ANNALES
POLONICI MATHEMATICI
87 (2005)

Nash cohomology of smooth manifolds

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Abstract. A Nash cohomology class on a compact Nash manifold is a mod 2 cohomology class whose Poincaré dual homology class can be represented by a Nash subset. We find a canonical way to define Nash cohomology classes on an arbitrary compact smooth manifold M. Then the Nash cohomology ring of M is compared to the ring of algebraic cohomology classes on algebraic models of M. This is related to three conjectures concerning algebraic cohomology classes.

1. Introduction. Let X be a compact nonsingular real algebraic set (in $\mathbb{R}^n$ for some n). A cohomology class in $H^k(X,\mathbb{Z}/2)$ is said to be *algebraic* if its Poincaré dual homology class in $H_*(X,\mathbb{Z}/2)$ can be represented by an algebraic subset of X. The set $H^k_{\mathrm{alg}}(X,\mathbb{Z}/2)$ of all algebraic cohomology classes in $H^k(X,\mathbb{Z}/2)$ is a subgroup, while the direct sum $H^*_{\mathrm{alg}}(X,\mathbb{Z}/2)$ of the $H^k_{\mathrm{alg}}(X,\mathbb{Z}/2)$, for $k\ge 0$, forms a subring of the cohomology ring $H^*(X,\mathbb{Z}/2)$. The reader can find a survey of properties and applications of $H^*_{\mathrm{alg}}(-,\mathbb{Z}/2)$ in [6].

Each compact smooth (of class C^∞) manifold M has an *algebraic model*, that is, M is diffeomorphic to a nonsingular real algebraic set [19] (cf. also [5, Theorem 14.1.10] and, for a weaker but influential result, [15]). We say that a subset E of $H^*(M,\mathbb{Z}/2)$ *admits an algebraic realization* if there exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X\to M$ such that $\varphi^*(E)\subseteq H^*_{\mathrm{alg}}(X,\mathbb{Z}/2)$ (when $E=\{v\}$ consists of one element, we simply say that v *admits an algebraic realization*). If E admits an algebraic realization, then so does the subring of $H^*(M,\mathbb{Z}/2)$ generated by E. The original goal of several researchers was to show that the whole ring $H^*(M,\mathbb{Z}/2)$ admits an algebraic realization, that is, M has an algebraic model X with $H^*_{\mathrm{alg}}(X,\mathbb{Z}/2) = H^*(X,\mathbb{Z}/2)$ (such a conjecture, motivated

2000 *Mathematics Subject Classification*: 14P20, 14P25, 14C25.
Key words and phrases: algebraic cohomology, Nash cohomology, algebraic model.

by far-reaching potential applications, was explicitly stated in [1]). However, since the publication of [4] it has been known that for some manifolds M this is impossible.

Denote by $A(M)$ the subring of $H^*(M,\mathbb{Z}/2)$ generated by the Stiefel–Whitney classes of all real vector bundles on M together with each cohomology class Poincaré dual to a homology class represented by a smooth submanifold of M. A very useful and important result is that $A(M)$ admits an algebraic realization [20, p. 93]. Already in [4] did the following conjecture appear.

Conjecture A. For any compact smooth manifold M, each subring of $H^*(M,\mathbb{Z}/2)$ which admits an algebraic realization is contained in $A(M)$ (equivalently, $H^*_{\mathrm{alg}}(X,\mathbb{Z}/2) \subseteq A(X)$ for any compact nonsingular real algebraic set X).

If $\dim M \leq 5$, then [18, Théorème II.26] implies $A(M) = H^*(M,\mathbb{Z}/2)$, and hence M has an algebraic model X with $H^*_{\mathrm{alg}}(X,\mathbb{Z}/2) = H^*(X,\mathbb{Z}/2)$. In order to survey known facts in higher dimensions, let us set $A^k(M) = A(M) \cap H^k(M,\mathbb{Z}/2)$ for $k \geq 0$. Note $A^k(M) = H^k(M,\mathbb{Z}/2)$ if either $k = 0, 1$ or $k \geq \frac{1}{2}\dim M$, and assuming $\dim M \leq 7$ also $A^3(M) = H^3(M,\mathbb{Z}/2)$ (cf. [18, Théorème II.26]). For any compact nonsingular real algebraic set X, one has $H^2_{\mathrm{alg}}(X,\mathbb{Z}/2) \subseteq A^2(X)$. The inclusion follows from [4] (cf. also [7] for an elementary proof). In particular, $H^*_{\mathrm{alg}}(X,\mathbb{Z}/2) \subseteq A(X)$ if $\dim X = 6$ or $\dim X = 7$, which means that *Conjecture A is true for all compact smooth manifolds of dimension* 6 *or* 7. This is nontrivial since for each $m \geq 6$, there is a compact smooth m-dimensional manifold M with $A^2(M) \neq H^2(M,\mathbb{Z}/2)$ (cf. [17] and Example 2.9 below, and also [4] for $m \geq 11$), which implies that no cohomology class in $H^2(M,\mathbb{Z}/2) \setminus A^2(M)$ admits an algebraic realization. In order to avoid a possible confusion, let us mention that [4] erroneously asserts that $A(M) = H^*(M,\mathbb{Z}/2)$ for $\dim M \leq 6$ is a consequence of [18]. However, [18] implies such an equality only for $\dim M \leq 5$.

It has recently been noticed that for each positive even integer k, there is a compact smooth manifold M having a cohomology class in $H^k(M,\mathbb{Z}/2)$ not admitting an algebraic realization (cf. [12] and Example 2.9). Whether analogous examples exist for k odd greater than 1 remains an open problem.

Conjecture A, if true, implies that $A(M)$ is the largest subring of $H^*(M,\mathbb{Z}/2)$ admitting an algebraic realization.

Conjecture B. For any compact smooth manifold M, there is a largest subring of $H^*(M,\mathbb{Z}/2)$ which admits an algebraic realization.

Although Conjecture B is rather unappealing, it allows us to identify and describe in a nice way the largest subring of $H^*(M,\mathbb{Z}/2)$ admitting an algebraic realization, leaving however open the possibility that this subring may be different from $A(M)$. To demonstrate this we need some preparation.

By a *Nash manifold* we shall mean an analytic submanifold of $\mathbb{R}^n$, for some n, which is also a semi-algebraic subset. A *Nash map* between Nash manifolds is an analytic map with semi-algebraic graph. A *Nash subset* of a Nash manifold is the set of common zeros of finitely many real-valued Nash functions. For basic properties of these objects we refer to [5].

Let N be a compact Nash manifold. An element of $H^*(N,\mathbb{Z}/2)$ is said to be a *Nash cohomology class* if its Poincaré dual homology class can be represented by a Nash subset of N. The set $H^k_{\mathrm{Nash}}(N,\mathbb{Z}/2)$ of all Nash cohomology classes in $H^k(N,\mathbb{Z}/2)$ is a subgroup, while the direct sum $H^*_{\mathrm{Nash}}(N,\mathbb{Z}/2)$ of the $H^k_{\mathrm{Nash}}(N,\mathbb{Z}/2)$ with $k \geq 0$ forms a subring of $H^*(N,\mathbb{Z}/2)$ (see Lemma 2.2). Clearly,

$$H^*_{\mathrm{alg}}(X,\mathbb{Z}/2) \subseteq H^*_{\mathrm{Nash}}(X,\mathbb{Z}/2)$$

for any compact nonsingular real algebraic set X.

Given a compact smooth manifold M, choose a Nash manifold N and a smooth diffeomorphism $\psi : M \to N$. One readily checks that the subring $\psi^*(H^*_{\mathrm{Nash}}(N,\mathbb{Z}/2))$ of $H^*(M,\mathbb{Z}/2)$, henceforth denoted $H^*_{\mathrm{Nash}}(M,\mathbb{Z}/2)$, does not depend on the choice of N and ψ (see Proposition 2.3). Observe that each subring of $H^*(M,\mathbb{Z}/2)$ admitting an algebraic realization is contained in $H^*_{\mathrm{Nash}}(M,\mathbb{Z}/2)$. In particular,

$$A(M) \subseteq H^*_{\mathrm{Nash}}(M,\mathbb{Z}/2).$$

CONJECTURE C. For any compact smooth manifold M, the subring $H^*_{\mathrm{Nash}}(M,\mathbb{Z}/2)$ of $H^*(M,\mathbb{Z}/2)$ admits an algebraic realization (equivalently, M has an algebraic model X with $H^*_{\mathrm{alg}}(X,\mathbb{Z}/2) = H^*_{\mathrm{Nash}}(X,\mathbb{Z}/2)$).

In Section 2 we prove some results which seem to support Conjecture C (see Theorems 2.5 and 2.7, Proposition 2.8, Example 2.9). However, there are only two nontrivial cases in which we can actually prove Conjecture C, namely for all compact smooth manifolds of dimension 6 or 7 (Corollary 1.3).

Obviously, Conjecture A implies Conjecture B, and Conjecture C implies Conjecture B. Other relationships between the conjectures under consideration are described in Propositions 1.1. and 1.2, whose proofs are contained in Section 3.

Given a smooth manifold M, we set $\overline{M} = (M \times \{0\}) \cup (M \times \{1\})$; thus $\overline{M}$ is simply a disjoint union of two copies of M. The unit circle will be denoted by S^1.

PROPOSITION 1.1. *Let M be a compact smooth manifold. If Conjecture B is true for either $M \times S^1$ or $\overline{M}$, then Conjecture C is true for M.*

In Section 3 we show that the full strength of Conjecture B is not needed in Proposition 1.1. To prove that Conjecture C is true for M it suffices to assume that $M \times S^1$ (resp. $\overline{M}$) has what we call property H^1 (resp. H^0), see Definition 3.1. This refinement is introduced with the hope that property H^i, $i = 0, 1$, will be easier to verify directly, thereby leading to a proof of Conjecture C.

PROPOSITION 1.2. *Let M be a compact smooth manifold. If Conjecture A is true for either $M \times S^1$ or $\overline{M}$, then $A(M) = H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2)$ and Conjectures A and C are true for M.*

An interesting consequence of Proposition 1.2 is the following result.

COROLLARY 1.3. *For all compact smooth manifolds M of dimension* 6 *or* 7, *one has $A(M) = H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2)$. In particular, Conjecture C is true for such manifolds.*

Proof. As we already demonstrated above, Conjecture A is true for all compact smooth manifolds M of dimension 6 or 7. Thus it is true for $\overline{M}$, and hence Proposition 1.2 implies Corollary 1.3. ∎

In Section 2 we will prove Corollary 1.3 in a more direct way.

It is clear that given a compact smooth manifold M, Conjecture C is true for M if and only if it is true for each connected component of M. Whether an analogous statement is true for Conjecture A or Conjecture B is not obvious. We only have the following weaker result.

COROLLARY 1.4. *Let M be a compact smooth manifold and let $M_1, \ldots, M_r$ be the connected components of M.*

- (i) *If Conjecture A* (*resp. Conjecture B*) *is true for M, then it is true for each M_i with $1 \le i \le r$.*
- (ii) *If Conjecture A* (*resp. Conjecture B*) *is true for each $M_i \times S^1$ with $1 \le i \le r$, then it is true for M.*

Proof. (i) is obvious, while (ii) follows from Proposition 1.2 for Conjecture A and from Proposition 1.1 for Conjecture B. ∎

In conclusion, we have the following diagram:

$$\begin{array}{ccc} \text{Conjecture A} & \Longrightarrow & \text{Conjecture B} \\ \Updownarrow & & \Updownarrow \\ A(-) = H^*_{\mathrm{Nash}}(-, \mathbb{Z}/2) & \Longrightarrow & \text{Conjecture C} \end{array}$$

in which each implication $\Phi \Rightarrow \Psi$ should be understood as follows: if Φ is true for all compact smooth manifolds, then Ψ is true for such manifolds. Furthermore, if one of the conjectures under consideration is true for all connected compact smooth manifolds, then it is true for all compact smooth manifolds.

2. Nash cohomology. Let N be a compact Nash manifold. Each d-dimensional Nash subset V of N (being an analytic subset) carries a unique fundamental homology class in $H_d(V,\mathbb{Z}/2)$, denoted here by $[V]$ (cf. [9]; since V is compact, we use the singular homology instead of the Borel–Moore homology used in [9]). We write $[V]_N$ for the image of $[V]$ under the homomorphism $H_d(V,\mathbb{Z}/2) \to H_d(N,\mathbb{Z}/2)$ induced by the inclusion map $V \hookrightarrow N$. Each element of $H_d(N,\mathbb{Z}/2)$ of the form $[V]_N$, for some d-dimensional Nash subset V of N, is said to be a *Nash homology class*. Since $[V]_N = [V_1]_N + \cdots + [V_r]_N$, where $V_1,\ldots,V_r$ are the irreducible components of V of dimension d, it follows that the set $H_d^{\mathrm{Nash}}(N,\mathbb{Z}/2)$ of all Nash homology classes in $H_d(N,\mathbb{Z}/2)$ is a subgroup. Elements of the subgroup

$$H^c_{\mathrm{Nash}}(N,\mathbb{Z}/2) = D_N^{-1}(H_d^{\mathrm{Nash}}(N,\mathbb{Z}/2))$$

of $H^c(N,\mathbb{Z}/2)$, where $c+d=\dim N$ and

$$D_N : H^c(N,\mathbb{Z}/2) \to H_d(N,\mathbb{Z}/2)$$

is the Poincaré duality isomorphism, are called *Nash cohomology classes*. We set

$$H^*_{\mathrm{Nash}}(N,\mathbb{Z}/2) = \bigoplus_{c\geq 0} H^c_{\mathrm{Nash}}(N,\mathbb{Z}/2).$$

LEMMA 2.1. *Let $f : L \to N$ be a continuous map between compact Nash manifolds. Then*

$$f^*(H^*_{\mathrm{Nash}}(N,\mathbb{Z}/2)) \subseteq H^*_{\mathrm{Nash}}(L,\mathbb{Z}/2).$$

Proof. Let v be an element of $H^c_{\mathrm{Nash}}(N,\mathbb{Z}/2))$. Then $D_N(v) = [V]_N$ for some Nash subset V of N. Let $\mathcal{S}$ be a stratification of V satisfying Whitney's condition (a) (cf. [5] or [13]) and let $g : L \to N$ be a smooth map homotopic to f and transverse to $\mathcal{S}$, that is, transverse to each stratum of $\mathcal{S}$. The set of all smooth maps from L into N transverse to $\mathcal{S}$ is open and dense in the space of all smooth maps (Whitney's condition (a) guarantees the openness, cf. [11, Proposition 3.6]). There is a Nash map $h : L \to N$ arbitrarily close to g, and hence homotopic to f. In particular, $f^* = h^*$ in cohomology. Furthermore, we may assume that h is transverse to $\mathcal{S}$. Thus

$$f^*(v) = h^*(v) = D_L^{-1}([h^{-1}(V)]_L),$$

where the last equality is a consequence of [9, Proposition 2.15]. Hence $f^*(v)$ belongs to $H^*_{\mathrm{Nash}}(L,\mathbb{Z}/2)$ and the proof is complete. ∎

LEMMA 2.2. *For any compact Nash manifold N, the set $H^*_{\mathrm{Nash}}(N,\mathbb{Z}/2)$ is a subring of the cohomology ring $H^*(N,\mathbb{Z}/2)$.*

Proof. We only have to show that $H^*_{\mathrm{Nash}}(N,\mathbb{Z}/2)$ is closed under cup product $\cup$. One readily sees that if v_1 and v_2 are in $H^*_{\mathrm{Nash}}(N,\mathbb{Z}/2)$, then the cross product $v_1 \times v_2$ is in $H^*_{\mathrm{Nash}}(N\times N,\mathbb{Z}/2)$. Since $v_1 \cup v_2 = \triangle^*(v_1\times v_2)$, where $\triangle : N \to N \times N$ is the diagonal map, Lemma 2.1 implies that $v_1 \cup v_2$ belongs to $H^*_{\mathrm{Nash}}(N, \mathbb{Z}/2)$. ∎

We can define the Nash cohomology of an arbitrary compact smooth manifold M. To this end, choose a Nash manifold N and a smooth diffeomorphism $\psi : M \to N$, and set

$$H^c_{\mathrm{Nash}}(M, \mathbb{Z}/2) = \psi^*(H^c_{\mathrm{Nash}}(N, \mathbb{Z}/2)),$$
$$H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2) = \bigoplus_{c \geq 0}(H^c_{\mathrm{Nash}}(M, \mathbb{Z}/2)).$$

PROPOSITION 2.3. *With notation as above*, $H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2)$ *does not depend on the choice of* N *and* ψ. *Moreover*, $H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2)$ *is a subring of the cohomology ring* $H^*(M, \mathbb{Z}/2)$.

Proof. Let L be another Nash manifold and let $\theta : M \to L$ be a smooth diffeomorphism. Then $\sigma = \psi \circ \theta^{-1} : L \to N$ is a smooth diffeomorphism. In view of Lemma 2.1,

$$\sigma^*(H^*_{\mathrm{Nash}}(N, \mathbb{Z}/2)) = H^*_{\mathrm{Nash}}(L, \mathbb{Z}/2).$$

This implies

$$\theta^*(H^*_{\mathrm{Nash}}(L, \mathbb{Z}/2)) = \psi^*(H^*_{\mathrm{Nash}}(N, \mathbb{Z}/2)),$$

which shows $H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2)$ is well defined. The fact that $H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2)$ is a subring of $H^*(M, \mathbb{Z}/2)$ follows immediately from Lemma 2.2. ∎

PROPOSITION 2.4. *If* $f : M \to P$ *is a continuous map between compact smooth manifolds*, *then*

$$f^*(H^*_{\mathrm{Nash}}(P, \mathbb{Z}/2)) \subseteq H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2).$$

Proof. The assertion follows from Lemma 2.1. ∎

Given a smooth manifold P, we let $\mathcal{N}_*(P)$ denote the unoriented bordism group of P (cf. [10]).

THEOREM 2.5. *Let* $f : M \to P$ *be a smooth map between compact smooth manifolds. Assume that the bordism class of* f *in* $\mathcal{N}_*(P)$ *is equal to the bordism class of a constant map from some compact smooth manifold into* P. *Then the subring* $f^*(H^*_{\mathrm{Nash}}(P, \mathbb{Z}/2))$ *of* $H^*(M, \mathbb{Z}/2)$ *admits an algebraic realization.*

Proof. Without loss of generality, we may assume P is a connected Nash manifold. Let $V_1, \ldots, V_r$ be Nash subsets of P such that $\{[V_1]_P, \ldots, [V_r]_P\}$ is the set of all Nash homology classes in $H_*(P, \mathbb{Z}/2)$. Let $\lambda_i : P \to \mathbb{R}$ be a Nash function with $\lambda_i^{-1}(0) = V_i$. By applying the Artin–Mazur theorem [5, Theorem 8.4.4] to the Nash map $\lambda = (\lambda_1, \ldots, \lambda_r) : P \to \mathbb{R}^r$, we obtain a nonsingular algebraic set Y, a connected component Y_0 of Y, a Nash diffeomorphism $\sigma : P \to Y_0$, and a regular map $\mu = (\mu_1, \ldots, \mu_r) : Y \to \mathbb{R}^r$

satisfying $\mu|Y_0 = \lambda \circ \sigma^{-1}$. Since σ is a diffeomorphism we get

$$H^*_{\mathrm{Nash}}(P, \mathbb{Z}/2) = \sigma^*(H^*_{\mathrm{Nash}}(Y_0, \mathbb{Z}/2)). \tag{1}$$

Next, $\sigma(V_1), \ldots, \sigma(V_r)$ are Nash subsets of Y_0 and $\{[\sigma(V_1)]_{Y_0}, \ldots, [\sigma(V_r)]_{Y_0}\}$ is the set of all Nash homology classes in $H_*(Y_0, \mathbb{Z}/2)$. Since $\sigma(V_i) = \mu_i^{-1}(0) \cap Y_0$ and $\mu_i^{-1}(0)$ is an algebraic subset of Y, we obtain $\sigma(V_i) = W_i \cap Y_0$, where W_i is the closure of $\sigma(V_i)$ in the Zariski topology on Y. In particular, W_i is an algebraic subset of Y of dimension $\dim \sigma(V_i)$ and $[\sigma(V_i)]_{Y_0}$ is the image of $[W_i]_Y$ under the homomorphism between the Borel–Moore homology groups

$$H^{\mathrm{BM}}_*(Y, \mathbb{Z}/2) \to H^{\mathrm{BM}}_*(Y_0, \mathbb{Z}/2) = H_*(Y_0, \mathbb{Z}/2)$$

induced by the inclusion map $e : Y_0 \hookrightarrow Y$ (Y may not be compact and therefore the Borel–Moore homology is required, cf. [9]). Consequently,

$$H^*_{\mathrm{Nash}}(Y_0, \mathbb{Z}/2) = e^*(H^*_{\mathrm{alg}}(Y, \mathbb{Z}/2)). \tag{2}$$

The bordism class of the smooth map $g = e \circ \sigma \circ f : M \to Y$ in $\mathcal{N}_*(Y)$ is equal to the bordism class of a constant map from some compact smooth manifold K into Y. We may assume that K is a nonsingular real algebraic set. It follows that there exist a nonsingular real algebraic set X, a smooth diffeomorphism $\varphi : X \to M$, and a regular map $h : X \to Y$ homotopic to $g \circ \varphi$ (cf. [2, Theorem 2.8.2]). Since $\varphi^* \circ g^* = (g \circ \varphi)^* = h^*$ in cohomology,

$$\varphi^*(g^*(H^*_{\mathrm{alg}}(Y, \mathbb{Z}/2)) = h^*(H^*_{\mathrm{alg}}(Y, \mathbb{Z}/2)) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2). \tag{3}$$

On the other hand, $g^* = (e \circ \sigma \circ f)^* = f^* \circ \sigma^* \circ e^*$, and hence, in view of (1) and (2), we get

$$g^*(H^*_{\mathrm{alg}}(Y, \mathbb{Z}/2)) = f^*(H^*_{\mathrm{Nash}}(P, \mathbb{Z}/2)),$$

which combined with (3) yields

$$\varphi^*(f^*(H^*_{\mathrm{Nash}}(P, \mathbb{Z}/2))) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2).$$

The last inclusion means that $f^*(H^*_{\mathrm{Nash}}(P, \mathbb{Z}/2))$ admits an algebraic realization. ∎

COROLLARY 2.6. *Let $f : M \to P$ be a smooth map between compact smooth manifolds. If the bordism class of f in $\mathcal{N}_*(P)$ is zero, then the subring $f^*(H^*_{\mathrm{Nash}}(P, \mathbb{Z}/2))$ of $H^*(M, \mathbb{Z}/2)$ admits an algebraic realization.*

Proof. If the bordism class of f in $\mathcal{N}_*(P)$ is zero, then M is the boundary of a compact smooth manifold with boundary, and hence the bordism class of any constant map from M into P is zero. It now suffices to apply Theorem 2.5. ∎

Our next result is in the style of Nash's original paper [15].

THEOREM 2.7. *For any connected compact smooth manifold M there is a nonsingular real algebraic set X such that*

(i) *X has exactly two connected components, each diffeomorphic to M*,
(ii) *for any smooth map $h : M \to X$ transforming M diffeomorphically onto a connected component of X, one has*

$$H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2) = h^*(H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)).$$

Proof. Let $F : M \times [0,1] \to M$ be the canonical projection. Setting $\overline{M} = (M \times \{0\}) \cup (M \times \{1\})$ we let $f : \overline{M} \to M$ denote the restriction of F. The bordism class of f in $\mathcal{N}_*(M)$ is zero and hence, by Corollary 2.6, the subring $f^*(H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2))$ of $H^*(\overline{M}, \mathbb{Z}/2)$ admits an algebraic realization. Let X be an algebraic model of $\overline{M}$ and let $\varphi : X \to \overline{M}$ be a smooth diffeomorphism satisfying

$$\varphi^*(f^*(H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2))) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2).$$

By construction, X has exactly two connected components

$$X_0 = \varphi^{-1}(M \times \{0\}), \qquad X_1 = \varphi^{-1}(M \times \{1\}),$$

each diffeomorphic to M. Thus (i) holds.

To show that (ii) is also satisfied we argue as follows. Let $e_i : X_i \hookrightarrow X$ be the inclusion map, $i = 0, 1$. Since $e_i^* \circ \varphi^* \circ f^* = (f \circ \varphi \circ e_i)^*$ and $f \circ \varphi \circ e_i : X_i \to M$ is a smooth diffeomorphism, we get

$$H^*_{\mathrm{Nash}}(X_i, \mathbb{Z}/2) = e_i^*(\varphi^*(f^*(H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2)))) \subseteq e_i^*(H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)),$$

which immediately yields

$$H^*_{\mathrm{Nash}}(X_i, \mathbb{Z}/2) = e_i^*(H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)).$$

The last equality implies (ii). ■

As we already noted in Section 1, for any compact smooth manifold M, one has $A(M) \subseteq H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2)$, and $A^k(M) = H^k(M, \mathbb{Z}/2)$ if either $k = 0, 1$ or $k \geq \frac{1}{2} \dim M$, and assuming $\dim M \leq 7$ also $A^3(M) = H^3(M, \mathbb{Z}/2)$. Hence $H^k_{\mathrm{Nash}}(M, \mathbb{Z}/2) = H^k(M, \mathbb{Z}/2)$ for k and $\dim M$ satisfying the same restrictions. We shall now identify two conditions which the Nash cohomology classes always satisfy and show how this leads to a construction of manifolds with $H^i_{\mathrm{Nash}}(M, \mathbb{Z}/2) \neq H^i(M, \mathbb{Z}/2)$ for some i.

Denote by $\varrho_M : H^*(M, \mathbb{Z}) \to H^*(M, \mathbb{Z}/2)$ the homomorphism induced by the epimorphism $\mathbb{Z} \to \mathbb{Z}/2$. Set

$$B^k(M) = \{v \in H^k(M, \mathbb{Z}/2) \mid v \cup v \text{ is in } \varrho_M(H^{2k}(M, \mathbb{Z}))\},$$
$$B(M) = \bigoplus_{k \geq 0} B^k(M).$$

Note (this is not important for our purposes) that $B(M)$ is a subring of $H^*(M, \mathbb{Z}/2)$.

PROPOSITION 2.8. *For any compact smooth manifold M,*

(i) $H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2) \subseteq B(M)$,
(ii) $H^2_{\mathrm{Nash}}(M, \mathbb{Z}/2) = A^2(M)$.

Proof. Let X be a compact nonsingular real algebraic set. It follows from [3, Theorem A(b)] that

$$H^*_{\mathrm{alg}}(X, \mathbb{Z}/2) \subseteq B(X),$$

while, as we already recalled in Section 1,

$$H^2_{\mathrm{alg}}(X, \mathbb{Z}/2) \subseteq A^2(X).$$

Hence (i) and (ii) follow from Theorem 2.7. ∎

We can now reprove Corollary 1.3 in a more direct way.

Proof of Corollary 1.3. Let M be a compact smooth manifold of dimension 6 or 7. We have $A^k(M) = H^k_{\mathrm{Nash}}(M, \mathbb{Z}/2) = H^k(M, \mathbb{Z}/2)$ for all $k \neq 2$. In view of Proposition 2.8(ii), $A(M) = H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2)$. Conjecture C is true for M since $A(M)$ admits an algebraic realization. ∎

We shall next demonstrate that Proposition 2.8(i) gives a nontrivial condition.

EXAMPLE 2.9. For any positive even integer k and any integer $m \geq 2k+2$, there exists an m-dimensional orientable connected compact smooth manifold M with $B^k(M) \neq H^k(M, \mathbb{Z}/2)$. Proposition 2.8(i) implies that an element u in $H^k(M, \mathbb{Z}/2) \setminus B^k(M)$ is not in $H^k_{\mathrm{Nash}}(M, \mathbb{Z}/2)$, and hence u does not admit an algebraic realization.

We can construct such a manifold M as follows. It is known that there is a 6-dimensional orientable connected compact smooth manifold N with $B^2(N) \neq H^2(N, \mathbb{Z}/2)$ (cf. [17, Lemmas 1, 2]). Choose a cohomology class v in $H^2(N, \mathbb{Z}/2) \setminus B^2(N)$. Let $\mathbb{P}^2(\mathbb{C})$ be the complex projective plane and let z be the generator of $H^2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z}/2) \cong \mathbb{Z}/2$. Let $P = \mathbb{P}^2(\mathbb{C}) \times \cdots \times \mathbb{P}^2(\mathbb{C})$ be the ℓ-fold product, where $2\ell = k-2$, and let $w = z \times \cdots \times z$ in $H^{k-2}(P, \mathbb{Z}/2)$ be the ℓ-fold cross product; if $\ell = 0$, we assume that P consists of one point and $w = 1$. Let Q be the unit $(m-(2k+2))$-sphere; if $m = 2k+2$, then by convention, Q consists of one point. Set $M = N \times P \times Q$ and $u = v \times w \times 1$. Then M is an orientable connected compact smooth manifold of dimension m, and u is a cohomology class in $H^k(M, \mathbb{Z}/2)$. Making use of Künneth's theorem in cohomology, one readily checks that u is not in $B^k(M)$.

REMARK 2.10. If M is a compact smooth manifold and r is an odd positive integer, then $B^r(M) = H^r(M, \mathbb{Z}/2)$. Indeed, for any cohomology class b in $H^r(M, \mathbb{Z}/2)$, one has $b \cup b = \mathrm{Sq}^r(b) = \mathrm{Sq}^1(\mathrm{Sq}^{r-1}(b))$, where Sq^i is the ith Steenrod square (cf. [16, p. 281] or [14, p. 182]) and each class in the image of Sq^1 belongs to $\varrho_M(H^*(M,\mathbb{Z}))$ (cf. [14, p. 182]). In particular, the construction in Example 2.9 cannot be repeated with k odd.

Remark 2.11. Example 2.9 implies that the Nash homology does not behave in a functorial manner. More precisely, there exists a Nash map $f: L \to N$ between compact Nash manifolds such that $f_*([L])$ is not in $H_*^{\mathrm{Nash}}(N,\mathbb{Z}/2)$. One constructs L, N, and f as follows. In view of Example 2.9, there is a compact Nash manifold N having a homology class z which is not in $H_*^{\mathrm{Nash}}(N,\mathbb{Z}/2)$. By [18], $z = f_*([L])$ for some compact smooth manifold L and smooth map $f: L \to N$. We may assume that L is a Nash manifold and f is a Nash map (cf. [5, Corollary 8.9.7]).

3. Proofs of Propositions 1.1 and 1.2. Conjecture B is equivalent to the following statement: For any compact smooth manifold M, if E_1 and E_2 are subsets of $H^*(M,\mathbb{Z}/2)$, each admitting an algebraic realization, then the union $E_1 \cup E_2$ admits an algebraic realization.

Definition 3.1. A compact smooth manifold M is said to have *property* H^i, where $i=0$ or $i=1$, if for any subset E of $H^*(M,\mathbb{Z}/2)$ admitting an algebraic realization, the union $E \cup H^i(M,\mathbb{Z}/2)$ admits an algebraic realization.

Since $H^i(M,\mathbb{Z}/2)$, with $i=0$ or $i=1$, always admits an algebraic realization, M has property H^i, provided Conjecture B is true for M. Note that M has property H^0 if and only if for any subring R of $H^*(M,\mathbb{Z}/2)$ admitting an algebraic realization and for any connected component M' of M, the subring $e^*(R)$ of $H^*(M',\mathbb{Z}/2)$, where $e: M' \hookrightarrow M$ is the inclusion map, admits an algebraic realization.

It is hoped that each compact smooth manifold has property H^i. This would be interesting in view of the next two results.

Proposition 3.2. *Let M be a compact smooth manifold. If $M \times S^1$ has property H^1, then Conjecture C is true for M.*

Proof. Suppose $M \times S^1$ has property H^1. Let $\pi: M \times S^1 \to M$ be the canonical projection. By Corollary 2.6, the subring $R = \pi^*(H^*_{\mathrm{Nash}}(M,\mathbb{Z}/2))$ of $H^*(M \times S^1,\mathbb{Z}/2)$ admits an algebraic realization. Property H^1 implies that there exist an algebraic model Y of $M \times S^1$ and a smooth diffeomorphism $\psi: Y \to M \times S^1$ satisfying

$$\psi^*(R \cup H^1(M \times S^1,\mathbb{Z}/2)) \subseteq H^*_{\mathrm{alg}}(Y,\mathbb{Z}/2). \tag{1}$$

Choose a point y_0 in S^1 and let $i: M \times \{y_0\} \hookrightarrow M \times S^1$ be the inclusion map. Since $i^* \circ \pi^* = (\pi \circ i)^*$ and the canonical projection $\pi \circ i: M \times \{y_0\} \to M$ is a smooth diffeomorphism, we get

$$H^*_{\mathrm{Nash}}(M \times \{y_0\},\mathbb{Z}/2) = i^*(\pi^*(H^*_{\mathrm{Nash}}(M,\mathbb{Z}/2))) = i^*(R).$$

Set $K = \psi^{-1}(M \times \{y_0\})$ and let $\theta : K \to M \times \{y_0\}$ be the restriction of ψ. Since θ is a smooth diffeomorphism,

$$H^*_{\mathrm{Nash}}(K, \mathbb{Z}/2) = \theta^*(H^*_{\mathrm{Nash}}(M \times \{y_0\}, \mathbb{Z}/2)) = \theta^*(i^*(R)). \tag{2}$$

We have $i \circ \theta = \psi \circ j$, where $j : K \hookrightarrow Y$ is the inclusion map, and hence $\theta^* \circ i^* = j^* \circ \psi^*$. In view of (2),

$$H^*_{\mathrm{Nash}}(K, \mathbb{Z}/2) = j^*(\psi^*(R)). \tag{3}$$

It follows from (1) that $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2) = H^1(Y, \mathbb{Z}/2)$. This implies that K can be approximated by nonsingular algebraic subsets of Y. More precisely, there is a smooth diffeomorphism $\sigma : Y \to Y$, which can be chosen arbitrarily close to the identity map, such that $X = \sigma^{-1}(K)$ is a nonsingular algebraic subset of Y (cf. [8, Theorem 3.1] or [5, Theorem 12.4.11]). The restriction $\tau : X \to K$ of σ is a smooth diffeomorphism and hence

$$H^*_{\mathrm{Nash}}(X, \mathbb{Z}/2) = \tau^*(H^*_{\mathrm{Nash}}(K, \mathbb{Z}/2)),$$

which in view of (3) yields

$$H^*_{\mathrm{Nash}}(X, \mathbb{Z}/2) = \tau^*(j^*(\psi^*(R))). \tag{4}$$

We may assume that σ is homotopic to the identity map of Y. In particular, σ^* is the identity homomorphism. Thus denoting by $e : X \hookrightarrow Y$ the inclusion map, we get $\sigma \circ e = j \circ \tau$ and $e^* = e^* \circ \sigma^* = \tau^* \circ j^*$, which in view of (4) implies

$$H^*_{\mathrm{Nash}}(X, \mathbb{Z}/2) = e^*(\psi^*(R)). \tag{5}$$

Combining (1) and (5), we obtain

$$H^*_{\mathrm{Nash}}(X, \mathbb{Z}/2) \subseteq e^*(H^*_{\mathrm{alg}}(Y, \mathbb{Z}/2)) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2),$$

where the last inclusion follows from the fact that $e : X \hookrightarrow Y$ is a regular map. Thus

$$H^*_{\mathrm{Nash}}(X, \mathbb{Z}/2) = H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$$

and the proof is complete since X is diffeomorphic to M. ∎

PROPOSITION 3.3. *Let M be a compact smooth manifold. If $\overline{M}$ has property H^0, then Conjecture C is true for M.*

Proof. Define $f : \overline{M} \to M$ by $f(x, i) = x$ for x in M and $i = 0, 1$. Clearly, the bordism class of f in $\mathcal{N}_*(M)$ is zero. By Corollary 2.6, the subring $R = f^*(H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2))$ of $H^*(\overline{M}, \mathbb{Z}/2)$ admits an algebraic realization. Define $e : M \to \overline{M}$ by $e(x) = (x, 0)$ for x in M. If $\overline{M}$ has property H^0, then the subring $e^*(R)$ of $H^*(M, \mathbb{Z}/2)$ admits an algebraic realization. Observing that $f \circ e : M \to M$ is the identity map, we get

$$H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2) = e^*(f^*(H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2))) = e^*(R),$$

which completes the proof. ∎

Proof of Proposition 1.1. If Conjecture B is true for $M \times S^1$ (resp. $\overline{M}$), then $M \times S^1$ (resp. $\overline{M}$) has property H^1 (resp. H^0). The proof is complete in view of Propositions 3.2 and 3.3. ■

Proof of Proposition 1.2. Suppose that Conjecture A is true for $M \times S^1$. Let $\pi : M \times S^1 \to M$ be the canonical projection. By Corollary 2.6, the subring $\pi^*(H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2))$ of $H^*(M \times S^1, \mathbb{Z}/2)$ admits an algebraic realization, and hence

$$\pi^*(H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2)) \subseteq A(M \times S^1).$$

Fix a point y_0 in S^1 and define $e : M \to M \times S^1$ by $e(x) = (x, y_0)$ for x in M. Since $e^* \circ \pi^* = (\pi \circ e)^*$ and $\pi \circ e : M \to M$ is the identity map, we get

$$H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2) = e^*(\pi^*(H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2))) \subseteq e^*(A(M \times S^1)) \subseteq A(M),$$

which implies $H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2) = A(M)$. Thus Proposition 1.2 is proved under the hypothesis that Conjecture A is true for $M \times S^1$ (recall that $A(M)$ admits an algebraic realization).

Suppose now that Conjecture A is true for $\overline{M}$. Then Conjecture B is true for $\overline{M}$. In view of Proposition 1.1, Conjecture C is true for M, which implies that it is also true for $\overline{M}$. Thus $A(\overline{M}) = H^*_{\mathrm{Nash}}(\overline{M}, \mathbb{Z}/2)$, which yields $A(M) = H^*_{\mathrm{Nash}}(M, \mathbb{Z}/2)$. The last equality means that Conjecture A is true for M. ■

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Reçu par la Rédaction le 2.6.2004 (1621)