},$$

where $\langle\,,\rangle$ denotes the usual scalar product and where, by the identification of $\overline{R}$ with a subfield of R, we may consider $\overline{r}$ as a polynomial map r on W. Let

$$\pi : N \to R^n, \qquad (x, a) \mapsto x + a,$$
$$\overline{\pi} : \overline{N} \to \overline{R}^n, \qquad (y, b) \mapsto y + b.$$

We may choose $\overline{r}$ in such a way that π and $\overline{\pi}$ map $U = U_r$ (and $\overline{U} = \overline{U}_r$) isomorphically onto a neighbourhood of W_{reg} (and $\overline{W}_{\mathrm{reg}}$ respectively). We have

$$r_*\varphi(y) = \int_{r^{-1}(y)} \varphi\, d\chi = \int_{\pi^{-1}r^{-1}(y)\cap U} \pi^*\varphi\, d\chi \quad \text{for } y \in \overline{W}.$$

Next, consider the map

$$d : N \to R \times W, \qquad (x, a) \mapsto (\langle x, x\rangle, a).$$

By [P-S] (cf. Theorem 2.4) the function $d_*\pi^*\varphi$ is algebraic. Therefore, we are reduced to the situation that φ lives on $R \times W$, and it remains to show that $r_*\varphi$ is generically algebraic on $\overline{W}$.

4. As before, we also consider $\overline{R} \times \overline{W}$. We may assume that $\varphi = \operatorname{sign}(f)$ with $f \in R[W][X]$. Since we need only show that $r_*\varphi$ is generically algebraic on $\overline{W}$, instead of looking at real points on $\overline{W}$ we look at orderings $\overline{\alpha} \in \operatorname{Spec}_{\mathrm{r}}(\overline{R}(\overline{W}))$, that is, we will show that $r_*^{\sim}\varphi|\operatorname{Spec}(\overline{R}(\overline{W}))$ is algebraic. As in step 1 we take f to be defined over B but with coefficients not all in m, and consider $\overline{f} \in \overline{R}[\overline{W}][X]$. We write $\overline{f} = X^k\overline{g}$ such that $\overline{g}$ does not vanish identically on $\overline{W}$. If $\overline{f} = \overline{g}$, then $r_*\varphi = \operatorname{sign}(\overline{g})$ on $\overline{W}$, up to a set of smaller dimension. So, in this case, we are done. Now, if $k > 0$, consider an ordering $\overline{\alpha} \in \operatorname{Spec}_{\mathrm{r}}(\overline{R}(\overline{W}))$. Recall the situation of Remark 4.5. There is a unique ordering $\alpha \in \operatorname{Spec}_{\mathrm{r}}(R(W))$ such that $\widetilde{\varrho}(\alpha) = \overline{\alpha}$. Consider the field $R_\alpha \supset R$ and its valuation ring

$$B_\alpha := \{x \in R_\alpha \mid \exists b \in B : x \le b\}.$$

Let m_α be the corresponding maximal ideal. Let $f = f_1 f_2^2$, where f_1, f_2 are mutually prime and square free (as elements of $R(W)[X]$). With these notations we have

$$(-1)^{\dim(\overline{W})}\varrho_*^{\sim}\varphi(\alpha) = \int\limits_{m_\alpha} \operatorname{sign}(f_1)\, d\chi - \#\{x \in m_\alpha \mid f_2(x) = 0\}.$$

By Proposition 4.3 and Remark 4.5 the second summand is algebraic on $\operatorname{Spec}_{\mathrm{r}}(\overline{R}(\overline{W}))$. For the first summand, we assume that $f = f_1$ and write $\overline{f} = X^k\overline{g}$ as above. Then, by Lemma 5.2,

$$\int\limits_{m_\alpha} \operatorname{sign}(f)\, d\chi = -\overline{g}(\alpha) - \sum_{2 \le 2k \le \deg(f)} Z_{2k,\alpha},$$

where

$$Z_{i,\alpha} := \sum_{x \in m_\alpha \cap V_{k,\alpha}} \operatorname{sign}(f^{(i)}(x)),$$

$$V_{k,\alpha} := \{x \in R_\alpha \mid f(x) = \cdots = f^{(k-1)}(x) = 0\}.$$

Now clearly $\overline{g}(\alpha)$ is algebraic on $\operatorname{Spec}_{\mathrm{r}}(\overline{R}(\overline{W}))$ and by Proposition 4.4 and Remark 4.5 this also holds for the functions $\alpha \mapsto Z_{2k,\alpha}$. ∎

6. Limits. For $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$, $r > 0$, let $B^n(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$, where $\|\ \|$ denotes the euclidean norm.

NOTATION 6.1. For a constructible function $\varphi : \mathbb{R}^n \to \mathbb{Z}$ we denote by $\widehat{\varphi} : \mathbb{R}^n \to \mathbb{Z}$ the function

$$x \mapsto \lim_{r\to 0} \int\limits_{B^n(x,r)} \varphi \, d\chi.$$

Clearly, $\widehat{\varphi}$ is again constructible. Moreover, it is well known that $\widehat{\widehat{\varphi}} = \varphi$ (see [Br3]) and if φ is algebraic, then so is $\widehat{\varphi}$. The latter follows, for instance, from [P-S] and Hardt's Theorem [Ha] (see also [MC-P]). Now let $(\varphi_m)_{m\in\mathbb{N}}$ be a sequence of constructible functions $\varphi_m : \mathbb{R}^n \to \mathbb{Z}$.

DEFINITION 6.2. We say that (φ_m) tends to $\psi : \mathbb{R}^n \to \mathbb{Z}$ and write $(\varphi_m) \to \psi$ if for all $x \in \mathbb{R}^n$ one has

$$\widehat{\psi}(x) = \lim_{r\to 0} \lim_{m\to\infty} \int\limits_{B^n(x,p)} \varphi_m \, d\chi.$$

For characteristic functions of closed convex sets this is the same as Hausdorff convergence, but in general, this is not true.

EXAMPLE 6.3. Consider the sequence of ellipses

$$E_m = \{x \in \mathbb{R}^2 \mid x_1^2 + m^2 x_2^2 = 1\},$$

which get more and more flat for increasing m. The Hausdorff limit of (E_m) is the interval $I = \{x \in \mathbb{R}^2 \mid x_1^2 \le 1,\ x_2 = 0\}$. Now let $\varphi_m = 1_{E_m}$. Then $(\varphi_m) \to \psi$ where

$$\psi(x) = \begin{cases} 2 & \text{for } x_1^2 < 1,\ x_2 = 0, \\ 1 & \text{for } x_1^2 = 1,\ x_2 = 0, \\ 0 & \text{else.} \end{cases}$$

Note that the φ_m are algebraic and so is ψ, but 1_I is not algebraic. This is a special case of a general result.

Consider $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$ as a family of spaces $\mathbb{R}^n$ with parameters in $\mathbb{R}^k$. We denote the variables in $\mathbb{R}^n$ by $X = (X_1, \dots, X_n)$ and those in $\mathbb{R}^k$ by $T = (T_1, \dots, T_k)$. For $t \in \mathbb{R}^k$ let $\mathbb{R}^n_t := \mathbb{R}^n \times \{t\} \subset \mathbb{R}^{n+k}$. Now let $\varphi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{Z}$ be constructible. Again we consider φ as a family $\{\varphi_t \mid t \in \mathbb{R}^k\}$ of constructible functions $\mathbb{R}^n \to \mathbb{Z}$ where $\varphi_t(x) := \varphi(x,t)$.

More generally, let $\alpha \in \mathrm{Spec}_{\mathrm{r}}(\mathbb{R}[T])$. Recall that we may represent α by a homomorphism $\alpha : \mathbb{R}[T] \to R_\alpha$ where $\mathbb{R} \subset R_\alpha$ is a real closed field. By model completeness of the theory of real closed fields, φ induces a constructible function $\varphi_\alpha : R_\alpha^n \to \mathbb{Z}$.

It may be helpful to "visualize" this. It is well known and used before that to α there corresponds an ultrafilter A of semialgebraic sets $S \subset \mathbb{R}^n$. Then R_α is represented by semialgebraic functions $\widetilde{f} : S \to \mathbb{R}$ for $S \in A$ where two such functions $\widetilde{f}$ and $\widetilde{g}$ are identified if they coincide on some $T \in A$. Now,

given $\varphi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{Z}$ it is clear that φ assigns eventually a well defined value to an n-tuple $f = (f_1, \ldots, f_n) \in R_\alpha^n$.

The canonical valuation ring B_α of R_α is represented by those $\widetilde{f}$ which are bounded on some $T \in A$, and the maximal ideal m_α by those $\widetilde{f}$ which are arbitrarily small on some $T \in A$. We may also write

$$B_\alpha = \{f \in R_\alpha \mid \exists r \in \mathbb{R} : |f| \leq r\},$$
$$m_\alpha = \{f \in R_\alpha \mid \forall r \in \mathbb{R}, r > 0 : |f| \leq r\}.$$

Note that the residue field is isomorphic to $\mathbb{R}$.

Now let (t_m) be a sequence in $\mathbb{R}^k$. Then we get a sequence $(\varphi_m) := (\varphi_{t_m})$ of constructible functions $\varphi_m : \mathbb{R}^n \to \mathbb{Z}$. So this is a sequence of constructible functions in the family $\mathbb{R}^n \times \mathbb{R}^k$. In this situation we have

THEOREM 6.4.

(a) *There is an element $\alpha \in \mathrm{Spec_r}(\mathbb{R}[T])$ with corresponding ultrafilter A such that for all semialgebraic sets $S \in A$ one has $t_m \in S$ for infinitely many m.*

(b) *Assume that $(\varphi_m) \to \psi$ and let α be as in* (a). *Then $\widehat{\psi} = \varrho_* \varphi_\alpha$, where ϱ is the reduction map according to the real closed field R_α with the valuation ring B_α. In particular, ψ is constructible.*

(c) *If φ is algebraic, then so is ψ.*

Proof. (a) The collection of all semialgebraic sets $S \subset \mathbb{R}^k$ for which $t_m \in S$ for almost all $m \in \mathbb{Z}$ forms a filter F. Let $A \supset F$ be any ultrafilter of semialgebraic sets in $\mathbb{R}^k$ and α the corresponding element in $\mathrm{Spec_r}(\mathbb{R}[T])$. Then clearly α has the required property.

(b) Let $x \in \mathbb{R}^n$ and $f \in R_\alpha$. Then $f - x \in m_\alpha$ if and only if for all $r > 0$, all $S \in A$ and all representatives $\widetilde{f}_i : S \to \mathbb{R}$ for f_i there exists $T \subset S$ with $T \in A$ such that $\widetilde{f}(t) \in B(x, r)$ for all $t \in T$. In particular, there are infinitely many t_m such that $\widetilde{f}(t_m) \in B(x, r)$. We have

$$\varrho_* \varphi(x) = \int\limits_{f - x \in m_\alpha} \varphi_\alpha(f)\, d\chi.$$

We may compute the right hand side by a finite procedure, say, a cell decomposition which still works on representatives defined over some $T \in A$. If we specialize the computation at some $t \in T$ we get the same value $\varrho_* \varphi(x)$ for the Euler integral on the one hand, but on the other hand, since there are infinitely many $t_m \in T$ for all $T \in A$ we get the value

$$\lim_{r \to 0} \lim_{m \to \infty} \int\limits_{B^n(x,r)} \varphi_m\, d\chi.$$

Thus $\varrho_* \varphi(x) = \widehat{\psi}(x)$.

(c) This follows from (b) and Theorem 5.1. ∎

REMARK 6.5. In the preceding theorem, the statement that the limit ψ of the sequence φ_n is again constructible should also be true in the o-minimal context.

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