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Bi-Lipschitz trivialization of the distance function to a stratum of a stratification

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Abstract. Given a Lipschitz stratification $\mathcal{X}$ that additionally satisfies condition (δ) of Bekka–Trotman (for instance any Lipschitz stratification of a subanalytic set), we show that for every stratum N of $\mathcal{X}$ the distance function to N is locally bi-Lipschitz trivial along N. The trivialization is obtained by integration of a Lipschitz vector field.

The existence of Lipschitz stratifications of complex analytic or real subanalytic sets ([5], [7], [8]) allows one to trivialize these sets locally along each stratum so that the trivialization is bi-Lipschitz. In this paper we show the following refinement of this result that answers positively a question posed to us by M. Ferrarotti and E. Fortuna.

Theorem 1. *Let $\mathcal{X}$ be a Lipschitz stratification of a locally closed subset $X \subset \mathbb{R}^n$ and let N be a stratum of $\mathcal{X}$. Suppose additionally that $\mathcal{X}$ satisfies condition (δ) of Bekka–Trotman along N. Then $\mathcal{X}$ can be trivialized locally along N in such a way that the distance to N is preserved by the trivialization and the trivialization is bi-Lipschitz.*

We explain in Remark 2 below what we precisely mean by bi-Lipschitz trivialization.

We always assume that the strata of $\mathcal{X}$ are C^2 subvarieties of $\mathbb{R}^n$. We say that $\mathcal{X}$ satisfies *condition (δ) of Bekka–Trotman along N* (see [1]) if for any $p \in N$, there are $c_0 > 0$ and $\varepsilon > 0$ such that for all $q \in X$ satisfying $\operatorname{dist}(p,q) < \varepsilon$, there is a unit vector $\mathbf{v} \in T_qS$, where S is the stratum containing q, such that

$$d\varrho(\mathbf{v}) \geq c_0, \tag{1}$$

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where ϱ denotes the distance function to N. Geometrically it means that the strata of $\mathcal{X}$ transversally intersect the levels of ϱ with the angle uniformly bounded away from zero. We note (cf. [5, Proposition 7.1]) that each Lipschitz stratification satisfies the Kuo–Verdier condition (w) (cf. [9]), and hence any Lipschitz stratification of a subanalytic set is Whitney regular. Whitney's (b)-regularity trivially implies condition (δ) of Bekka–Trotman.

One may ask whether any subanalytic function f on X can be trivialized along strata of a stratification of X so that the trivialization is bi-Lipschitz. For this general question the answer is definitely negative, even for families of analytic function germs of two complex or real variables (see [2], [3]). Besides the distance to a stratum and similar distance-like functions (see Remark 7 below), it is not clear to us for what other types of functions the answer to this question is positive.

The proof of Theorem 1 is fairly elementary and uses the method introduced in [5]. The author thanks T. Mostowski for interesting discussions concerning the problem.

Proof of Theorem 1. Fix a stratum N and $p \in N$. Let $k = \dim N$. Denote by X^i the union of strata of dimension $\le i$ and by $d_i(q)$ the distance from q to X^i. The open ball centred at q and of radius r will be denoted by $B(q,r)$. As above, the distance function to N will be denoted by ϱ. We shall work in a small neighbourhood U of p in X that we will make smaller if necessary. All vector fields will be tangent to strata.

We will be sometimes sloppy about constants. In general, $\varepsilon > 0$ will be used to denote very small constants, M very large positive constants, $C > 0$ will denote a universal constant and $L > 0$ a universal constant for the Lipschitz condition. Our main tool will be the extension property of Lipschitz vector fields (cf. [7, Remark 1.3], [8, Proposition 1.3], [5, Proposition 1.1] and [7, Lemma 1.7]).

The main idea of the proof is simple. Given a system of Lipschitz vector fields $\mathbf{e}_1, \ldots, \mathbf{e}_k$ defined in a neighbourhood of p in N that form a basis of T_pN. By Gram–Schmidt orthonormalization we may suppose that for each p' close to p, $\mathbf{e}_1(p'), \ldots, \mathbf{e}_k(p')$ is an orthonormal basis of $T_{p'}N$. We will show that there exist extensions of $\mathbf{e}_1, \ldots, \mathbf{e}_k$ to Lipschitz vector fields $\widehat{\mathbf{e}}_1, \ldots, \widehat{\mathbf{e}}_k$, defined on U, that satisfy

$$d\varrho(\widehat{\mathbf{e}}_i) \equiv 0 \quad \text{for } i = 1, \ldots, k.$$

Let $\pi : U \to N$ denote the closest point projection. Using the argument of Proposition 1.1 of [5] or Lemma 1.7 of [7], we may then change the vector fields $\widehat{\mathbf{e}}_i$ so that they additionally satisfy

$$d\pi(\widehat{\mathbf{e}}_i) = \mathbf{e}_i, \qquad i = 1, \ldots, k.$$

Then the theorem follows by integration of these vector fields as in loc. cit.

REMARK 2. By *local bi-Lipschitz triviality* of ϱ along N we mean the following. Let $p_0 \in N$ and let U be a small neighbourhood of p_0 in X. Define $U_N = U \cap N$ and $U_p = \pi^{-1}(p) \cap U$ for $p \in U_N$. Let ϱ_p be the restriction of ϱ to U_p. Then for every p_0 there is such a U and a bi-Lipschitz homeomorphism $\Phi : U_{p_0} \times U_N \to U$ such that $\pi(\Phi(x,y)) = y$ and $\varrho(\Phi(x,y)) = \varrho_p(x)$.

By adding the open stratum $\mathbb{R}^n \setminus X$ we may, moreover, assume that the above trivialization is the restriction to X of a local bi-Lipschitz trivialization of the ambient space $\mathbb{R}^n$.

Now we present the details.

LEMMA 3. *There are a neighbourhood U of p and positive constants ε, c, C, L such that for any $q_0 \in U \setminus N$ there is a Lipschitz vector field $\mathbf{v}$ defined on $B(q_0, \varepsilon\varrho(q_0))$, with Lipschitz constant $L\varrho(q_0)^{-1}$, such that for all $q \in B(q_0, r)$,*

$$\|\mathbf{v}(q)\| \le C, \tag{2}$$

$$d\varrho(\mathbf{v}(q)) \ge c. \tag{3}$$

Proof. First fix an arbitrary small neighbourhood U of p in $\mathbb{R}^n$. It will be replaced later by a smaller neighbourhood of p. Let $q_0 \in U \setminus N$, S the stratum containing q_0, and let $\mathbf{v}_0 \in T_{q_0}S$ be a vector satisfying condition (1). The vector field $\mathbf{v}$ of the statement of the lemma will satisfy $\mathbf{v}(q_0) = \mathbf{v}_0$.

Assume now that $q_0 \in X^j \setminus X^{j-1}$. We abbreviate $\varrho(q_0)$ by ϱ_0. The proof is by induction on $j = k+1, \ldots, \dim X$. First we explain the inductive step. Let ε', c', C', L' be the constants for which the lemma holds for each $q_0 \in X^{j-1} \cap U$. Fix a large $M > 0$, in particular we require $M^{-1} \le \varepsilon'/3$.

CASE 1: Suppose that for all $i = k, \ldots, j-2$,

$$d_i(q_0) \le M d_{i+1}(q_0). \tag{4}$$

Then $\varrho_0 \le M^{n-k} d_{j-1}(q_0)$. The vector field on $X^{j-1} \cup \{q_0\}$ that is identically equal to zero on X^{j-1} and to $\mathbf{v}_0$ at q_0 is Lipschitz with Lipschitz constant $\varrho_0^{-1} M^{n-k}$. By [7] and [8] this vector field can be extended to a Lipschitz vector field on X with constant $\varrho_0^{-1} M^{n-k} L_{\mathcal{S}}$, where $L_{\mathcal{S}}$ denotes a universal constant depending only on the Lipschitz stratification. An easy computation shows that this extension has the required property on $B(q_0, \varepsilon\varrho_0)$, provided $\varepsilon > 0$ is chosen sufficiently small ($\varepsilon \le \frac{1}{2}(M^{n-k}L_{\mathcal{S}})^{-1} c_0$ would do, where c_0 is given by Bekka–Trotman's condition).

Let $k' > k$ be the minimum dimension of strata S such that $p \in \overline{S} \setminus S$. Note that if $j = k'$ then we are necessarily in Case 1. Thus Case 1 gives also the initial step of the induction.

CASE 2: Set $j_0 = i_0 + 1$, where i_0 is the largest $i \in \{k, \ldots, j-2\}$ for which (4) fails. That is, (4) holds for $i \ge j_0$ but

$$d_{j_0-1}(q_0) > M d_{j_0}(q_0).$$

Fix $q' \in X^{j_0} \setminus X^{j_0-1}$ such that $\|q_0 - q'\| = d_{j_0}(q_0)$. We have

$$\|q_0 - q'\| = d_{j_0}(q_0) < M^{-1} d_{j_0-1}(q_0) \le M^{-1}\varrho_0.$$

Thus if $M^{-1} \le \varepsilon'/3$ then $B(q_0, \varepsilon'\varrho_0/3) \subset B(q', \varepsilon'\varrho(q'))$ and the vector field constructed for q' works as well for q_0. ∎

Next we construct the desired vector fields $\widehat{\mathbf{e}}_i$ locally near each $q_0 \in U \setminus N$. Let $\widetilde{\mathbf{e}}_i$ be arbitrary Lipschitz extensions of $\mathbf{e}_i$ onto U. Define, on $B(q_0, \varepsilon\varrho(q_0))$,

$$\widehat{\mathbf{e}}_i := \widetilde{\mathbf{e}}_i - \frac{d\varrho(\widetilde{\mathbf{e}}_i)}{d\varrho(\mathbf{v})}\,\mathbf{v}, \tag{5}$$

where $\mathbf{v}$ is given by Lemma 3. Clearly $d\varrho(\widehat{\mathbf{e}}_i) \equiv 0$. We show that the $\widehat{\mathbf{e}}_i$ are Lipschitz extensions of $\mathbf{e}_i$. We use the following obvious lemma.

Lemma 4. *Let f_1, f_2 be two bounded Lipschitz functions with Lipschitz constants L_1, L_2 and bounded by C_1, C_2 respectively. Then the product $f_1 f_2$ is Lipschitz with constant $C_1L_2 + L_1C_2$. If moreover $|f_1| \ge c_1$ then $1/f_1$ is Lipschitz with constant $L_1 c_1^{-2}$.*

To use the lemma we need to establish the Lipschitz constants and the bounds for the vector fields $\mathbf{v}, d\varrho(\mathbf{v}), d\varrho(\widetilde{\mathbf{e}})$.

For $\mathbf{v}$ they are given in Lemma 3. Since ϱ is a distance function, $|d\varrho(\mathbf{v})| \le \|\mathbf{v}\| \le C$. Moreover, the second order partial derivatives of ϱ can be universally bounded by a multiple of ϱ^{-1}. (If N is a C^2 submanifold then ϱ^2 is of class C^2 in a neighbourhood of p.) Consequently, by Lemma 4, the Lipschitz constant of $d\varrho(\mathbf{v})$ can be universally bounded by a multiple of $\varrho(q_0)^{-1}$.

Finally,

$$d\varrho(q)(\widetilde{\mathbf{e}}_i(q)) = d\varrho(q)(\widetilde{\mathbf{e}}_i(q) - \mathbf{e}_i(\pi(q))) + d\varrho(q)(\mathbf{e}_i(\pi(q))).$$

Since π is the projection to the closest point, the last summand on the right hand side is equal to zero. On the other hand, $\|\widetilde{\mathbf{e}}_i(q) - \mathbf{e}_i(\pi(q))\| \le L\varrho(q)$, where L denotes the Lipschitz constant of $\widetilde{\mathbf{e}}_i$. Therefore $d\varrho(q)(\widetilde{\mathbf{e}}_i(q))$ is bounded by $L\varrho(q)$ and is Lipschitz (with a universal Lipschitz constant). This concludes the argument showing that the vector fields $\widehat{\mathbf{e}}_i$ of (5) are Lipschitz.

Note also that $\widehat{\mathbf{e}}_i$, extended by $\mathbf{e}_i$ to $N \cup B(q_0, \varepsilon\varrho(q_0))$, is Lipschitz. We use the following lemma to glue the Lipschitz vector fields thus constructed.

Lemma 5 (after Lemma 3.1 of [4]). *Given $\alpha > 0$. There is $M > 0$ and an* (*infinite*) *family of functions $\varphi_m \ge 0$ on $U \setminus N$ such that*

(1) *for each $x \in U \setminus N$ only finitely many $\varphi_m(x) \neq 0$,*
(2) $\sum_m \varphi_m \equiv 1$,
(3) *for all $m \in \mathbb{N}$,* $\operatorname{diam}\operatorname{supp}\varphi_m \le \alpha \operatorname{dist}(\operatorname{supp}\varphi_m, N)$,
(4) *each φ_m is Lipschitz with constant* $M(\operatorname{dist}(\operatorname{supp}\varphi_m, N))^{-1}$.

Thus choosing q_m so that $\operatorname{supp}\varphi_m \subset B(q_m, \varepsilon\varrho(q_m))$ and choosing all constants appropriately we construct

$$\widehat{\mathbf{e}}_i = \sum \varphi_m \widehat{\mathbf{e}}_{i,m},$$

where $\widehat{\mathbf{e}}_{i,m}$ denote the vector fields constructed above in $B(q_m, \varepsilon\varrho(q_m))$. These vector fields have the desired properties. This ends the proof of Theorem 1. ∎

Note that in Lemma 3 we may additionally require that $d\pi(\mathbf{v}) \equiv 0$. Indeed, we may write

$$d\pi(\mathbf{v}(q)) = \sum_i \lambda_i(q)\mathbf{e}_i(\pi(q)).$$

The functions λ_i thus defined are Lipschitz with a universal constant. Then

$$\widehat{\mathbf{v}} = \mathbf{v} - \sum_i \lambda_i \widehat{\mathbf{e}}_i$$

has the desired properties.

Corollary 6. *There exists a Lipschitz vector field* $\mathbf{w}$ *defined on* $U \setminus N$ *such that* $d\pi(\mathbf{w}) \equiv 0$ *and for all* $q \in U \setminus N$,

$$\|\mathbf{w}(q)\| \le C\varrho(q), \qquad d\varrho(\mathbf{w}(q)) \ge c\varrho(q).$$

Proof. Locally on $B(q_0, \varepsilon\varrho_0)$ we put $\mathbf{w} := \varrho\widehat{\mathbf{v}}$. Then we glue these vector fields using Lemma 5 as above. ∎

Remark 7. The above argument allows us to trivialize along N functions more general than the distance function to the stratum N.

More precisely, suppose $\mathcal{X}$ is a Lipschitz stratification of $X \subset \mathbb{R}^n$ such that Whitney's condition (b) is satisfied along a stratum N. Let U be an open subset of X and let $f : U \to \mathbb{R}$. We suppose that for every $p \in U \cap N$ there is a neighbourhood V of p in $\mathbb{R}^n$ and a C^2 function $\varphi : V \to \mathbb{R}^{n-k}$ such that $f = \|\varphi\|$ on $V \cap X$, 0 is a regular value of φ, and $N = \varphi^{-1}(0)$. Then f is locally bi-Lipschitz trivial along $U \cap N$.

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