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A note on Bézout's theorem

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Abstract. We present a version of Bézout's theorem basing on the intersection theory in complex analytic geometry. Some applications for products of surfaces and curves are also given.

1. Introduction. In the intersection theory in complex analytic geometry constructed in [T] and completed by [AR], [R_1], [R_2], [R_3], [R_4], [R_5], for a system $X_1, \ldots, X_p$ of irreducible analytic subsets of a complex manifold N, one can define the analytically costructible (upper semicontinuous in the Zariski topology) function

$$d : N \ni x \mapsto d(x) := d(X_1, \ldots, X_p)(x) \in \mathbb{N},$$

which assigns to each point x in N the *multiplicity of intersection* of the sets $X_1, \ldots, X_p$ at the point x (see also [A], [ATW] and [CKT]). By definition, the *intersection product* of these sets is the unique analytic cycle $X_1 \bullet \cdots \bullet X_p$ defined by the equality

$$\nu(X_1 \bullet \cdots \bullet X_p, x) = d(X_1, \ldots, X_p)(x) \quad \text{for } x \in N,$$

where $\nu(C, x)$ denotes the degree of the analytic cycle C at x. One can extend this definition in the natural way to the case of analytic cycles. Some details of this construction will be given in the next section of this paper.

One can observe that for a system of analytic cycles $C_1, \ldots, C_p$ on N and each point $a \in N$ we have "by definition" the Bézout type equality

$$\nu(C_1 \bullet \cdots \bullet C_p, a) = d(C_1, \ldots, C_p)(a),$$

saying that the multiplicity of the intersection product of our system coincides with the intersection multiplicity at the point. Usually it is possible to

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find relations between $d(C_1, \ldots, C_p)(a)$ and the product $\nu(C_1, a) \cdots \nu(C_p, a)$ of the multiplicities of the cycles, and so we obtain inequalities or equalities extending the classical Bézout's theorem to the case of analytic sets.

The organization of this paper is as follows. Section 2 is of preparatory nature; we collect together basic constructions and facts on intersection theory in complex analytic geometry. In Section 3 our main results are proved, and then used, in Section 4, to present some applications for products of surfaces and curves. Some comments on the presented version of Bézout's theorem can be found in [N$_2$] and [N$_1$].

2. Intersection theory. In this paper *analytic* means complex analytic, and *manifold* means a second-countable complex manifold. An *analytic cycle* on a manifold M is a formal sum

$$C = \sum_{j \in J} \alpha_j Y_j,$$

where $\alpha_j \neq 0$ for $j \in J$ are integers and $\{Y_j\}_{j \in J}$ is a locally finite family of nonempty, pairwise distinct, irreducible analytic subsets of the manifold M. The zero cycle $C = 0$ is defined by $J = \emptyset$. The family of all analytic cycles on N will be considered with the natural structure of $\mathbb{Z}$-module.

The analytic set $\bigcup_{j \in J} Y_j$ is called the *support* of the cycle C. The sets Y_j are called the *components* of C with *multiplicities* α_j, $j \in J$. We say that the cycle C is *positive* if $\alpha_j > 0$ for all $j \in J$. If all the components of C have the same dimension k, C will be called a *k-cycle*. For a cycle C and an open subset V of M we can define in the natural way the *restriction* $C \cap V$ of C to V (cf. [R$_3$]). If $\varphi : M \to M'$ is a biholomorphism of manifolds then we define the *image* $\varphi(C)$ of C by

$$\varphi(C) = \sum_{j \in J} \alpha_j \varphi(Y_j).$$

Now, let M be an m-dimensional manifold and let Y be a pure k-dimensional analytic subset of M. For $x \in N$ we denote by $\nu(Y, x)$ the degree of Y at the point x [D, p. 194]. This degree is equal to the classical algebraic Samuel multiplicity, and the so-called Lelong number of Y at x. In this paper we will consider a natural extension of this definition to the case of an arbitrary analytic cycle. Namely, if $C = \sum_{j \in J} \alpha_j Y_j$ is an analytic cycle on M, then the sum

$$\nu(C, x) = \sum_{j \in J} \alpha_j \nu(Y_j, x)$$

is well defined and we call it the *degree* of the cycle C at the point x.

For the cycle C there exists a unique decomposition

$$C = C_{(m)} + C_{(m-1)} + \cdots + C_{(0)},$$

where $C_{(j)}$ is a j-cycle for $j = 0, \dots, m$. We define the *extended degree* of C at x by the formula

$$\widetilde{\nu}(C, x) = (\nu(C_{(m)}, x), \dots, \nu(C_{(0)}, x)) \in \mathbb{Z}^{m+1}.$$

Denote by $\nu(C)$ and $\widetilde{\nu}(C)$ the functions

$$\nu(C) : M \ni x \mapsto \nu(C, x) \in \mathbb{Z}, \quad \widetilde{\nu}(C) : M \ni x \mapsto \widetilde{\nu}(C, x) \in \mathbb{Z}^{m+1}.$$

Observe that $\nu(C, x) = |\widetilde{\nu}(C, x)|$, where $|\mu|$ denotes the sum of the coordinates of $\mu \in \mathbb{Z}^{m+1}$.

Let M be an m-dimensional manifold and let S be a closed s-dimensional submanifold of M. For an arbitrary analytic cycle $C = \sum_{j \in J} \alpha_j Y_j$ in M the part of C supported by S is defined as

$$C^S = \sum_{j \in J,\, Y_j \subset S} \alpha_j Y_j.$$

Observe that every analytic cycle has the decomposition $C = C^S + (C - C^S)$. If C is positive, then so are both parts of this decomposition.

Let U be an open subset of M such that $U \cap S \neq \emptyset$. Denote by $\mathcal{H}(U)$ the set of all $\mathcal{H} := (H_1, \dots, H_{m-s})$ satisfying the following conditions:

(1) H_j is a smooth hypersurface in U containing $U \cap S$ for $j = 1, \dots, m-s$,
(2) $\bigcap_{j=1}^{m-s} T_x(H_j) = T_x S$ for each $x \in U \cap S$.

For a given analytic subset Z of M of pure dimension k we denote by $\mathcal{H}(U, Z)$ the set of all $\mathcal{H} \in \mathcal{H}(U)$ such that $((U \setminus S) \cap Z) \cap H_1 \cap \dots \cap H_j$ is an analytic subset of $U \setminus S$ of pure dimension $k - j$ (or empty) for $j = 1, \dots, m-s$.

For every $\mathcal{H} = (H_1, \dots, H_{m-s}) \in \mathcal{H}(U, Z)$ an analytic cycle $Z \cdot \mathcal{H}$ in $S \cap U$ is defined by the following algorithm (cf. [T], see also [F]), where in each step we have only proper intersections, and so the intersection product is given by the classical theory (cf. [D], [W$_1$]). For $\mathcal{H}$ define

$$d := \min\Big\{ j \in \{0, 1, \dots, m-s\} : ((U \setminus S) \cap Z) \cap \bigcap_{i=1}^{j} H_i = \emptyset \Big\},$$

and consider the following

Algorithm 1.

Step 0: Let $Z_0 = Z \cap U$. Then $Z_0 = Z_0^{S \cap U} + (Z_0 - Z_0^{S \cap U})$.
Step 1: Let $Z_1 = (Z_0 - Z_0^{S \cap U}) \cdot H_1$ and $Z_1 = Z_1^{S \cap U} + (Z_1 - Z_1^{S \cap U})$.
Step 2: Let $Z_2 = (Z_1 - Z_1^{S \cap U}) \cdot H_2$ and $Z_2 = Z_2^{S \cap U} + (Z_2 - Z_2^{S \cap U})$.
. . . .
Step d: Let $Z_d = (Z_{d-1} - Z_{d-1}^{S \cap U}) \cdot H_d$. We have $Z_d = Z_d^{S \cap U}$.

We call a positive analytic cycle $Z \cdot \mathcal{H} = Z_0^{S \cap U} + Z_1^{S \cap U} + \dots + Z_d^{S \cap U}$ in $S \cap U$ the *result* of the above algorithm.

Definition 2. For $c \in S$ we call

$$\widetilde{g}(c) = \widetilde{g}(Z,S)(c) := \min_{\mathrm{lex}}\{\widetilde{\nu}(Z \cdot \mathcal{H}, c) : \mathcal{H} \in \mathcal{H}(U,Z),\ c \in U\} \in \mathbb{N}^{s+1}$$

the *extended index of intersection* and $g(c) = |\widetilde{g}(c)|$ the *index of intersection* of Z with the submanifold S at the point c.

For a system of irreducible analytic sets $X_1, \ldots, X_p$ on a complex manifold N we can consider the analytic set $Z = X_1 \times \cdots \times X_p$ in $M = N^p$ and $S = \Delta_N$ the diagonal of N^p. By [T, Thm. 6.2] the function

$$d : N \ni x \mapsto d(x) := d(X_1, \ldots, X_p)(x) = g(X_1 \times \cdots \times X_p, \Delta_N)(x, \ldots, x) \in \mathbb{N}$$

is analytically costructible and by [T, Prop. 2.1] we can state the following

Definition 3. The *intersection product* of the system $X_1, \ldots, X_p$ is the unique analytic cycle $X_1 \bullet \cdots \bullet X_p$ such that $\nu(X_1 \bullet \cdots \bullet X_p, x) = d(x)$ for $x \in N$.

One can extend this definition in the natural way to the case of analytic cycles (see [T, Def. 6.4]). Namely, let

$$C_k = \sum_{j_k \in J_k} \alpha_{j_k}^{(k)} X_{j_k}^{(k)}, \qquad k = 1, \ldots, p,$$

be analytic cycles on a manifold N.

Definition 4. The *intersection product* of the cycles $C_1, \ldots, C_p$ is defined by

$$C_1 \bullet \cdots \bullet C_p = \sum_{j_k \in J_k,\, k=1,\ldots,p} \alpha_{j_1}^{(1)} \cdots \alpha_{j_p}^{(p)} X_{j_1}^{(1)} \bullet \cdots \bullet X_{j_p}^{(p)}.$$

We conclude this section with a useful theorem which follows directly from our definitions and [AR, Cor. 5].

Theorem 5. *If $C_1, \ldots, C_p$ is a system of cycles on a complex manifold N, V is an open subset of N and $\varphi : V \to N'$ is a biholomorphism, then*

$$\varphi(C_1 \cap V) \bullet \cdots \bullet \varphi(C_p \cap V) = \varphi((C_1 \bullet \cdots \bullet C_p) \cap V).$$

3. Bézout's theorem. In this section an analytic cycle $A = \sum_{j \in J} \alpha_j X_j$ in $\mathbb{C}^n$ is called *homogeneous* (resp. *projective*) if all sets X_j are cones in $\mathbb{C}^n$ (resp. projective varieties).

Theorem 6. *Let $C_1, \ldots, C_p$ be homogeneous cycles. Then the cycle $C_1 \bullet \cdots \bullet C_p$ is homogeneous and*

$$\deg(C_1 \bullet \cdots \bullet C_p) = \deg C_1 \cdots \deg C_p.$$

Proof. We maintain the notation of the previous section. It suffices to prove the theorem for irreducible cones $X_1, \ldots, X_p$. Observe that for $Z = X_1 \times \cdots \times X_p$,

$$d(X_1, \ldots, X_p)(0) = g(Z, \Delta_{\mathbb{C}^n})(0, \ldots, 0) = |\widetilde{g}(Z, \Delta_{\mathbb{C}^n})(0)|.$$

By [AR, Cor. 3] there exists a system $\mathcal{H} = (H_1, \dots, H_{m-s})$ of hyperplanes such that $\widetilde{g}(Z, \Delta_{\mathbb{C}^n})(0) = \widetilde{\nu}(Z \cdot \mathcal{H}, 0)$. The classical Bézout's theorem used in each step of Algorithm 1 gives

$$|\widetilde{g}(Z, \Delta_{\mathbb{C}^n})(0)| = \nu(Z, 0) = \deg X_1 \cdots \deg X_p.$$

Then we have the equality

$$\nu(X_1 \bullet \dots \bullet X_p, 0) = d(X_1, \dots, X_p)(0) = \deg X_1 \cdots \deg X_p.$$

It is easy to check that $X_1 \bullet \dots \bullet X_p$ is homogeneous, so

$$\nu(X_1 \bullet \dots \bullet X_p, 0) = \deg X_1 \bullet \dots \bullet X_p$$

and the theorem follows. ∎

For a projective cycle $C = \sum_{j \in J} \alpha_j Y_j$ we define the cycle $\widehat{C} := \sum_{j \in J} \alpha_j \widehat{Y}_j$, where $\widehat{Y}_j$ denotes the cone over the variety Y_j.

In the case of the analytic intersection product we have the following version of Bézout's theorem.

THEOREM 7. *Let $C_1, \dots, C_p$ be projective cycles and let α_0 be the multiplicity of the point $\{0\}$ in the intersection product $\widehat{C}_1 \bullet \dots \bullet \widehat{C}_p$. Then*

$$\deg(C_1 \bullet \dots \bullet C_p) = \deg C_1 \cdots \deg C_p - \alpha_0.$$

Proof. Since the intersection product is linear (see [AR, Cor. 5]), we can assume that $C_1, \dots, C_p$ are varieties of dimensions $d_1, \dots, d_p$ respectively.

We choose a point $Q := (1 : q_1 : \ldots : q_n) \in C_1 \cap \ldots \cap C_p$. To simplify our notation we put $V := (\{1\} \times \mathbb{C}^n)^p \subset \mathbb{C}^{p(n+1)}$ and $\mathcal{Q} := (1, q_1, \dots, q_n)$. The Grassmannian of affine r-planes through the point $\mathcal{Q}$ is denoted by $\mathrm{G}^r_{\mathcal{Q}}(\mathbb{C}^{p(n+1)})$.

By [AR, Cor. 3] the set of affine hyperplanes H containing $\mathcal{Q}$, meeting the space V properly and satisfying the condition

$$\nu\Big(\Big(\prod_{i=1}^{p} \widehat{C}_i \cdot H\Big)^{\Delta}, Q\Big) = \Big(\widetilde{g}\Big(\prod_{i=1}^{p} \widehat{C}_i, \Delta\Big)(Q)\Big)_1,$$

where $(v)_1$ stands for the first coordinate of a vector v, is an open and dense subset of $\mathrm{G}^{p(n+1)-1}_{\mathcal{Q}}(\mathbb{C}^{p(n+1)})$. Since the map

$$H \mapsto H \cap V \in \mathrm{G}^{p(n+1)-2}_{\mathcal{Q}}(V)$$

restricted to the set of hyperplanes that do not contain V is continuous, we can assume that

$$\nu\Big(\Big(\prod_{i=1}^{p} C_i \cdot (H \cap V)\Big)^{\Delta}, \mathcal{Q}\Big) = \Big(\widetilde{g}\Big(\prod_{i=1}^{p} C_i, \Delta\Big), Q\Big)_1.$$

Proceeding inductively we construct a system $\mathcal{H} = (H_1, \dots, H_{pn-\sum d_i})$ of affine hypersurfaces such that

$$\widetilde{\nu}\Big(\prod \widehat{C}_i \cdot \mathcal{H}, \mathcal{Q}\Big) = \widetilde{g}\Big(\prod \widehat{C}_i, \Delta\Big)(\mathcal{Q}),$$
$$\widetilde{\nu}\Big(\prod C_i \cdot \mathcal{H}|_V, \mathcal{Q}\Big) = \widetilde{g}\Big(\prod C_i, \Delta\Big)(Q),$$

where $\mathcal{H}|_V := (H_1 \cap V, \dots, H_{pn-\sum d_i} \cap V)$. Here, by abuse of notation, both the diagonal in $(\mathbb{C}^n)^p$ and the one in $(\mathbb{P}^n)^p$ are denoted by Δ. Moreover, we identify Q, resp. $\mathcal{Q}$, with the corresponding point in Δ.

Applying [TW, Thm. 2.2] in each step of Algorithm 1 we get

$$\widetilde{\nu}\Big(\prod \widehat{C}_i \cdot \mathcal{H}, \mathcal{Q}\Big) = \widetilde{\nu}\Big(\prod C_i \cdot \mathcal{H}|_V, \mathcal{Q}\Big)$$

which implies

$$\widetilde{g}\Big(\prod \widehat{C}_i, \Delta\Big)(\mathcal{Q}) = \widetilde{g}\Big(\prod C_i, \Delta\Big)(Q).$$

As an immediate consequence we obtain the equality

$$\widehat{C}_1 \bullet \dots \bullet \widehat{C}_p = \sum_{j \ge 1} \alpha_j \widehat{Y}_j + \alpha_0 \cdot \{0\},$$

where $C_1 \bullet \dots \bullet C_p := \sum_{j\ge 1} \alpha_j Y_j$. By Theorem 6, we have

$$\deg(C_1 \bullet \dots \bullet C_p) = \deg C_1 \cdots \deg C_p - \alpha_0. \ \blacksquare$$

4. Applications. To show an application of Theorem 6 we analyze the product of an algebraic surface and a curve. We start with the following lemma:

LEMMA 8. *Let S be a projective surface and let $C \subset S$ be a curve that has no common component with* $\mathrm{Sing}(S)$. *Then*

$$\widehat{S} \bullet \widehat{C} = \widehat{C} + \sum_{i=1}^{r} \alpha_i \,\widehat{a}_i + \alpha \cdot \{0\},$$

where

(1) $a_i \in \mathrm{Sing}(S) \cup \mathrm{Sing}(C)$ *for* $i = 1, \dots, r$,
(2) $\alpha_i \ge (\nu(S, a_i) - 1)\, \nu(C, a_i)$ *for* $i = 1, \dots, r$,
(3) $\alpha \ge 0$.

Proof. (1) It is obvious that all components of the cycle $\widehat{S} \bullet \widehat{C}$ are cones. If $a \in \mathrm{Reg}(S) \cap \mathrm{Reg}(C)$, then the germ $\widehat{C}_a$ can be cut out from $\widehat{S}_a$ by one smooth hypersurface that meets $\widehat{S}_a$ transversally along $\widehat{C}_a$. The latter implies that $\widehat{C}$ appears in the intersection product with multiplicity one and (1) holds.

(2) Since $g(\widehat{S}\times\widehat{C},\Delta)(1,a_i,1,a_i)=\nu(C,a_i)+\alpha_i$, it suffices to prove the inequality

$$g(\widehat{S}\times\widehat{C},\Delta)(1,a_i,1,a_i)\ge \nu(S,a_i)\cdot\nu(C,a_i)\,.$$

The latter is an immediate consequence of [W_2, Property 3].

(3) To prove that $\alpha\ge 0$ choose $\mathcal{H}:=(H_1,\ldots)\in\mathcal{H}(U,\widehat{S}\times\widehat{C})$, where U is a neighbourhood of 0, such that

$$\widetilde{\nu}((\widehat{S}\times\widehat{C})\cdot\mathcal{H},0)=\widetilde{g}(\widehat{S}\times\widehat{C},\Delta)(0)\,.$$

By [AR, Corollary 3] we can assume H_i to be hyperplanes. Let Z_i, Z_i^{Δ} denote the results of the ith step of Algorithm 1 applied to $\mathcal{H},\widehat{S}\times\widehat{C}$, Δ, where Δ is the diagonal in $(\mathbb{C}^{n+1})^2$.

One can see that $Z_1^{\Delta}=Z_2^{\Delta}=0$. Thus

$$Z_3=(H_1\cap H_2\cap H_3)\cdot(\widehat{S}\times\widehat{C})=j(\widehat{C})+\sum\beta_kC_k\,,$$

where $j:\mathbb{C}^{n+1}\ni x\mapsto(x,x)\in\Delta$ and C_k are two-dimensional cones.

Observe that $j(\widehat{a}_i)\subset\bigcup C_k$ for each a_i such that $\alpha_i>0$. Now

$$Z_4=H_4\cdot\Big(\sum\beta_kC_k\Big)=\sum\beta_k'l_k$$

where l_k are lines. Consequently

$$\widetilde{g}(\widehat{S}\times\widehat{C},\Delta)(0)=\Big(\deg(C),\sum_{l_k\subset\Delta}\beta_k',\sum_{l_k\not\subset\Delta}\beta_k'\Big).$$

Choose $b\in\widehat{a}_i\cap U$. Observe that if $\widehat{a}_i\ne l_k$ for all k, then $\alpha_i=0$. Indeed, we have $\mathcal{H}\in\mathcal{H}(U,\widehat{S}\times\widehat{C})$, which yields the equality $g(b)=\nu(\widehat{C},b)$ and implies that $\widehat{C}$ is the only component of the cycle $\widehat{S}\bullet\widehat{C}$ passing through b.

If $\alpha_i>0$ then $\widehat{a}_i$ is one of the lines l_k, say $\widehat{a}_i=l_{k_0}$, and

$$\widetilde{g}(\widehat{S}\times\widehat{C},\Delta)(b)\le_{\mathrm{lex}}(\nu(\widehat{C},b),\beta_{k_0}',0).$$

Since $(\widetilde{g}(\widehat{S}\times\widehat{C},\Delta)(b))_1\ge\nu(\widehat{C},b)$, we get $\alpha_i\le\beta_{k_0}'$ and

$$\alpha=g(\widehat{S}\times\widehat{C},\Delta)(0)-\nu\Big(\widehat{C}+\sum\alpha_i\widehat{a}_i,0\Big)\ge\sum_{l_k\not\subset\Delta}\beta_k'\ge 0.\ \blacksquare$$

Now we can state

Corollary 9. *Let S be a projective surface and let $C\subset S$ be a curve that has no common component with* $\mathrm{Sing}(S)$. *Then*

$$\sum_{a\in\mathrm{Sing}(S)\cap C}(\nu(S,a)-1)\cdot\nu(C,a)\le(\deg(S)-1)\cdot\deg(C)\,.$$

Proof. We maintain the notation of Lemma 8. By Theorem 6 we have

$$\deg(\widehat{S}\bullet\widehat{C})=\deg(S)\cdot\deg(C),$$

so Lemma 8 yields

$$\deg(S)\cdot\deg(C)=\deg(C)+\sum\alpha_i+\alpha.$$

By Lemma 8(2) we get

$$\sum_{a\in \mathrm{Sing}(S)\cap C}(\nu(S,a)-1)\cdot\nu(C,a)\le(\deg(S)-1)\cdot\deg(C). \ \blacksquare$$

REMARK 10. It is natural to ask to what extent the bounds from Lemma 8 and Corollary 9 are sharp. This amounts to the question how to control the behaviour of the multiplicity of the point 0 in the intersection product of cones. Let us point out that the latter can be expressed with the help of the Hilbert function (see [AR], [AM]) and computed using various computer algebra systems.

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