Whitney triangulations of semialgebraic sets

by Masahiro Shiota (Nagoya)

Abstract. A compact semialgebraic set admits a semialgebraic triangulation such that the family of open simplexes forms a Whitney stratification and is compatible with a finite number of given semialgebraic subsets.

A semialgebraic triangulation of a compact semialgebraic set $X$ is a pair of a simplicial complex $K$ and a semialgebraic homeomorphism $f: |K| \to X$ such that for each $\sigma \in K$, $f(\text{Int}\, \sigma)$ is a $C^\omega$ manifold and $f|_{\text{Int}\, \sigma}$ is a $C^\omega$ diffeomorphism onto $f(\text{Int}\, \sigma)$, where $|K|$ denotes the underlying polyhedron of $K$. As a consequence of the theory developed in [S2] we have:

**Theorem.** A compact semialgebraic set $X$ admits a semialgebraic triangulation $(K, f)$ such that $\{f(\text{Int}\, \sigma): \sigma \in K\}$ is a Whitney stratification of $X$ and the triangulation is compatible with a finite number of given semialgebraic subsets $Y_i$ of $X$.

When we work in an o-minimal structure, let the above diffeomorphisms be of class $C^r$ for $r \in \mathbb{N}$. Then the arguments below go through. Most of the following maps and manifolds are of class $C^2$. However, it is possible to work in the $C^1$ category. See [S2] for that case.

A (semialgebraic) $C^2$ tube about a (semialgebraic) $C^2$ manifold $M \subset \mathbb{R}^n$ is a triple $T = (|T|, \tau, \varrho)$, where $|T|$ is a (semialgebraic) neighborhood of $M$ in $\mathbb{R}^n$, $\tau: |T| \to M$ is a (semialgebraic) submersive $C^2$ retraction, and $\varrho$ is a nonnegative (semialgebraic) $C^2$ function on $|T|$ such that $\varrho^{-1}(0) = M$ and each point $x$ of $M$ is a unique and nondegenerate critical point of the restriction of $\varrho$ to $\tau^{-1}(x)$. The ambient space of $M$ is not necessarily a Euclidean space. It may be a (semialgebraic) $C^2$ manifold possibly with boundary. We also define a controlled (semialgebraic) $C^2$ tube system for a (semialgebraic) $C^2$ stratification in the same way as in the $C^\infty$ case (see [G-al] for the $C^\infty$ case).

2000 Mathematics Subject Classification: Primary 14P10.

Key words and phrases: semialgebraic sets, Whitney stratifications, triangulations.
Let \( \{M_k\}_{k=1,2,\ldots} \) and \( \{N_k\} \) be Whitney \( C^2 \) stratifications of sets \( M \) and \( N \), respectively, in \( \mathbb{R}^n \). Let \( \{T^M_k = (|T^M_k|, \tau^M_k, \varphi^M_k)\} \) and \( \{T^N_k = (|T^N_k|, \tau^N_k, \varphi^N_k)\} \) be controlled \( C^2 \) tube systems for \( \{M_k\} \) and \( \{N_k\} \) respectively. Let \( \varphi: M \to N \) be a homeomorphism such that \( \varphi|_{M_k} \) is a \( C^2 \) diffeomorphism onto \( N_k \) for each \( k \). We call \( \varphi \) compatible with \( \{T^M_k\} \) and \( \{T^N_k\} \) if for each \( k \), \( \varphi(M \cap |T^M_k|) = N \cap |T^N_k| \) and
\[
\tau^N_k \circ \varphi = \varphi \circ \tau^M_k \quad \text{and} \quad \varphi^N_k \circ \varphi = \varphi^M_k \quad \text{on} \quad M \cap |T^M_k|.
\]

Let \( f \) and \( f_l \), \( l = 1,2,\ldots, \) be semialgebraic \( C^0 \) functions on a semialgebraic set \( M \). We say that \( f_l \to f \) in the \( C^0 \) topology as \( l \to \infty \) if for any positive semialgebraic \( C^0 \) function \( g \) on \( M \) there exists \( l_0 \in \mathbb{Z} \) such that \(|f_l - f| < g\) for \( l \geq l_0 \). Assume \( M \) is a semialgebraic \( C^2 \) manifold and \( f_l \) and \( f \) are semialgebraic \( C^2 \) functions. Then we say that \( f_l \to f \) in the \( C^2 \) topology if \( f_l \to f \) and the differentials \( df_l \to df, \quad ddf_l \to ddf \) in the \( C^0 \) topology. In the same way we define the \( C^0 \) topology on the set of semialgebraic \( C^0 \) maps between semialgebraic sets and the \( C^2 \) topology on the set of semialgebraic \( C^2 \) maps between semialgebraic \( C^2 \) manifolds.

The key to the proof of the theorem is the following lemma, which does not hold without the assumption of semialgebraicness nor without the one that \( M_k \) are semialgebraically diffeomorphic to Euclidean spaces (see [S2, p. 238 and Remark II.7.4]).

**Lemma (Uniqueness of controlled semialgebraic tube system, [S2, Theorem II.7.3]).** Let \( \{M_k\} \) and \( \{N_k\} \) be finite semialgebraic Whitney \( C^2 \) stratifications of locally compact semialgebraic sets \( M \) and \( N \), respectively, such that all strata are semialgebraically \( C^2 \) diffeomorphic to Euclidean spaces. Let \( \{T^M_k\} \) and \( \{T^N_k\} \) be controlled semialgebraic \( C^2 \) tube systems for \( \{M_k\} \) and \( \{N_k\} \) respectively. Let \( \varphi: M \to N \) be a semialgebraic homeomorphism such that \( \varphi|_{M_k} \) is a \( C^2 \) diffeomorphism onto \( N_k \) for each \( k \). Then shrinking \( |T^M_k| \) and \( |T^N_k| \), we can modify \( \varphi \) to be compatible with \( \{T^M_k\} \) and \( \{T^N_k\} \).

**Proof of Theorem.** Let \( X \subset \mathbb{R}^n \). By the usual triangulation theorem for semialgebraic sets we have a finite number of semialgebraic \( C^0 \) imbeddings \( \pi_\alpha: B^\alpha \to X, \alpha \in A, \) such that each \( B^\alpha \) is the unit ball (always with center 0) in a Euclidean space, \( \pi_\alpha|_{\text{Int } B^\alpha} \) is a \( C^\omega \) imbedding,

\[
\pi_\alpha(\text{Int } B^\alpha) \cap \pi_{\alpha'}(\text{Int } B^\alpha') = \emptyset \quad \text{for} \quad \alpha \neq \alpha' \in A,
\]

\( \{\pi_\alpha(\text{Int } B^\alpha): \alpha \in A\} \) is a Whitney stratification of \( X \) compatible with \( \{Y_i\} \), \( \pi^{-1}_\alpha(\pi_{\alpha'}(\text{Int } B^\alpha)) \) is a \( C^\omega \) manifold for \( \alpha \in A \) and \( \alpha' \in A_\alpha = \{\alpha' \in A: \pi_{\alpha'}(B^\alpha') \subset \pi_\alpha(B^\alpha)\} \), the restriction of \( \pi_\alpha \) to the manifold is a \( C^\omega \) imbedding, and \( \{\pi^{-1}_\alpha(\pi_{\alpha'}(\text{Int } B^\alpha')): \alpha' \in A_\alpha\} \) is a Whitney stratification of \( B^\alpha \) for each \( \alpha \). We then call \( \{\pi_\alpha: \alpha \in A\} \) a semialgebraic \( C^\omega \) ball decomposition of \( X \) (cf. [R-S]). Let the ambient space of \( B^\alpha \) always be \( B^\alpha \) itself.
Set $d_\alpha = \dim B^\alpha$ and

$$C_\alpha = \pi_\alpha(B^\alpha), \quad \partial C_\alpha = \pi_\alpha(\partial B^\alpha), \quad \Int C_\alpha = \pi_\alpha(\Int B^\alpha) \quad \text{for } \alpha \in A,$$

$$B_\alpha^\alpha = \pi_\alpha^{-1}(C_\alpha), \quad \partial B_\alpha^\alpha = \pi_\alpha^{-1}(\partial C_\alpha), \quad \Int B_\alpha^\alpha = \pi_\alpha^{-1}(\Int C_\alpha) \quad \text{for } \alpha' \in A_\alpha.$$ 

Then $\{\Int C_\alpha: \alpha \in A\} \text{ and } \{\Int B_\alpha^\alpha: \alpha' \in A_\alpha\}$ for each $\alpha \in A$ are Whitney stratifications of $X$ and $B^\alpha$ respectively.

Let $0 * Y$ denote the cone with vertex 0 in some $B^\alpha$ and base $Y \subset \partial B^\alpha$. Let $\{T_\alpha = (\|T_\alpha\|, \tau_\alpha, \varrho_\alpha): \alpha \in A\}$ and $\{T_\alpha^\alpha = (\|T_\alpha^\alpha\|, \tau_\alpha^\alpha, \varrho_\alpha^\alpha): \alpha' \in A_\alpha\}$ for each $\alpha \in A$ be controlled semialgebraic $C^2$ tube systems for $\{\Int C_\alpha: \alpha \in A\}$ and $\{\Int B_\alpha^\alpha: \alpha' \in A_\alpha\}$ respectively. Then we can choose the latter so that

$$\left(\tau_\alpha^\alpha\right)^{-1}(x) = |T_\alpha^\alpha| \cap 0 * (\partial B^\alpha \cap \left(\tau_\alpha^\alpha\right)^{-1}(x))$$

for $x \in \Int B_\alpha^\alpha$, $\alpha' \in A_\alpha - \{\alpha\}$, $\alpha \in A$

by the usual method of construction of controlled tube systems (see e.g. [G-al]). Moreover, we can modify $\{\tau_\alpha\}$ so that each $\tau_\alpha: B^\alpha \to C_\alpha$ is compatible with $\{T_\alpha^\alpha: \alpha' \in A_\alpha\}$ and $\{T_\alpha: \alpha' \in A_\alpha\}$ by the lemma. Here $\pi_\alpha|_{\Int B_\alpha^\alpha}$ are no longer of class $C^\omega$ but $C^2$. Hence $\{\pi_\alpha: \alpha \in A\}$ is called a semialgebraic $C^2$ ball decomposition of $X$. By loosening the condition of compatibility we will $C^\omega$ smooth later.

In the following arguments we subdivide $\{\Int C_\alpha\}$, which works by the following fact:

(*) Let $\pi_\alpha$, $A$, $A_\alpha$, $B^\alpha$, $B_\alpha^\alpha$, $T_\alpha$ and $T_\alpha^\alpha$ be given as above (i.e., $\{\pi_\alpha\}$ is a semialgebraic $C^2$ ball decomposition of $X$, (1) is satisfied and $\pi_\alpha$ are compatible). Let $\{\pi^\alpha: B^\alpha \to X: \tilde{\alpha} \in \tilde{A}\}$ be a second semialgebraic $C^2$ ball decomposition of $X$ but without tube systems as yet. Define $\tilde{A}_\alpha$, $B_\alpha^\alpha$ and $C_\alpha$ in the same way. Let $m \in Z$. Assume

$$A \cap \tilde{A} = \{\alpha \in A: d_\alpha > m\} = \{\tilde{\alpha} \in \tilde{A}: d_{\tilde{\alpha}} > m\},$$

$$B^\alpha = B_\alpha^\alpha \quad \text{for } \alpha = \tilde{\alpha} \in A \cap \tilde{A},$$

the maps $\pi_\alpha: B^\alpha \to C_\alpha$ and $\pi^\alpha: B_\alpha^\alpha \to C_\alpha$ coincide for $\alpha = \tilde{\alpha} \in A \cap \tilde{A}$, and $\{\Int C_\alpha: \alpha \in \tilde{A}, d_{\alpha} \leq m\}$ is compatible with $\{\Int C_\alpha: \alpha \in A, d_\alpha \leq m\}$. Then fixing $\pi^\alpha$ for $\tilde{\alpha} \in \tilde{A}$ with $d_{\tilde{\alpha}} > m$ and $\Im \pi^\alpha$, $d_{\tilde{\alpha}} \leq m$, and modifying only $\pi^\alpha$, $d_{\tilde{\alpha}} \leq m$, we have controlled semialgebraic $C^2$ tube systems $\{T_\alpha^\alpha: \tilde{\alpha} \in \tilde{A}\}$ for $\{\Int C_\alpha: \tilde{\alpha} \in \tilde{A}\}$ and $\{T_\alpha^\alpha: \tilde{\alpha}' \in \tilde{A}_\alpha\}$ for $\{\Int B_\alpha^\alpha: \tilde{\alpha}' \in \tilde{A}_\alpha\}$ for each $\tilde{\alpha} \in \tilde{A}$ with the same properties as $\{T_\alpha\}$ and $\{T_\alpha^\alpha\}$—by (1) and compatibility of $\pi^\alpha$—such that

$$T_\alpha^\alpha = \begin{cases} T_\alpha & \text{for } \alpha = \tilde{\alpha} \in A \cap \tilde{A}, \\ T_\alpha|_{T_\alpha^\alpha} & \text{for } \alpha \in A, \tilde{\alpha} \in \tilde{A} \text{ with } C_\alpha \subset C_{\tilde{\alpha}}, d_\alpha = d_{\tilde{\alpha}} = m, \end{cases}$$

for $\alpha \in A$, $\tilde{\alpha} \in \tilde{A}$ with $C_\alpha \subset C_{\tilde{\alpha}}$, $d_\alpha = d_{\tilde{\alpha}} = m$. 

**Whitney triangulations of semialgebraic sets** 239
and for each $\tilde{\alpha} = \alpha \in A \cap \tilde{A}$,

$$(2') \quad T_{\tilde{\alpha}}^\alpha = \begin{cases} T_{\alpha}^\alpha & \text{for } \tilde{\alpha} = \alpha \in A \cap \tilde{A}, \tilde{\alpha}' = \alpha' \in A_\alpha \cap \tilde{A}_\tilde{\alpha}, \\ T_{\alpha}^\alpha|_{T_{\tilde{\alpha}}^\alpha} & \text{for } \tilde{\alpha} = \alpha \in A \cap \tilde{A}, \alpha' \in A_\alpha, \tilde{\alpha}' \in \tilde{A}_\tilde{\alpha} \ w i t h \ B_{\alpha'}^\alpha \subset B_{\alpha'}^\alpha, \ d_{\alpha'} = d_{\alpha'}. \end{cases}$$

**Proof of (1).** It suffices to find a controlled semialgebraic $C^2$ tube system \(\{T_{\tilde{\alpha}}^\alpha : \tilde{\alpha} \in \tilde{A}\}\) for \(\{\text{Int } C_{\tilde{\alpha}} : \tilde{\alpha} \in \tilde{A}\}\) such that (2) holds and

\[
(3) \quad \tau_{\tilde{\alpha}}' = \tau_{\tilde{\alpha}} \circ \tau_{\alpha}, \quad \varrho_{\tilde{\alpha}} = \varrho_\alpha + \varrho_{\tilde{\alpha}} \circ \tau_{\alpha} \quad \text{on } |T_{\tilde{\alpha}}^\alpha| \quad \text{for } \alpha \in A, \ \tilde{\alpha} \in \tilde{A} \ w i t h \ \text{Int } C_{\tilde{\alpha}} \subset \text{Int } C_\alpha, \ d_{\alpha} < m.
\]

Indeed, assume there exists such \(\{T_{\tilde{\alpha}}^\alpha\}\). Using it, we need to construct \(\{T_{\tilde{\alpha}}^\alpha : \tilde{\alpha}' \in \tilde{A}_\tilde{\alpha}\}\) for each $\tilde{\alpha} \in \tilde{A}$ as required in (1).

For $\tilde{\alpha} \in A \cap \tilde{A}$ and $\tilde{\alpha}' \in \tilde{A}_\tilde{\alpha}$, define $T_{\tilde{\alpha}}^\alpha$ by

\[
(4) \quad |T_{\tilde{\alpha}}^\alpha| = \pi_\alpha^{-1}(|T_{\tilde{\alpha}}^\alpha|), \quad \tau_{\tilde{\alpha}}^\alpha = \pi_\alpha^{-1} \circ \tau_{\alpha} \circ \pi_\alpha, \quad \varrho_{\tilde{\alpha}}^\alpha = \varrho_\alpha \circ \tau_{\alpha}.
\]

Then for each $\tilde{\alpha} \in A \cap \tilde{A}$, \(\{T_{\tilde{\alpha}}^\alpha : \tilde{\alpha}' \in \tilde{A}_\tilde{\alpha}\}\) is a controlled semialgebraic $C^2$ tube system for \(\{\text{Int } B_{\alpha'}^\alpha : \tilde{\alpha}' \in \tilde{A}_\tilde{\alpha}\}\). Indeed, $|T_{\tilde{\alpha}}^\alpha|$ is a semialgebraic neighborhood of Int $B_{\alpha'}^\alpha$ in $B_{\alpha'}^\alpha$; $\tau_{\tilde{\alpha}}^\alpha : |T_{\tilde{\alpha}}^\alpha|$ into $\text{Int } B_{\alpha'}^\alpha$ is a semialgebraic retraction and of class $C^2$ because $\pi_\alpha^{-1} \mid \text{Int } C_{\alpha'}$ is of class $C^2$ and because

\[
\tau_{\alpha}' \circ \pi_\alpha \equiv \tau_{\tilde{\alpha}}' \circ \tau_{\alpha} \circ \pi_\alpha \quad \text{compatibility of } \tau_{\alpha}' \circ \pi_\alpha \ circ \ \tau_{\alpha}'
\]

where $\alpha \in A$ and $\alpha' \in A_\alpha$ with $\alpha = \tilde{\alpha} \ \text{Int } C_{\tilde{\alpha}} \subset \text{Int } C_{\alpha'}$, and because $\pi_\alpha \mid \text{Int } B_{\alpha'}^\alpha : \text{Int } B_{\alpha'}^\alpha \rightarrow \text{Int } C_{\alpha'}$ is a $C^2$ diffeomorphism; $\varrho_{\alpha}'$ is a nonnegative semialgebraic $C^0$ function on $|T_{\tilde{\alpha}}^\alpha|$: $(\varrho_{\alpha}'^{-1})(0) = \text{Int } B_{\alpha'}^\alpha$; $\varrho_{\alpha}'$ is of class $C^2$ and each $x \in \text{Int } B_{\alpha'}^\alpha$ is a unique and nondegenerate critical point of $\varrho_{\alpha}'^{-1}(x)$ because

\[
\varrho_{\alpha}' = \varrho_{\tilde{\alpha}}' \circ \pi_{\tilde{\alpha}} \equiv \frac{1}{2} \varrho_{\alpha} \circ \tau_{\alpha}' \circ \pi_{\tilde{\alpha}} = \varrho_{\alpha} \circ \tau_{\alpha}' \circ \pi_{\tilde{\alpha}} + \varrho_{\tilde{\alpha}} \circ \tau_{\alpha}' \circ \pi_{\tilde{\alpha}}
\]

Moreover, by (4), $\pi_{\tilde{\alpha}}$ is compatible with $\{T_{\tilde{\alpha}}^\alpha : \tilde{\alpha}' \in \tilde{A}_\tilde{\alpha}\}$ and $\{T_{\tilde{\alpha}}^\alpha : \tilde{\alpha}' \in \tilde{A}_\tilde{\alpha}\}$ for each $\tilde{\alpha} \in A \cap \tilde{A}$, and clearly (2)' is satisfied by (2) since $\pi_{\alpha}$ is compatible with $\{T_{\alpha}^\alpha : \alpha' \in A_\alpha\}$ and $\{T_{\alpha'} : \alpha' \in A_\alpha\}$.

For $\tilde{\alpha} \in \tilde{A} - A$, i.e., $d_{\tilde{\alpha}} \leq m$, let $\{T_{\tilde{\alpha}}^\alpha : \tilde{\alpha}' \in \tilde{A}_\tilde{\alpha}\}$ be an arbitrary controlled semialgebraic $C^2$ tube system for $\{\text{Int } B_{\alpha'}^\alpha : \tilde{\alpha}' \in \tilde{A}_\tilde{\alpha}\}$ with (1). Then by the Lemma we can modify $\pi_{\tilde{\alpha}}$ and assume it is compatible with $\{T_{\tilde{\alpha}}^\alpha : \tilde{\alpha}' \in \tilde{A}_\tilde{\alpha}\}$ and $\{T_{\tilde{\alpha}}^\alpha : \tilde{\alpha}' \in \tilde{A}_\tilde{\alpha}\}$. 
Thus \( \{ T_{\tilde{\alpha}}^m \} \) together with \( \{ T_{\tilde{\alpha}} \} \) fulfills the requirements in (\( \ast \)), and hence we will construct only \( \{ T_{\tilde{\alpha}} \} \) with (2) and (3).

Order the elements of \( A \) as \( \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots \) so that \( d_{\tilde{\alpha}_k} \leq d_{\tilde{\alpha}_{k+1}} \) for any \( k \). Let \( m' \in \mathbb{Z} \). Assume by induction we have a controlled semialgebraic \( C^2 \) tube system \( \{ T_{\tilde{\alpha}_1}, \ldots, T_{\tilde{\alpha}_{m'-1}} \} \) for \( \{ \text{Int } C_{\tilde{\alpha}_1}, \ldots, \text{Int } C_{\tilde{\alpha}_{m'-1}} \} \) with (2) and (3). Let \( \alpha_k \in A \) be such that \( \text{Int } C_{\tilde{\alpha}_k} \subset \text{Int } C_{\alpha_k} \) for each \( k \). Then we only need to find a semialgebraic \( C^2 \) tube \( T_{\tilde{\alpha}_{m'}}\) about \( \text{Int } C_{\tilde{\alpha}_{m'}} \) such that (2) and (3) hold for \( \alpha = \alpha_{m'} \) and \( \tilde{\alpha} = \tilde{\alpha}_{m'} \), and

\[
\begin{align*}
\tau_{\tilde{\alpha}_k} \circ \tau_{\tilde{\alpha}_m'} &= \tau_{\tilde{\alpha}_k} \circ \tau_{\alpha_m'}; \\
\varrho_{\tilde{\alpha}_k} \circ \tau_{\tilde{\alpha}_m'} &= \varrho_{\tilde{\alpha}_k} \circ \tau_{\alpha_m'}, \quad \text{for } k < m'
\end{align*}
\]

because \( \{ T_{\tilde{\alpha}_{m'}}, T_{\tilde{\alpha}_k} \} \) is controlled for each \( k < m' \) as follows:

\[
\begin{align*}
\tau_{\tilde{\alpha}_k} \circ \tau_{\tilde{\alpha}_m'} &= \tau_{\tilde{\alpha}_k} \circ \tau_{\tilde{\alpha}_m'} \circ \tau_{\alpha_m'}; \\
\varrho_{\tilde{\alpha}_k} \circ \tau_{\tilde{\alpha}_m'} &= \varrho_{\tilde{\alpha}_k} \circ \tau_{\tilde{\alpha}_m'} \circ \tau_{\alpha_m'}, \quad \text{controlledness of } \{ T_{\tilde{\alpha}} \}
\end{align*}
\]

The construction of \( T_{\tilde{\alpha}_{m'}} \) is easy. If \( d_{\tilde{\alpha}_{m'}} \geq m \), \( T_{\tilde{\alpha}_{m'}} \) is defined by (2). Then (5) is clearly satisfied. Assume \( d_{\tilde{\alpha}_{m'}} < m \). Let \( \{ T_{\tilde{\alpha}_{m'}} = \{ (T'_{\tilde{\alpha}_{m'}}, \tau_{\tilde{\alpha}_{m'}}, \varrho_{\tilde{\alpha}_{m'}} ) \} \), \( T_{\tilde{\alpha}_k}: k < m' \} \) be a controlled semialgebraic \( C^2 \) tube system for \( \{ C_{\tilde{\alpha}_k} : k \leq m' \} \), whose existence is shown in the usual construction of a controlled tube system (see, e.g., [G-al]). (Here we shrink \( |T_{\tilde{\alpha}_k}|, k < m' \), if necessary.)

Set

\[
|T_{\tilde{\alpha}_{m'}}| = |T_{\alpha_{m'}}| \cap |T'_{\tilde{\alpha}_{m'}}|, \quad \tau_{\tilde{\alpha}_{m'}} = \tau'_{\tilde{\alpha}_{m'}} \circ \tau_{\alpha_{m'}}, \quad \varrho_{\tilde{\alpha}_{m'}} = \varrho_{\alpha_{m'}} + \varrho'_{\tilde{\alpha}_{m'}} \circ \tau_{\alpha_{m'}}.
\]

Then (3) and (5) are satisfied. Thus (\( \ast \)) holds.

Let \( \{ \pi_\alpha : \alpha \in A \} \) be again a semialgebraic \( C^2 \) ball decomposition of \( X \) with controlled semialgebraic \( C^2 \) tube systems \( \{ T_\alpha : \alpha \in A \} \) and \( \{ T'_\alpha : \alpha' \in A_\alpha \} \) for each \( \alpha \in A \) such that (1) holds and \( \pi_\alpha \) is compatible with \( \{ T'_\alpha : \alpha' \in A_\alpha \} \) and \( \{ T_\alpha : \alpha' \in A_\alpha \} \). We show good properties of such \( \{ \pi_\alpha \}, \{ T_\alpha \} \) and \( \{ T'_\alpha \} \). Set

\[
0 \ast_\alpha Y = \pi_\alpha (0 \ast \pi_\alpha^{-1}(Y)) \quad \text{for } Y \subset \partial C_\alpha,
\]

\[
J = \{(\alpha_1, \ldots, \alpha_l) \in A^l : l \in \mathbb{N}, \alpha_1 \in A_{\alpha_2}, \ldots, \alpha_{l-1} \in A_{\alpha_l}, \alpha_1 \neq \cdots \neq \alpha_l\},
\]

\[
U_J = \text{Int } 0 \ast_\alpha (\cdots \ast_\alpha (0 \ast_\alpha C_{\alpha_1} \cdots)) \quad \text{for } J = (\alpha_1, \ldots, \alpha_l) \in J, l > 1,
\]

\[
U_{x,J} = \text{Int } 0 \ast_\alpha (\cdots \ast_\alpha (0 \ast_\alpha x \cdots)) \quad \text{for same } J \text{ and } x \in \text{Int } C_{\alpha_1},
\]

\[
U_\alpha = \text{Int } C_\alpha, \quad U_{x,\alpha} = x \quad \text{for } \alpha \in A \text{ and } x \in \text{Int } C_\alpha,
\]

where \( 0 \ast_\alpha (\cdots \ast_\alpha (0 \ast_\alpha C_{\alpha_1} \cdots)) \) and \( 0 \ast_\alpha (\cdots \ast_\alpha (0 \ast_\alpha x \cdots)) \) are semialgebraically homeomorphic to balls of dimension \( d_\alpha + l - 1 \) and \( l - 1 \), respec-
tively, and \( \text{Int} \) stands for open balls. Note that \( U_J \) and \( U_{x,J} \) are \( C^2 \) manifolds. For \( J = (\alpha_1, \ldots, \alpha_l) \in \mathcal{J} \), let \( p_J: U_J \to \text{Int} C_{\alpha_1} \) denote the semialgebraic \( C^2 \) submersion such that \( x \in U_{p_J(x),J} \). We also define \( U_J^\alpha \subset B^\alpha \), \( U_{x,J}^\alpha \subset B^\alpha \) and \( p_J^\alpha: U_J^\alpha \to \text{Int} B_{\alpha_1}^\alpha \) to be \( \pi_{\alpha}^{-1}(U_J) \), \( \pi_{\alpha}^{-1}(U_{\pi_\alpha(x),J}) \) and \( \pi_{\alpha}^{-1} \circ p_J \circ \pi_{\alpha} \), respectively, for \( J = (\alpha_1, \ldots, \alpha_l) \in \mathcal{J} \) and \( \alpha \in A \) and \( x \in \text{Int} B_{\alpha_1}^\alpha \). Note that \( U_J, p_J, U_J^\alpha \) and \( p_J^\alpha \) depend only on \( \{\pi_{\alpha}\} \) but not on \( \{T_\alpha\}, \{T_\alpha^\alpha\} \).

Then by (1) and compatibility of \( \{\pi_{\alpha}: \alpha \in A\} \) we have

(6) \[ p_J = \tau_{\alpha_1} \text{ on } U_J \cap |T_{\alpha_1}|, \quad p_J^\alpha = \tau_{\alpha_1}^\alpha \text{ on } U_J^\alpha \cap |T_{\alpha_1}^\alpha|. \]

Moreover, as shown below, we can suppose the following condition (7) is satisfied, which will allow a “derived” subdivision of \( \{\text{Int} C_\alpha\} \) also to be a Whitney stratification.

(7) \( \{U_J, \text{Int} C_{\alpha_1}\} \) (resp. \( \{U_J^\alpha, \text{Int} B_{\alpha_1}^\alpha\} \)) satisfies the Whitney condition.

Note that (7) depends on \( C_{\alpha_1} \) but not on \( \pi_{\alpha_1} \), hence fixing \( C_{\alpha_1} \) we can change \( \pi_{\alpha_1} \) arbitrarily when (7) is satisfied and when we need to keep the property, and that (6) and (7) imply that:

(8) \[ \{p_J^{-1}(M_1), M_2\} \text{ (resp. } \{(p_J^\alpha)^{-1}(M_1), M_2\}) \text{ is a Whitney stratification for a Whitney } C^2 \text{ stratification } \{M_1, M_2\} \text{ in Int } C_{\alpha_1} \text{ (resp. Int } B_{\alpha_1}^\alpha) \text{ or for a } C^2 \text{ submanifold } M_1 = M_2 \text{ of Int } C_{\alpha_1} \text{ (resp. Int } B_{\alpha_1}^\alpha) \text{.} \]

We modify \( \{\pi_{\alpha}\} \) so that (7) is satisfied by downward induction. Let \( 0 \leq m \leq n \) be an integer. Assume we have a semialgebraic \( C^2 \) ball decomposition \( \{\pi_{\alpha}: \alpha \in A\} \) of \( X \) and controlled semialgebraic \( C^2 \) tube systems \( \{T_\alpha: \alpha \in A\} \) for \( \{C_\alpha: \alpha \in A\} \) and \( \{T_\alpha^\alpha: \alpha' \in A_{\alpha}\} \) for each \( \alpha \in A \) such that (1) is satisfied for any \( \alpha \in A \) and \( \alpha' \in A_{\alpha} \), any \( \pi_{\alpha} \) is compatible with \( \{T_\alpha^\alpha: \alpha' \in A_{\alpha}\} \) and \( \{T_\alpha: \alpha' \in A_{\alpha}\} \), and (7) is satisfied for any \( J = (\alpha_1, \ldots, \alpha_l) \in \mathcal{J} \) with \( l > 1 \) and \( d_{\alpha_1} > m \). Then we need to find a semialgebraic \( C^2 \) ball decomposition \( \{\pi_{\tilde{\alpha}}: \tilde{\alpha} \in \tilde{A}\} \) and controlled semialgebraic \( C^2 \) tube systems \( \{T_{\tilde{\alpha}}: \tilde{\alpha} \in \tilde{A}\} \) for \( \{C_{\tilde{\alpha}}: \tilde{\alpha} \in \tilde{A}\} \) and \( \{T_{\tilde{\alpha}}^\tilde{\alpha}: \tilde{\alpha}' \in \tilde{A}_{\tilde{\alpha}}\} \) for \( \{B_{\tilde{\alpha}}^\tilde{\alpha}: \tilde{\alpha}' \in \tilde{A}_{\tilde{\alpha}}\} \) with the same properties as \( \{\pi_{\alpha}\}, \{T_\alpha\} \) and \( \{T_\alpha^\alpha\} \) and, moreover, such that (7) is satisfied also for \( \tilde{J} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_l) \in \tilde{J} \) with \( l > 1 \) and \( d_{\tilde{\alpha}_1} = m \), where \( C_{\tilde{\alpha}}, \tilde{A}_{\tilde{\alpha}}, B_{\tilde{\alpha}}^\tilde{\alpha}, \) and \( \tilde{J} \) are defined as before. We further require, as in (\(*\)),

\[ A \cap \tilde{A} = \{\alpha \in A: d_{\alpha} > m\} = \{\tilde{\alpha} \in \tilde{A}: d_{\tilde{\alpha}} > m\}, \]

\[ B^\alpha = B^{\tilde{\alpha}} \text{ for } \alpha = \tilde{\alpha} \in A \cap \tilde{A}, \]

\[ \pi_{\tilde{\alpha}}: B^\alpha \to C_\alpha \text{ and } \pi_{\tilde{\alpha}}: B^{\tilde{\alpha}} \to C_{\tilde{\alpha}} \text{ coincide for } \alpha = \tilde{\alpha} \in A \cap \tilde{A}, \text{ and } \{\text{Int } C_{\tilde{\alpha}}: \tilde{\alpha} \in \tilde{A}, d_{\tilde{\alpha}} \leq m\} \text{ is compatible with } \{\text{Int } C_\alpha: \alpha \in A, d_{\alpha} \leq m\}. \]

The construction of such \( \{\pi_{\tilde{\alpha}}\}, \{T_{\tilde{\alpha}}\} \) and \( \{T_{\tilde{\alpha}}^\tilde{\alpha}\} \) is clear by (\(*\)) and the following well known fact:
FACT. Fix $J = (\alpha_1, \ldots, \alpha_l) \in \mathcal{J}$ and $\alpha \in A$ with $l > 1$, $\alpha_l \in A_{\alpha}$ and $d_{\alpha_1} = m$. The subset of $\text{Int} C_{\alpha_1}$ (resp. $\text{Int} B_{\alpha_1}^\alpha$) consisting of points where \{U_j, \text{Int} C_{\alpha_1}\} (resp. \{U_j^\alpha, \text{Int} B_{\alpha_1}^\alpha\}) does not satisfy the Whitney condition is semialgebraic and of dimension smaller than $m$.

Next we will $C^\omega$ smooth $\pi_\alpha|_{\text{Int} B^\alpha}$. We assume $X$ is the unit ball in $\mathbb{R}^n$ for simplicity of notation and without loss of generality. Preparatory to smoothing, set

$$\varepsilon * \alpha Y = \pi_\alpha \{t\pi_\alpha^{-1}(x); 1 - \varepsilon \leq t < 1, x \in Y\}$$

for $\varepsilon \in ]0, 1[$, $\alpha \in A$, $Y \subset \partial C_\alpha$, $U_j(\varepsilon) = \{\varepsilon_1 * \alpha_1 (\cdots * \alpha_3 (\varepsilon_2 * \alpha_2 \text{Int} C_{\alpha_1}) \cdots);\varepsilon_2, \ldots, \varepsilon_l \in ]0, 1[; \varepsilon_2 + \cdots + \varepsilon_l = \varepsilon\}$

for $J = (\alpha_1, \ldots, \alpha_l) \in \mathcal{J}$, $l > 1$, $U_\alpha(\varepsilon) = \text{Int} C_\alpha$, $W_{\alpha_1}(\varepsilon) = \bigcup_{J = (\alpha_1, \ldots, \alpha_l) \in \mathcal{J}} U_J(\varepsilon)$ for each $\alpha_1 \in A$,

and define naturally $U_j^\alpha(\varepsilon)$ and $W_{\alpha_1}(\varepsilon)$ in $B^\alpha$. Then for each $\alpha \in A$ and $J = (\alpha_1, \ldots, \alpha_l) \in \mathcal{J}$ with $\alpha_l \in A_{\alpha}$, we have $U_j(\varepsilon) \subset U_j^\alpha(\varepsilon)$ (resp. $U_j^\alpha(\varepsilon) \subset U_j^\alpha(\varepsilon')$) for $0 < \varepsilon < \varepsilon' < 1$, $\bigcup_{\varepsilon \in ]0, 1]} U_j(\varepsilon) = U_j$ (resp. $\bigcup_{\varepsilon \in ]0, 1]} U_j^\alpha(\varepsilon) = U_j^\alpha$), $W_{\alpha_1}(\varepsilon)$ (resp. $W_{\alpha_1}(\varepsilon)$) is a neighborhood of $\text{Int} C_{\alpha_1}$ in $X$ (resp. $\text{Int} B_{\alpha_1}$ in $B^\alpha$) called the $\varepsilon$-neighborhood of $\text{Int} C_{\alpha_1}$ (resp. $\text{Int} B_{\alpha_1}$), and the map $q_{\alpha_1} : W_{\alpha_1}(\varepsilon) \rightarrow \text{Int} C_{\alpha_1}$ (resp. $q_{\alpha_1}^\alpha : W_{\alpha_1}(\varepsilon) \rightarrow \text{Int} B_{\alpha_1}^\alpha$) defined to be $p_j$ on $U_j(\varepsilon)$ (resp. $p_j^\alpha$ on $U_j^\alpha(\varepsilon)$) is proper. (If we define $W_{\alpha_1}(\varepsilon)$ and $q_{\alpha_1}$ with $U_j$ in place of $U_j(\varepsilon)$, then $q_{\alpha_1}$ is not proper for $l > 1$. This is the reason why we apply $U_j(\varepsilon)$ and not $U_j$.)

For smoothing we drop compatibility of $\pi_\alpha$ and weaken (6) as follows:

$$(6)' \quad \text{For } \alpha \in A \text{ and } \alpha_1 \in A_{\alpha}, q_{\alpha_1} : W_{\alpha_1}(\varepsilon) \rightarrow \text{Int} C_{\alpha_1} \text{ (resp. } q_{\alpha_1}^\alpha : W_{\alpha_1}(\varepsilon) \rightarrow \text{Int} B_{\alpha_1}^\alpha \text{)} \text{ is a submersive } C^2 \text{ retraction for some } \varepsilon.$$

Namely, we are in the situation that \{\pi_\alpha : \alpha \in A\} is a semialgebraic $C^2$ ball decomposition of $X$, \{\varepsilon_0, \alpha \in A\} and \{\varepsilon_0^\alpha, \alpha' \in A_{\alpha}\} for each $\alpha \in A$ are controlled semialgebraic $C^2$ tube systems for \{C_\alpha : \alpha \in A\} and \{B_\alpha^\alpha : \alpha' \in A_{\alpha}\}, respectively, and (1), (6)' and (7) are satisfied, where the definitions of $U_j$ etc. are not changed. We saw (8) under the conditions (6) and (7). But we can replace (6) by (6)' there. Hence (8) is now also satisfied.

Next we can assume $\pi_\alpha|_{\text{Int} B^\alpha}$ are of class $C^\omega$ as follows. First choose \{\pi_\alpha\} so that $\text{Int} C_{\alpha}$ are $C^\omega$ manifolds, which is possible by the above arguments. Let $m \in \mathbb{Z}$. Inductively suppose $\pi|_{\text{Int} B^\alpha}$ are of class $C^\omega$ for $d_\alpha > m$, and let $\alpha_0 \in A$ with $d_{\alpha_0} = m$. By the approximation theorem of $[S_1]$, $\pi_{\alpha_0}|_{\text{Int} B^{\alpha_0}}$ can be approximated by a semialgebraic $C^\omega$ map $\tilde{\pi}_{\alpha_0} : \text{Int} B^{\alpha_0} \rightarrow \text{Int} C_{\alpha_0}$ in the
$C^2$ topology. Then $\hat{\pi}_{\alpha_0}$ is a diffeomorphism (see [S1]). Extend it to $\partial B_{\alpha_0}$ by setting $\hat{\pi}_{\alpha_0} = \pi_{\alpha_0}$ there. Then $\hat{\pi}_{\alpha_0} : B_{\alpha_0} \to C_{\alpha_0}$ is a homeomorphism by the definition of the $C^0$ topology, and $\{\pi_\alpha : \alpha \in A, \alpha \neq \alpha_0\} \cup \{\hat{\pi}_{\alpha_0}\}$, $\{T_\alpha : \alpha \in A\}$ and $\{T_\alpha' : \alpha' \in A\}$ still satisfy (1), (6)' and (7) as shown below.

(1) is clear because we do not change $\{T_\alpha\}$; (6)' is also trivial if $C_{\alpha_1} \notin \partial C_{\alpha_0}$ (resp. $B_{\alpha_1} \notin \partial B_{\alpha_0}$).

Assume $C_{\alpha_1} \subset \partial C_{\alpha_0}$ in (6)', and that (6)' holds for $\varepsilon$. Let $\hat{\rho}_J$, $\hat{\alpha}$, and $\hat{W}_\alpha(\varepsilon)$ be defined by $\{\pi_\alpha : \alpha \in A, \alpha \neq \alpha_0\} \cup \{\hat{\pi}_{\alpha_0}\}$ in the same way as $\rho_J$, $q_\alpha$, and $W_\alpha(\varepsilon)$, and let $\tilde{\varepsilon} \in ]0,1[ \text{ be so close to } 0 \text{ that } \hat{W}_{\alpha_1}(\tilde{\varepsilon}) \subset W_{\alpha_1}(\varepsilon)$. Then

$$\{x \in \hat{W}_{\alpha_1}(\tilde{\varepsilon}) : q_{\alpha_1}(x) \neq \hat{q}_{\alpha_1}(x)\} \subset W_{\alpha_0}(\tilde{\varepsilon}) (= \hat{W}_{\alpha_0}(\tilde{\varepsilon})) .$$

Hence it suffices to see that the map

$$(q_{\alpha_1} - \hat{q}_{\alpha_1})|_{W_{\alpha_0}(\tilde{\varepsilon}) \cap \hat{W}_{\alpha_1}(\tilde{\varepsilon})} : W_{\alpha_0}(\tilde{\varepsilon}) \cap \hat{W}_{\alpha_1}(\tilde{\varepsilon}) \to \mathbb{R}^n$$

is close to the zero map in the $C^2$ topology. By the definitions of $q_{\alpha_1}$ and $\hat{q}_{\alpha_1}$ we have

$$q_{\alpha_1} \circ q_{\alpha_0} = q_{\alpha_1}, \quad \hat{q}_{\alpha_1} \circ \hat{q}_{\alpha_0} = \hat{q}_{\alpha_1} \quad \text{on } W_{\alpha_0}(\tilde{\varepsilon}) \cap \hat{W}_{\alpha_1}(\tilde{\varepsilon}), \quad \hat{q}_{\alpha_0} = q_{\alpha_0} .$$

Hence

$$(q_{\alpha_1} - \hat{q}_{\alpha_1})|_{W_{\alpha_0}(\tilde{\varepsilon}) \cap \hat{W}_{\alpha_1}(\tilde{\varepsilon})} = (q_{\alpha_1} - \hat{q}_{\alpha_1})|_{\hat{W}_{\alpha_1}(\tilde{\varepsilon}) \cap \text{Int } C_{\alpha_0}} \circ q_{\alpha_0}|_{W_{\alpha_0}(\tilde{\varepsilon}) \cap \hat{W}_{\alpha_1}(\tilde{\varepsilon})} .$$

Therefore, we only need to see that

$$(q_{\alpha_1} - \hat{q}_{\alpha_1})|_{\hat{W}_{\alpha_1}(\tilde{\varepsilon}) \cap \text{Int } C_{\alpha_0}} : \hat{W}_{\alpha_1}(\tilde{\varepsilon}) \cap \text{Int } C_{\alpha_0} \to \mathbb{R}^n$$

is close to the zero map in the $C^2$ topology since $q_{\alpha_0}$ is a proper semialgebraic $C^2$ retraction (for a proper semialgebraic $C^2$ map between semialgebraic $C^2$ manifolds $\varphi : M_1 \to M_2$, the pull back by $\varphi$: \{semialgebraic $C^2$ functions on $M_2$\} $\to$ \{semialgebraic $C^2$ functions on $M_1$\} is continuous). However, that is clear because

$$q_{\alpha_1}|_{\hat{W}_{\alpha_1}(\tilde{\varepsilon}) \cap \text{Int } C_{\alpha_0}} = \pi_{\alpha_0} \circ q_{\alpha_1}^0 \circ \pi_{\alpha_0}^{-1}|_{\hat{W}_{\alpha_1}(\tilde{\varepsilon}) \cap \text{Int } C_{\alpha_0}} ,$$

$$\hat{q}_{\alpha_1}|_{\hat{W}_{\alpha_1}(\tilde{\varepsilon}) \cap \text{Int } C_{\alpha_0}} = \pi_{\alpha_0} \circ q_{\alpha_1}^0 \circ \hat{\pi}_{\alpha_0}^{-1}|_{\hat{W}_{\alpha_1}(\tilde{\varepsilon}) \cap \text{Int } C_{\alpha_0}} ,$$

which holds by the definitions of $q_{\alpha_1}$ and $\hat{q}_{\alpha_1}$, though $\pi_{\alpha_0}$ or $\hat{\pi}_{\alpha_0}$ is not necessarily compatible with $\{T_\alpha\}$ and $\{T_\alpha'\}$.

In the same way, we see that also $q_{\alpha_1}^\alpha$ is a submersive $C^2$ retraction when $B_{\alpha_1} \subset \partial B_{\alpha_0}$. (Note that after replacing $\pi_{\alpha_0}$ by $\hat{\pi}_{\alpha_0}$ we cannot preserve (6) nor compatibility of $\pi_{\alpha_0}$) By the same reason (7) is kept. Therefore, we assume $\pi_{\alpha_0}|_{\text{Int } B_{\alpha}}$ are all of class $C^\omega$.

Now we define a semialgebraic triangulation $(K, f)$ of $X$, as required in the theorem. Let the vertices of $K$ correspond to elements of $A$ and be denoted by $\{\sigma_\alpha : \alpha \in A\}$, and let vertices $\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_i}$ span a simplex of $K$
if and only if \((\alpha_1, \ldots, \alpha_l) \in J\) for some permutation \(\gamma\) of \(\{1, \ldots, l\}\). Then \(K\) is a well defined simplicial complex called the \textit{dual complex} of \(\{C_\alpha\}\) (see [R-S]). For \(J = (\alpha_1, \ldots, \alpha_l) \in J\), let \(\sigma_J\) denote the simplex spanned by \(\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_l}\). Thus \(K = \{\sigma_J; J \in J\}\). Let \(m \in \mathbb{Z}\) and set

\[
J_m = \{(\alpha_1, \ldots, \alpha_l) \in J; d_{\alpha_l} \leq m\}, \quad K_m = \{\sigma_J; J \in J_m\}.
\]

Note \(K_m\) is a subcomplex of \(K\). By induction assume we already have a semialgebraic triangulation \((K_m, f_m)\) of \(\bigcup_{d_\alpha \leq m} C_\alpha\) such that \(f_m|_{\text{Int } J}\) is a \(C^\omega\) diffeomorphism onto \(U_\pi\alpha_1(0), J\) for each \(J = (\alpha_1, \ldots, \alpha_l) \in J_m\). Then we need to extend \((K_m, f_m)\) to a semialgebraic triangulation \((K_{m+1}, f_{m+1})\) of \(\bigcup_{d_\alpha \leq m+1} C_\alpha\). For \(J = (\alpha_1, \ldots, \alpha_l) \in J_{m+1} - J_m\), i.e., \(d_{\alpha_l} = m + 1\), define

\[
f_{m+1}(t_1\sigma_{\alpha_1} + \cdots + t_l\sigma_{\alpha_l}) = \pi_{\alpha_1}((1 - t_l)\pi_{\alpha_l}^{-1}(f_m(t_1\sigma_{\alpha_1}/(1 - t_l) + \cdots + t_{l-1}\sigma_{\alpha_{l-1}}/(1 - t_{l-1}))))
\]

for \(t_1, \ldots, t_l \in [0, 1]\) with \(t_1 + \cdots + t_l = 1\) and \(t_l \neq 1\).

Then \((K_{m+1}, f_{m+1})\) a semialgebraic triangulation of \(\bigcup_{d_\alpha \leq m+1} C_\alpha\), and we obtain a semialgebraic triangulation \((K, f)\) as required.

It remains to prove that \(\{U_\pi\alpha_1(0), J; J = (\alpha_1, \ldots, \alpha_l) \in J\}\) is a Whitney stratification. We show that, moreover,

\[
\text{(9)} \quad \{U_\pi\alpha_1(0), J, U_\pi\alpha'_1(0), J'\} \quad \text{(resp. } \{U_\pi\alpha_1(0), J, U_\pi\alpha'_1(0), J'\}\) satisfies the Whitney condition for \(J = (\alpha_1, \ldots, \alpha_l), J' = (\alpha'_1, \ldots, \alpha'_{l'}), J \in J\) and \(\alpha, \alpha' \in A\) with \(l > 1\), \(U_\pi\alpha_1(0), J \subset \text{Cl } U_\pi\alpha'_1(0), J' - U_\pi\alpha'_1(0), J'\) and \(\alpha, \alpha' \in A\).

Here we also argue by induction. Let \(m \in \mathbb{Z}\). Assume inductively (9) is proved for \(d_{\alpha_{l'}} < m\), and let \(d_{\alpha_{l'}} = m\). There are two possibilities \(\alpha_l = \alpha'_{l'}\) or \(\alpha_l \neq \alpha'_{l'}\).

\textit{Case of } \(\alpha_l = \alpha'_{l'}\). Set \(J_0 = (\alpha_1, \ldots, \alpha_{l-1})\) and \(J'_0 = (\alpha'_1, \ldots, \alpha'_{l'-1})\). Then \(U_\pi\alpha_1(0), J_0 \subset \text{Cl } U_\pi\alpha'_1(0), J'_0 - U_\pi\alpha'_1(0), J'_0\) and \(d_{\alpha_{l'-1}} < m\). Hence \(\{U_\pi\alpha_1(0), J_0, U_\pi\alpha'_1(0), J'_0\}\) (resp. \(\{U_\pi\alpha_1(0), J_0, U_\pi\alpha'_1(0), J'_0\}\) satisfies the Whitney condition by hypothesis when \(l - 1 > 1\), and trivially when \(l - 1 = 1\). In particular, so does \(\{U_\pi\alpha_1(0), J_0, U_\pi\alpha_1(0), J'_0\}\). Then \(\{U_\pi\alpha_1(0), J_0, U_\pi\alpha_1(0), J'_0\}\) is a Whitney stratification by (1). Moreover, (9) holds since \(\pi_{\alpha_1}|_{\text{Int } B^{\alpha}}\) : \(\text{Int } B^{\alpha} \to \text{Int } C_\alpha\) is a \(C^\omega\) diffeomorphism and \(U_\pi\alpha_1(0), J, U_\pi\alpha'_1(0), J' \subset \text{Int } B^{\alpha}\).

\textit{Case of } \(\alpha_l \neq \alpha'_{l'}\). Set \(J'_0 = (\alpha'_1, \ldots, \alpha'_{l'-1})\) as above. Then \(J = J'_0\) or \(U_\pi\alpha_1(0), J \subset \text{Cl } U_\pi\alpha'_1(0), J'_0 - U_\pi\alpha'_1(0), J'_0\), and hence \(U_\pi\alpha_1(0), J\) is equal to \(U_\pi\alpha'_1(0), J'_0\) or \(\{U_\pi\alpha_1(0), J, U_\pi\alpha'_1(0), J'_0\}\) (resp.
\( \{ U_\alpha^{(0)}, J, U_\alpha^{(0)'} \} \) is a Whitney stratification. In both cases, (9) follows from (8). This completes the proof. ■

References


Graduate School of Mathematics
Nagoya University
Chikusa, Nagoya, 464-8602
Japan
E-mail: shiota@math.nagoya-u.ac.jp

*Reçu par la Rédaction le 24.5.2005* (1623)