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The Łojasiewicz exponent of subanalytic sets

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Abstract. We prove that the infimum of the regular separation exponents of two subanalytic sets at a point is a rational number, and it is also a regular separation exponent of these sets. Moreover, we consider the problem of attainment of this exponent on analytic curves.

1. Introduction. Let $\mathcal{M}$ be a finite-dimensional, real analytic manifold countable at infinity, ϱ be a distance function on $\mathcal{M}$ induced by a Riemannian metric on $\mathcal{M}$, and let $X, Y \subset \mathcal{M}$ be closed subanalytic sets. In the theory of semi-analytic and subanalytic sets ([2], [15], [22], [24], [25]), an important role is played by the fact (proved by Łojasiewicz in [22] and [25]) that X and Y are regularly separated at any x_0. Namely:

Theorem 1.1. *For any $x_0 \in X \cap Y$ there exist $\nu > 0$ and $C > 0$ such that for some neighbourhood $\Omega \subset \mathcal{M}$ of x_0,*

$$\varrho(x, X) + \varrho(x, Y) \geq C\varrho(x, X \cap Y)^\nu \quad \text{for } x \in \Omega. \tag{S}$$

If additionally $x_0 \in \overline{X \setminus Y}$, then $\nu \geq 1$ and (S) *is equivalent to*

$$\varrho(x, Y) \geq C'\varrho(x, X \cap Y)^\nu \quad \text{for } x \in \Omega' \cap X, \tag{S'}$$

where $C' > 0$ and Ω' is a neighbourhood of x_0.

Note that the condition $x_0 \notin \overline{X \setminus Y}$ leads to the trivial cases $\nu = 0$ in (S′) and $\nu = 0$ or $\nu = 1$ in (S), provided we put $0^0 = 0$.

In this paper we investigate the smallest exponent ν satisfying (S). Bochnak and Risler ([3, Corollary 2]) proved:

Theorem 1.2. *For a fixed relatively compact neighbourhood Ω of $x_0 \in X \cap Y$,*

$$\mathcal{L}_\Omega(X, Y) := \inf\{\nu \in \mathbb{R} : \exists_{C>0}\ \forall_{x\in\Omega}\ \varrho(x, X) + \varrho(x, Y) \geq C\varrho(x, X \cap Y)^\nu\}$$

is a rational number.

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The exponent ν satisfying (S) for some Ω and $C > 0$ is called a *regular separation exponent* of X and Y at x_0. The infimum of all regular separation exponents of X and Y at x_0 is called the *Łojasiewicz exponent of* X, Y *at* x_0 and denoted by $\mathcal{L}_{x_0}(X,Y)$. It is easy to see that

$$\mathcal{L}_{x_0}(X,Y) = \inf\{\mathcal{L}_{\Omega}(X,Y) : \Omega \text{ a relatively compact neighbourhood of } x_0\}.$$

We shall prove the following generalisation of Theorem 1.2.

THEOREM 1.3. *Let* $x_0 \in X \cap Y$. *Then* $\mathcal{L}_{x_0}(X,Y) \in \mathbb{Q}$, *and* (S) *holds for* $\nu = \mathcal{L}_{x_0}(X,Y)$, *some* $C > 0$ *and a neighbourhood* Ω *of* x_0.

The exponent $\mathcal{L}_{x_0}(X,Y)$ is attained on an analytic curve, namely, we have

THEOREM 1.4. *Let* $x_0 \in X \cap Y$ *and* $x_0 \in \overline{\mathcal{M} \setminus (X \cap Y)}$. *Then for any neighbourhood* Ω *of* x_0 *there exists an analytic curve* $\varphi : [0,r) \to \Omega$ *such that* $\varphi(0) \in X \cap Y$, $\varphi((0,r)) \subset \Omega \setminus (X \cap Y)$ *and for some constant* $C' > 0$,

$$\varrho(\varphi(t),X) + \varrho(\varphi(t),Y) \le C'\varrho(\varphi(t),X \cap Y)^{\mathcal{L}_{x_0}(X,Y)}, \quad t \in [0,r).$$

The above two theorems will follow from analogous results in a slightly more general situation. Namely, for three subanalytic sets $X, Y, Z \subset \mathcal{M}$ such that $X \cap Y \subset Z$, we define a regular separation exponent of Y and Z on X at a point $x_0 \in X \cap Y$ to be any real positive ν such that

$$(\#) \qquad \varrho(x,Y) \ge C\varrho(x,Z)^{\nu} \qquad \text{for } x \in X \cap \Omega,$$

where $C > 0$ and Ω is a neighbourhood of x_0. The infimum of all such exponents ν will be denoted by $\mathcal{L}_{x_0}(X;Y,Z)$. If $Z = X \cap Y$ then obviously $\mathcal{L}_{x_0}(X;Y,Z) = \mathcal{L}_{x_0}(X,Y)$, provided $x_0 \in \overline{X \setminus Y}$. The main result of this paper is the following

THEOREM 1.5. *Let* X, Y, Z *be closed subanalytic subsets of* $\mathcal{M}$ *with* $X \cap Y \subset Z$, *and let* $x_0 \in X \cap Y$.

(i) *We have* $\mathcal{L}_{x_0}(X;Y,Z) \in \mathbb{Q}$, *and* (#) *holds for* $\nu = \mathcal{L}_{x_0}(X;Y,Z)$, *some* $C > 0$ *and a neighbourhood* Ω *of* x_0.

(ii) *If* $x_0 \in \overline{X \setminus Z}$, *then* $\mathcal{L}_{x_0}(X;Y,Z)$ *is attained on an analytic curve, i.e. for any neighbourhood* $\widetilde{\Omega}$ *of* x_0 *there exists an analytic curve* $\varphi : [0,r) \to X \cap \widetilde{\Omega}$ *such that* $\varphi((0,r)) \subset X \setminus Z$ *and* $\varphi(0) \in X \cap Y$, *and for some constant* $C_1 > 0$,

$$C_1\varrho(\varphi(t),Y) \le \varrho(\varphi(t),Z)^{\mathcal{L}_{x_0}(X;Y,Z)} \quad \text{for } t \in [0,r).$$

The proof of the above theorem will be given in Section 2. Unfortunately, in Theorems 1.4 and 1.5, we cannot require that $\varphi(0) = x_0$ (see Example 2.5). This observation shows that, in the proof of Theorem 1.5, it does not suffice to apply the Curve Selection Lemma. We have to use another tool, the notion of Lipschitz stratification introduced by T. Mostowski [26] (see also [27]–[29]).

Sections 3 and 4 are devoted to applications of Theorem 1.5.

Let X, Y be subanalytic sets, $F : X \to Y$ be a subanalytic mapping and $\Gamma(F)$ be the graph of F. Let $\mathcal{L}_{x_0}(F) := \mathcal{L}_{(x_0,y_0)}(\Gamma(F); X \times \{y_0\}, V \times Y)$, where $x_0 \in X$, $y_0 = F(x_0)$, and $V = F^{-1}(y_0)$. Theorem 1.5 implies that $\mathcal{L}_{x_0}(F)$ is the smallest exponent ν satisfying the following *fundamental Łojasiewicz inequality*:

$$\varrho(F(x), y_0) \geq C\varrho(x, V)^{\nu}, \quad x \in X \cap \Omega, \tag{Ł}$$

for some neighbourhood Ω of x_0, and $C > 0$ (Corollary 3.1). The inequality (Ł) plays an important role in singularity theory ([2], [19], [22], [23], [32]), and in the solution of the division problem in distribution theory ([16], [21]).

For two subanalytic mappings $F : X \to Y$, $g : X \to Z$, where X, Y, Z are subanalytic sets, and $x_0 \in X$, $y_0 = F(x_0)$, $z_0 = g(x_0)$, we show that the number $\mathcal{L}_{x_0}(F/g) := \mathcal{L}_{(x_0,y_0,z_0)}(\Gamma(F,g); X \times \{y_0\} \times Z, X \times Y \times \{z_0\})$ is the smallest exponent ν satisfying

$$\varrho(F(x), y_0) \geq C\varrho(g(x), z_0)^{\nu}, \quad x \in X \cap \Omega, \tag{LT}$$

in a neighbourhood Ω of x_0 for some $C > 0$, provided $F^{-1}(y_0) \subset g^{-1}(z_0)$ (Corollary 4.1). In particular, we obtain the Lejeune-Jalabert and Teissier result stating that in the complex analytic case $\mathcal{L}_{x_0}(F/g) \in \mathbb{Q}$ ([20, Corollary 6.4], cf. [3]). We collect some relations between $\mathcal{L}_{x_0}(F)$ and $\mathcal{L}_{x_0}(F/g)$ in Remark 4.3.

If X is a semi-algebraic set and $F : X \to \mathbb{R}^m$ is a semi-algebraic mapping, then the set $\{\mathcal{L}_x(F) : x \in V\}$ is finite, where $V = F^{-1}(0)$ (Corollary 2.7). So, the number $\mathcal{L}(F) = \max_{x \in V} \mathcal{L}_x(F)$ is the smallest exponent ν for which (Ł) holds at each $x_0 \in V$. In Theorem 3.5 we prove that there exists a rational number l such that

$$|F(x)|(1 + |x|)^l \geq C\varrho(x, V)^{\mathcal{L}(F)} \quad \text{in } X \tag{JKS}$$

for some constant $C > 0$. Moreover, the infimum $l_\infty(F)$ of all such exponents l is also a rational number and satisfies (JKS), provided $\mathcal{L}(F) > 0$ and $X \setminus V$ is unbounded (if X is compact, then (JKS) holds for $l = 0$). Theorem 3.5 is a generalisation of the Ji, Kollár and Shiffman result to the semi-algebraic case ([17, Theorem 5 and Corollary 6], see also [4], [9], [10], [18]). In the case when V is finite, (JKS) is also important in the polynomial mappings theory (Remark 3.6).

In Section 4 we consider the notion of separation of two mappings. In particular we give a version of (JKS) for two mappings (Theorem 4.5, cf. [12]).

2. Separation of subanalytic sets. We recall some notions. A subset of a linear space M is called *semi-algebraic* when it is defined by a finite alternative of finite systems of inequalities $P > 0$ or $P \geq 0$, where P are polynomials on M. A set $E \subset \mathcal{M}$ is called *semi-analytic* if every point of $\mathcal{M}$ has a neighbourhood Ω such that $E \cap \Omega$ is defined by a finite alternative of finite systems of inequalities $f > 0$ or $f \geq 0$, where $f : \Omega \to \mathbb{R}$ are analytic functions. The set E is called a *subanalytic subset* of $\mathcal{M}$ if every point $x \in \mathcal{M}$ has a neighbourhood Ω such that $E \cap \Omega$ is the image under the projection map $\mathcal{M} \times \mathbb{R}^k \to \mathcal{M}$ of a semi-analytic relatively compact subset of $\mathcal{M} \times \mathbb{R}^k$ (where k depends on x). For the basic properties of semi-analytic and subanalytic sets see for instance [2], [11], [15], [22], [25].

For $A \subset \mathcal{M}$, we denote by $\varrho(\cdot, A)$ the distance function to A, i.e. $\varrho(z, A) = \inf_{x \in A} \varrho(z, x)$ if $A \neq \emptyset$, and $\varrho(z, \emptyset) = 1$.

In the remainder of this section, X, Y, Z are closed subanalytic sets in $\mathcal{M}$. We start with some remarks on $\mathcal{L}_{x_0}(X; Y, Z)$.

Remark 2.1. (a) If $x_0 \in X \cap Y \subset Z$ and $X \cap \Omega \subset Z$ for some neighbourhood Ω of x_0, then obviously, for any $C, \nu > 0$ the inequality (#) holds in Ω. So, $\mathcal{L}_{x_0}(X; Y, Z) = 0$. In order to omit this trivial case, we will assume $x_0 \in \overline{X \setminus Z}$.

(b) Obviously, $\mathcal{L}_{x_0}(X, Y) = \mathcal{L}_{x_0}(Y, X)$. However, we cannot require that $\mathcal{L}_{x_0}(X; Y, Z) = \mathcal{L}_{x_0}(Y; X, Z)$. Indeed, for $X = \{(x_1, x_2) : x_2 = 0\}$ and $Y = Z = \{(x_1, x_2) : x_1 = x_2\}$ we easily obtain $\mathcal{L}_0(X; Y, Z) = 1$ and $\mathcal{L}_0(Y; X, Z) = 0$.

Since the exponent $\mathcal{L}_{x_0}(X; Y, Z)$ has a local character, the proof of Theorem 1.5 can be carried out in the case of subanalytic sets in an open set G of a finite-dimensional real linear space M. This exponent does not depend on the choice of the norm, so we will use the Euclidean norm $|\cdot|$. Set $B(x_0, R) = \{x \in M : |x - x_0| < R\}$, where $x_0 \in M$ and $R > 0$.

Lemma 2.2. *Let Z be a closed subanalytic subset of G and $x_0 \in Z$. Let $R > 0$ be such that $B = \overline{B(x_0, R)} \subset G$. Then*

$$A = \{(x, z) \in B \times (Z \cap B) : \varrho(x, Z \cap B) = |x - z|\}$$

is a nonempty compact and subanalytic set in $G \times M$. Moreover, if Z is a semi-algebraic (and closed) subset of M, then the set $\{(x, z) \in M \times Z : \varrho(x, Z) = |x - z|\}$ is semi-algebraic.

Proof. Since $\{(x, z, w) \in B \times (Z \cap B) \times (Z \cap B) : |x - z| > |x - w|\}$ is a subanalytic and relatively compact subset of $G \times M^2$, its projection

$$E = \{(x, z) \in B \times (Z \cap B) : \exists_{w \in Z \cap B}\ |x - z| > |x - w|\}$$

onto $B \times M$ is subanalytic. Moreover, $A = [B \times (Z \cap B)] \setminus E$, and hence, by the Gabrielov Theorem on Complement ([13, Theorem 1], [25, IV.4]), the set A is subanalytic. The proof is analogous when Z is semi-algebraic (by using the Tarski–Seidenberg Theorem, see [1, Theorem 2.3.4]). ∎

A curve $\varphi : [0,r) \to \mathcal{M}$, where $r > 0$, is called *analytic* if φ has an analytic extension $\psi : (r',r) \to \mathcal{M}$, where $r' < 0$. If $\mathcal{M} = M$, then in a neighbourhood of 0, φ is the sum of a power series of the form

$$\varphi(t) = \alpha_p t^p + \alpha_{p+1} t^{p+1} + \cdots, \qquad \alpha_i \in M,\ p \in \mathbb{Z},\ p \geq 0.$$

If $\varphi \neq 0$, then we may assume that $\alpha_p \neq 0$. The number p is called the *order* of φ and denoted by $\operatorname{ord}\varphi$. Additionally we put $\operatorname{ord} 0 = \infty$.

LEMMA 2.3. *Let $x_0 \in \overline{X \cap Y} \subset Z$, where $x_0 \in \overline{X \setminus Z}$, and let $B = \overline{B(x_0,R)}$, where $R > 0$ and $\overline{B(x_0,2R)} \subset G$. Then there exist analytic curves $\varphi : [0,r) \to X \cap B$, $\varphi_1 : [0,r) \to Y$, and $\psi : [0,r) \to Z$, where $r > 0$, such that*

(i) $\varphi((0,r)) \subset X \setminus Z$,
(ii) $\varphi(0) = \varphi_1(0) = \psi(0)$,
(iii) *there exists $C > 0$ such that*

$$\varrho(x,Y) \geq C\varrho(x,Z)^{\nu} \quad \textit{for } x \in B \cap X, \tag{1}$$

where

$$\nu = \frac{\operatorname{ord}(\varphi - \varphi_1)}{\operatorname{ord}(\varphi - \psi)}, \tag{2}$$

(iv) *the smallest exponent ν for which* (1) *holds is defined by* (2); *moreover, there exist $C_1 > 0$ and $t_0 \in (0,r)$ such that*

$$\varrho(\varphi(t),Y) \leq C_1 \varrho(\varphi(t),Z)^{\nu} \quad \textit{for } t \in [0,t_0]. \tag{3}$$

Proof. For $x \in B$ and $E \subset G$ we have $\varrho(x,E) = \varrho(x, E \cap \overline{B(x_0,2R)})$. Thus we may assume that $X \subset B$ and $Y, Z \subset \overline{B(x_0,2R)}$. Let

$$V = \{(x,y,z) \in X \times Y \times Z : \varrho(x,Y) = |x-y| \ \wedge\ \varrho(x,Z) = |x-z|\},$$
$$U = \{((x,y,z),(a,b,c)) \in V \times V : |a-c| = |x-z| \ \wedge\ |x-y| > |a-b|\}.$$

By Lemma 2.2, the sets V and U are subanalytic and relatively compact in $G \times M^2$ and $G \times M^5$, respectively. Then the projection $W = \{(x,y,z) \in V : \exists_{(a,b,c)\in V}\ ((x,y,z),(a,b,c)) \in U\}$ of U is a subanalytic set. So, the complement $\Gamma = V \setminus W$ is subanalytic. Obviously,

$$\Gamma = \{(x,y,z) \in V : \varrho(x,Y) = \inf\{\varrho(a,Y) : a \in X \cap B \wedge \varrho(a,Z) = \varrho(x,Z)\}\}.$$

Since B and $\overline{B(x_0,2R)}$ are compact sets, for any $a \in (X \cap B) \setminus Z$ there exists $(x,y,z) \in \Gamma$ such that $\varrho(x,Z) = \varrho(a,Z)$. By the assumption $x_0 \in \overline{X \setminus Z}$, there exists $x_1 \in X \cap Y$ such that (x_1,x_1,x_1) is an accumulation point of Γ. Consequently, by the Curve Selection Lemma ([25, IV.3]), there exists an analytic curve $(\varphi,\varphi_1,\psi) : [0,r) \to \Gamma$, where $r > 0$, such that φ, φ_1, ψ satisfy (i) and (ii).

For φ, φ_1, ψ chosen above, let the number ν be given by (2). Obviously, $\nu \in \mathbb{Q}$ and $\nu > 0$. By the definition of ν, there exist $t_0 \in (0,r)$ and $C_1, C_2 > 0$

such that

$$(4)\quad |\varphi(t)-\varphi_1(t)| \le C_1|\varphi(t)-\psi(t)|^\nu \le C_2|\varphi(t)-\varphi_1(t)| \quad \text{for } t\in[0,t_0].$$

By the definition of Γ we see that (4) implies (3).

Set $\varepsilon=\varrho(\varphi(t_0),Z)$; we have $\varepsilon>0$. Take any $x\in(X\setminus Z)\cap B$ such that $\varrho(x,Z)<\varepsilon$. Then there exists $t\in(0,t_0)$ such that $\varrho(x,Z)=\varrho(\varphi(t),Z)$. So, from (3) and the definition of Γ,

$$(5)\qquad \varrho(x,Y)\ge\varrho(\varphi(t),Y)\ge\frac{C_1}{C_2}\,\varrho(\varphi(t),Z)^\nu=\frac{C_1}{C_2}\,\varrho(x,Z)^\nu.$$

Since $\{x\in X\cap B:\varrho(x,Z)\ge\varepsilon\}$ is compact, by (5), diminishing $C=C_1/C_2$ if necessary, we obtain (1) for $x\in X\cap B$. This gives (iii). The remaining condition in (iv) immediately follows from (3) and (4), because (4) holds only for ν given by (2). ∎

By a *stratification* of a subset $X\subset\mathcal{M}$ we mean a decomposition of X into a disjoint locally finite union

$$(6)\qquad X=\bigcup S_\alpha,$$

where the subsets S_α are called *strata*, such that each S_α is a connected embedded submanifold of $\mathcal{M}$, and each $(\overline{S}_\alpha\setminus S_\alpha)\cap X$ is the union of some strata of dimension smaller than $\dim S_\alpha$.

The set X with stratification (6) is called *locally bi-Lipschitz trivial along each stratum* if for each stratum S_α and each $x\in S_\alpha$ there exist: a neighbourhood $U\subset\mathcal{M}$ of x, a submanifold $\mathcal{N}$ of U transverse to S_α at x and of dimension complementary to $\dim S_\alpha$, and a bi-Lipschitz homeomorphism (i.e. Lipschitz homeomorphism with Lipschitz inverse)

$$(7)\qquad \Psi: X\cap U\to(S_\alpha\cap U)\times(\mathcal{N}\cap X).$$

In [28] and [29] Parusiński showed the existence of a *Lipschitz stratification* of subanalytic sets, and proved that any Lipschitz stratification of X ensures locally bi-Lipschitz triviality of X along each stratum ([28, Theorem 1.9], and [29, Lipschitz Isotopy Lemma, Theorem 1.6]). From these results we obtain:

LEMMA 2.4. *Let $X=X_1\cup\cdots\cup X_k$, where $X_1,\dots,X_k$ are compact subanalytic subsets of M. Then there exists a stratification $X=\bigcup S_\alpha$ of X such that each $X_1,\dots,X_k$ is a union of some strata S_α, and X is locally bi-Lipschitz trivial along each stratum. In particular, for each stratum S_α and any $y,z\in S_\alpha$ there exist neighbourhoods Ω_y, Ω_z of y, z, respectively, and a bi-Lipschitz homeomorphism $\Phi: X\cap\Omega_y\to X\cap\Omega_z$ which preserves $X_1,\dots,X_k$, i.e.*

$$(8)\qquad \Phi(X_i\cap\Omega_y)=X_i\cap\Omega_z \quad \textit{for } i=1,\dots,k.$$

Proof. By Theorems 1.4 and 1.6 in [29] there exists a stratification $X = \bigcup S_\alpha$ of X such that each $X_1, \ldots, X_k$ is the union of some strata S_α, and X is locally bi-Lipschitz trivial along each stratum S_α. Take any stratum S_α which contains at least two points. Let $x \in S_\alpha$ and let Ψ be a bi-Lipschitz homeomorphism of the form (7). One can assume that Ψ is defined by a Lipschitz flow obtained by integrating a Lipschitz vector field tangent to strata of X (see proof of Theorem 1.6 in [29] and proof of Proposition 1.1 in [26]). Then $\Psi(X_i \cap U) = (S_\alpha \cap U) \times (\mathcal{N} \cap X_i)$ for $i = 1, \ldots, k$. Thus, for any $y \in S_\alpha \cap U$ we easily get (8). Since S_α is connected, we obtain the assertion. ∎

Proof of Theorem 1.5. Without loss of generality we may assume that $x_0 \in \overline{X \setminus Z}$. By Lemma 2.4, one can assume that there exists a stratification $X \cup Y \cup Z = \bigcup_\alpha S_\alpha$ such that each of the sets $X \cap Y$, X, Y, Z is a union of some strata S_α, and $X \cup Y \cup Z$ is locally bi-Lipschitz trivial along each stratum.

Take any $x_0 \in X \cap Y$ and let $S_{\alpha_1}, \ldots, S_{\alpha_k}$ be all the strata for which $x_0 \in \overline{S}_\alpha$. Let $R > 0$ be such that $\overline{B(x_0, 2R)} \subset G$ and

$$(X \cup Y \cup Z) \cap \overline{B(x_0, 2R)} = (S_{\alpha_1} \cup \cdots \cup S_{\alpha_k}) \cap \overline{B(x_0, 2R)}.$$

Let $\varphi : [0, r) \to X \cap \overline{B(x_0, R)}$ be an analytic curve for which there exist analytic curves φ_1, ψ such that the assertion of Lemma 2.3 holds. Let ν be as in (2). Then ν is a rational number and satisfies (1). Hence, it suffices to prove that

$$\nu = \mathcal{L}_{x_0}(X; Y, Z). \tag{9}$$

In accordance with (1), it suffices to prove that for any $0 < R_1 < R$ there exists a continuous curve $\kappa : [0, \varepsilon) \to X \cap B_1$, where $\varepsilon > 0$ and $B_1 = \overline{B(x_0, R_1)}$, such that $\kappa((0, \varepsilon)) \subset X \setminus Z$, $\kappa(0) \in X \cap Y$, and for some $C_1, C_2 > 0$,

$$C_1 \varrho(\kappa(t), Y) \le \varrho(\kappa(t), Z)^\nu \le C_2 \varrho(\kappa(t), Y) \quad \text{for } t \in [0, \varepsilon). \tag{10}$$

Take any $0 < R_1 < R$. Let $x = \varphi(0)$, and let $x \in S_{\alpha_i}$. Then $S_{\alpha_i} \subset X \cap Y$ and there exists $y_0 \in S_{\alpha_i}$ such that $|y_0 - x_0| < R_1/2$. By Lemma 2.4, there exist neighbourhoods Ω_1, Ω_2 of x and y_0, respectively, where $\Omega_2 \subset B_1$, and a bi-Lipschitz homeomorphism $\Phi : (X \cup Y \cup Z) \cap \Omega_1 \to (X \cup Y \cup Z) \cap \Omega_2$ such that $\Phi(X \cap \Omega_1) = X \cap \Omega_2$ and $\Phi(Y \cap \Omega_1) = Y \cap \Omega_2$ and $\Phi(Z \cap \Omega_1) = Z \cap \Omega_2$. Moreover, there exists $0 < \varepsilon < t_0$ such that $\varphi([0, \varepsilon)) \subset \Omega_1$. Put $\kappa(t) = \Phi(\varphi(t))$ for $t \in [0, \varepsilon)$. Since Φ is a bi-Lipschitz homeomorphism,

$$D_1 \varrho(\varphi(t), Y) \le \varrho(\kappa(t), Y) \le D_2 \varrho(\varphi(t), Y),$$
$$D_1 \varrho(\varphi(t), Z) \le \varrho(\kappa(t), Z) \le D_2 \varrho(\varphi(t), Z)$$

for $t \in [0, \varepsilon)$ and some $D_1, D_2 > 0$. Then by (3) we obtain (10) and, as a consequence, (9). This gives (i). Assertion (ii) follows from the above and Lemma 2.3. ∎

EXAMPLE 2.5. In Theorems 1.4 and 1.5(ii), we cannot require that $\varphi(0) = x_0$. Indeed, let $x_0 = 0 \in \mathbb{R}^3$ and

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 = x_2 x_3\}, \quad Y = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = 0\}.$$

Then $X \cap Y = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. Let $Z = X \cap Y$.

By Theorem 1.1, $\mathcal{L}_0(X, Y) = \mathcal{L}_0(X; Y, Z)$. Note that $\mathcal{L}_0(X; Y, Z) = 2$. Indeed, we may use the polycylindric norm in $\mathbb{R}^3$. Let $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \max\{|x_1|, |x_2|, |x_3|\} < \varepsilon\}$, $0 < \varepsilon < 1$. For any $x = (x_1, x_2, x_3) \in X \cap \Omega$ we have $\varrho(x, Y) = |x_2|$ and $\varrho(x, Z) = \max\{|x_1|, |x_2|\}$. If $\varrho(x, Z) = |x_2|$, then

$$\varrho(x, Y) = |x_2| \geq |x_2|^2 = \varrho(x, Z)^2.$$

If $\varrho(x, Z) = |x_1|$, then

$$\varrho(x, Y) = |x_2| \geq |x_2 x_3| = |x_1|^2 = \varrho(x, Z)^2.$$

So, $\mathcal{L}_0(X; Y, Z) \leq 2$. On the other hand, taking the curve $\varphi : [0, \varepsilon/2) \ni t \mapsto (t, (2/\varepsilon)t^2, \varepsilon/2) \in X \cap \Omega$, we have

$$\varrho(\varphi(t), Y) = \frac{2}{\varepsilon}\, t^2 = \frac{2}{\varepsilon}\, \varrho(\varphi(t), Z)^2.$$

Hence $\mathcal{L}_0(X; Y, Z) \geq 2$. Summing up, $\mathcal{L}_0(X, Y) = \mathcal{L}_0(X; Y, Z) = 2$.

We shall show that the exponent $\mathcal{L}_0(X, Y)$ is not attained on any analytic curve φ such that $\varphi(0) = 0$. Assume to the contrary that for some analytic curve $\varphi = (\varphi_1, \varphi_2, \varphi_3) : [0, r) \to \mathbb{R}^3$, where $\varphi(0) = 0$, $\varphi((0, r)) \subset \mathbb{R}^3 \setminus Z$, we have $\varrho(\varphi(t), X) + \varrho(\varphi(t), Y) \leq C\varrho(\varphi(t), Z)^2$ for $t \in [0, r)$, where $C > 0$. Then

$$(11) \quad \varrho(\varphi(t), X) \leq C\varrho(\varphi(t), Z)^2, \quad \varrho(\varphi(t), Y) \leq C\varrho(\varphi(t), Z)^2 \quad \text{for } t \in [0, r).$$

Since $\varrho(\varphi(t), Y) = |\varphi_2(t)|$, $\varrho(\varphi(t), Z) = \max\{|\varphi_1(t)|, |\varphi_2(t)|\}$ for $t \in [0, r)$, from (11) we may assume $\varrho(\varphi(t), Z) = |\varphi_1(t)|$ for $t \in [0, r)$. So, (11) gives

$$(12) \qquad 0 < \operatorname{ord} \varphi_1 < \infty \quad \text{and} \quad \operatorname{ord} \varphi_2 \geq 2 \operatorname{ord} \varphi_1.$$

By (11), the origin is an accumulation point of the subanalytic set

$$E = \{(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6 : x_1 \neq 0 \ \wedge \ y_1^2 = y_2 y_3 \\ \wedge \ \max\{|x_2|, |x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\} \leq C|x_1|^2\}.$$

So, by the Curve Selection Lemma we may assume that there exists an analytic curve $\psi = (\psi_1, \psi_2, \psi_3) : [0, r) \to X$ such that $\psi(0) = 0$ and $(\varphi(t), \psi(t)) \in E$ for $t \in (0, r)$. Then

$$|\varphi_i(t) - \psi_i(t)| \leq C|\varphi_1(t)|^2 \quad \text{for } t \in [0, r),$$

and so $\operatorname{ord}(\varphi_i - \psi_i) \geq 2 \operatorname{ord} \varphi_1$, $i = 1, 2$. Since $0 < \operatorname{ord} \varphi_1 < 2 \operatorname{ord} \varphi_1$, wee have $\operatorname{ord} \psi_1 = \operatorname{ord} \varphi_1$. Moreover, $\psi_1^2 = \psi_2 \psi_3$ and $\psi(0) = 0$, hence $\operatorname{ord} \psi_2 < 2 \operatorname{ord} \psi_1$. Therefore, $\operatorname{ord} \varphi_2 = \operatorname{ord} \psi_2 < \infty$. Hence and from (12),

$$\operatorname{ord} \varphi_2 \geq 2 \operatorname{ord} \varphi_1 = 2 \operatorname{ord} \psi_1 > \operatorname{ord} \psi_2 = \operatorname{ord} \varphi_2.$$

This is impossible.

COROLLARY 2.6. *Let X, Y, Z be compact subanalytic subsets of a finite-dimensional real linear space M such that $X \cap Y \subset Z$. Then there exists a stratification*

$$X \cap Y = \bigcup S_\alpha \tag{13}$$

such that for each stratum S_α, the function

$$S_\alpha \ni x \mapsto \mathcal{L}_x(X; Y, Z) \tag{14}$$

is constant. In particular, the function $X \cap Y \ni x \mapsto \mathcal{L}_x(X; Y, Z)$ is upper semi-continuous.

Proof. By Lemma 2.4, one can assume that there exists a stratification $X \cup Y \cup Z = \bigcup_\alpha S_\alpha$ such that each of $X \cap Y$, X, Y, Z is a union of some strata S_α, and $X \cup Y \cup Z$ is locally bi-Lipschitz trivial along each stratum.

Let $S_\alpha \subset X \cap Y$. Take any $z, w \in S_\alpha$. Then there exist neighbourhoods Ω_1, Ω_2 of z, w respectively and a bi-Lipschitz homeomorphism $\Phi : (X \cup Y \cup Z) \cap \Omega_1 \to (X \cup Y \cup Z) \cap \Omega_2$ such that $\Phi(X \cap \Omega_1) = X \cap \Omega_2$, $\Phi(Y \cap \Omega_1) = Y \cap \Omega_2$ and $\Phi(Z \cap \Omega_1) = Z \cap \Omega_2$. If $X \cap \Omega \subset Z$ for some neighbourhood Ω of z, then $X \cap \Phi(\Omega \cap \Omega_1) \subset Z$, so $\mathcal{L}_z(X; Y, Z) = \mathcal{L}_w(X; Y, Z) = 0$. Assume that $z, w \in \overline{X \setminus Z}$. By Theorem 1.5, one can assume that

$$\varrho(x, Y) \geq C_1 \varrho(x, Z)^{\mathcal{L}_z(X;Y,Z)} \quad \text{for } x \in X \cap \Omega_1,$$
$$\varrho(x, Y) \geq C_2 \varrho(x, Z)^{\mathcal{L}_w(X;Y,Z)} \quad \text{for } x \in X \cap \Omega_2,$$

where $C_1, C_2 > 0$ are some constants. Since Φ is a bi-Lipschitz homeomorphism, we have $\mathcal{L}_z(X; Y, Z) = \mathcal{L}_w(X; Y, Z)$, and so the function (14) is constant. ■

COROLLARY 2.7. *For any closed semi-algebraic subsets X, Y, Z of a linear space M such that $X \cap Y \subset Z$, the set $\{\mathcal{L}_x(X; Y, Z) : x \in X \cap Y\}$ is finite.*

Proof. Let $B = \{z \in M : |z| < 1\}$. The mapping

$$H : B \ni z \mapsto \frac{z}{1 - |z|^2} \in M$$

is a diffeomorphism. The inverse of H is of the form

$$H^{-1}(w) = w \frac{2}{1 + \sqrt{1 + 4|w|^2}}.$$

Moreover, H and H^{-1} are semi-algebraic and locally bi-Lipschitz homeomorphisms. Let $E = H^{-1}(X)$, $W = H^{-1}(Y)$, $V = H^{-1}(Z)$. Then $E \cup W \cup V$ is a semi-algebraic set, and in consequence, $A = \overline{E} \cup \overline{W} \cup \overline{V}$ is a compact semi-algebraic set. By Corollary 2.6, there exists a stratification $\overline{E} \cap \overline{W} = \bigcup S_\alpha$ such that for each stratum S_α the function $S_\alpha \cap B \ni z \mapsto \mathcal{L}_z(E; W, V)$ is constant, and the number of strata S_α is finite. Since $X = H(E)$, $Y = H(W)$,

$Z = H(V)$, $X \cap Y = H(E \cap W) = \bigcup H(S_\alpha \cap B)$ and H is locally bi-Lipschitz homeomorphism, it follows that for any $H(S_\alpha \cap B) \neq \emptyset$ the mapping $H(S_\alpha \cap B) \ni x \mapsto \mathcal{L}_x(X; Y, Z)$ is constant. This gives the assertion. ∎

By the i-th *skeleton* of the stratification (6) we mean $X^i = \bigcup_{\dim S_\alpha \le i} S_\alpha$. The stratification (6) of a complex analytic subset X of a complex analytic manifold $\mathcal{M}$ is called *complex analytic* if all the skeletons X^i are complex analytic subsets of $\mathcal{M}$. The stratification (6) of a complex algebraic subset X of a complex linear space M is called *complex algebraic* if all the skeletons X^i are complex algebraic subsets of M and the number of strata S_α is finite.

REMARK 2.8. Parusiński (in [28, Theorems 2.4 and 2.6]) proved that for a complex analytic set X, we may require that the locally Lipschitz stratification (6) is complex analytic, and for a complex algebraic set, we may require that the stratification is complex algebraic. Hence, in Corollary 2.6, for complex analytic sets X, Y, Z, we may require that the stratification (13) is complex analytic, and for complex algebraic sets X, Y, Z, we may require that it is complex algebraic.

REMARK 2.9. Let X, Y, Z be closed subanalytic subsets of an open subset $G \subset \mathbb{R}^n$ such that $X \cap Y \subset Z$. Under a given locally Lipschitz stratification

$$X \cup Y \cup Z = \bigcup S_\alpha$$

such that $X \cap Y$ is a union of some strata S_α, for any $x_0 \in X \cap Y$ such that $x_0 \in \overline{X \setminus Z}$, we may determine a neighbourhood Ω of x_0 on which the inequality (#) holds for $\nu = \mathcal{L}_{x_0}(X; Y, Z)$.

Indeed, let $S_{\alpha_1}, \ldots, S_{\alpha_k}$ be all the strata for which $x_0 \in \overline{S}_\alpha$. Take any $R > 0$ such that $B = \overline{B(x_0, R)} \subset G$ and $(X \cup Y \cup Z) \cap B = (S_{\alpha_1} \cup \cdots \cup S_{\alpha_k}) \cap B$. Put $\Omega = \{x \in \mathbb{R}^n : |x - x_0| < R/2\}$. Under the notation of the proof of Theorem 1.5, we obtain (1) for $\nu = \mathcal{L}_{x_0}(X; Y, Z)$. For $x \in X$ such that $|x - x_0| < R/2$, we have $\varrho(x, Y) = \varrho(x, Y \cap B)$ and $\varrho(x, Z) = \varrho(x, Z \cap B)$. Thus, by (1), we obtain $\varrho(x, Y) \ge C\varrho(x, Z)^{\mathcal{L}_{x_0}(X; Y, Z)}$ for $x \in X \cap \Omega$.

3. Łojasiewicz exponent of a mapping. Let X, Y be closed subanalytic subsets of $\mathcal{M}$, and let $F : X \to Y$ be a *subanalytic mapping*, i.e. a continuous mapping with subanalytic graph $\Gamma(F)$.

From Theorem 1.5 we get (cf. [3, Corollary 1], [20, Corollary 6.4]):

COROLLARY 3.1. *Let $x_0 \in X$, $y_0 = F(x_0)$, and $V = F^{-1}(y_0)$. The number $\mathcal{L}_{x_0}(F) = \mathcal{L}_{(x_0, y_0)}(\Gamma(F); X \times \{y_0\}, V \times Y)$ is the smallest exponent ν satisfying* (Ł) *for some $C > 0$ and a neighbourhood Ω of x_0.*

Proof. For any $x \in X$ we have $\varrho(F(x), y_0) = \varrho((x, F(x)), X \times \{y_0\})$ and $\varrho(x, V) = \varrho((x, F(x)), V \times Y)$. So, condition (Ł) is equivalent to the inequality $\varrho(z, X \times \{y_0\}) \geq C\varrho(z, V \times Y)^\nu$, $z \in \Gamma(F) \cap \Delta$, where Δ is a neighbourhood of (x_0, y_0). Thus, Theorem 1.5 gives the assertion. ∎

The number $\mathcal{L}_{x_0}(F)$ is called the *Łojasiewicz exponent* of F at x_0. From Corollaries 3.1 and 2.6, we immediately obtain

COROLLARY 3.2. *Let $V = F^{-1}(y_0)$, where $y_0 \in Y$. The function $V \ni x \mapsto \mathcal{L}_x(F)$ is upper semi-continuous.*

REMARK 3.3. Let $V = F^{-1}(F(x_0))$ and let $x_0 \in \overline{X \setminus V}$. By Corollary 3.1 and Theorem 1.5, the exponent $\mathcal{L}_{x_0}(F)$ is attained on an analytic curve, i.e. for any neighbourhood Ω of x_0 there exist $C_1 > 0$ and an analytic curve $\varphi : [0, r) \to X \cap \Omega$ such that $\varphi(0) \in V$, $\varphi((0, r)) \subset X \setminus V$ and $\varrho(F(\varphi(t)), y_0) \leq C_1\varrho(\varphi(t), V)^{\mathcal{L}_{x_0}(F)}$ for $t \in [0, r)$. We cannot require that $\varphi(0) = x_0$. Indeed, it suffices to consider the set X from Example 2.5 and the projection map $F(x_1, x_2, x_3) = x_2$.

In the remainder of this section, $F : X \to \mathbb{R}^m$ is a *semi-algebraic mapping*, i.e. a continuous mapping defined on a semi-algebraic set $X \subset \mathbb{R}^n$ with semi-algebraic graph $\Gamma(F)$. We assume that X is closed. Let $V = F^{-1}(0)$.

According to Corollaries 2.7 and 3.1, the set $\{\mathcal{L}_x(F) : x \in V\}$ is finite, so we may define a special regular separation exponent of F,

$$\mathcal{L}(F) = \max\{\mathcal{L}_x(F) : x \in V\} \quad \text{if } V \neq \emptyset.$$

Additionally, we put $\mathcal{L}(F) = 0$ if $V = \emptyset$.

REMARK 3.4. (a) Obviously, $\mathcal{L}(F) = 0$ if and only if for each connected component W of X either $V \cap W = W$ or $V \cap W = \emptyset$.

(b) From the definition of $\mathcal{L}(F)$, it follows that if X is a compact set, then there exists $C > 0$ such that $|F(x)| \geq C\varrho(x, V)^{\mathcal{L}(F)}$ for $x \in X$.

In the considerations at infinity we will use the notion of curves meromorphic at infinity instead of analytic curves. A curve $\varphi : [a, \infty) \to \mathbb{R}^k$, where $a \in \mathbb{R}$, is called *meromorphic at infinity* if φ is the sum of a Laurent series of the form

$$\varphi(t) = \alpha_p t^p + \alpha_{p-1} t^{p-1} + \cdots, \qquad \alpha_i \in \mathbb{R}^k,\ p \in \mathbb{Z}.$$

If $\varphi \neq 0$, then we may assume that $\alpha_p \neq 0$. The number p is called the *degree* of φ and denoted by $\deg \varphi$. Additionally, we put $\deg 0 = -\infty$.

THEOREM 3.5. *If $X \setminus V$ is an unbounded set, then for any $\nu \in \mathbb{Q}$ such that $\nu \geq \mathcal{L}(F)$, there exists a unique $l \in \mathbb{Q}$ such that for some constant $C > 0$,*

$$|F(x)|(1 + |x|)^l \geq C\varrho(x, V)^\nu \quad \textit{for } x \in X, \tag{15}$$

and for some curve $\varphi : [a,\infty) \to X \setminus V$ *meromorphic at infinity, with* $\deg\varphi > 0$,

$$|F(\varphi(t))|(1+|\varphi(t)|)^l \le C'\varrho(\varphi(t),V)^\nu, \quad t \in [a,\infty), \tag{16}$$

where $C' > 0$ *is a constant.*

Proof. For any $r > 0$, the set $\{x \in X : |x| \le r\}$ is compact, so there exists $C_r > 0$ such that

$$C_r\varrho(x,V)^\nu \le |F(x)| \quad \text{for } x \in X,\ |x| \le r. \tag{17}$$

Observe that the set

$$W = \{w \in X \setminus V : \forall_{x\in X}\ (|w| = |x| \Rightarrow 2\varrho(w,V)^\nu|F(x)| \ge \varrho(x,V)^\nu|F(w)|)\}$$

is unbounded. Indeed, since $X \setminus V$ is unbounded, for any sufficiently large $r > 0$ the set

$$A = \{\varrho(x,V)^\nu/|F(x)| : |x| = r \wedge x \in X \setminus V\}$$

is nonempty, and $A \subset (0,\infty)$. Thus, from (17) we get $0 < \sup A \le 1/C_r$, and therefore, there exists $w \in X \setminus V$ such that $|w| = r$ and

$$\frac{\varrho(w,V)^\nu}{|F(w)|} > \frac{1}{2}\sup A.$$

This implies that $w \in W$. In consequence, W is an unbounded set.

Since ν is a rational number, by Lemma 2.2 and the Tarski–Seidenberg Theorem we conclude that W is a semi-algebraic set. Moreover, W is unbounded, thus, by the Curve Selection Lemma at infinity, there exists a curve $(\varphi,\varphi_1) : [a,\infty) \to \Gamma(F|_W)$ meromorphic at infinity such that $\deg\varphi > 0$, $\varphi_1 = F(\varphi)$, $\deg\varphi_1 \in \mathbb{Z}$. If $V \ne \emptyset$, then by Lemma 2.2 we may assume that $\varrho(\varphi(t),V) = |\varphi(t) - \psi(t)|$ for $t \in [a,\infty)$, where $\psi : [a,\infty) \to V$ is a curve meromorphic at infinity. If $V = \emptyset$, we put $\psi = \varphi + 1$. Let

$$l = \frac{\nu\deg(\varphi-\psi) - \deg F(\varphi)}{\deg\varphi}.$$

Obviously $l \in \mathbb{Q}$. Moreover, there exist $C', C'' > 0$ and $R > 0$, where $R = |\varphi(t_0)|$ for some t_0, such that for any $t \in [a,\infty)$ satisfying $|\varphi(t)| > R$, we have

$$2C''\frac{\varrho(\varphi(t),V)^\nu}{|F(\varphi(t))|} \le (1+|\varphi(t)|)^l \le C'\frac{\varrho(\varphi(t),V)^\nu}{|F(\varphi(t))|}. \tag{18}$$

Take any $x \in X \setminus V$ such that $|x| > R$. Since $\deg\varphi > 0$, we have $|x| = |\varphi(t)|$ for some $t \in [a,\infty)$. By the definition of W and from (18),

$$C''\frac{\varrho(x,V)^\nu}{|F(x)|} \le 2C''\frac{\varrho(\varphi(t),V)^\nu}{|F(\varphi(t))|} \le (1+|\varphi(t)|)^l = (1+|x|)^l. \tag{19}$$

Let $C = \min\{C'', C_R\min\{1,(1+R)^l\}\}$. Then (19) gives (15) for $x \in X$ such that $|x| > R$. Since $(1+|x|)^l \ge \min\{1,(1+R)^l\}$ for $|x| \le R$, (17) gives (15) for $x \in X$ such that $|x| \le R$. Summing up, (15) holds in X. Moreover, (16) immediately follows from (18). ∎

For any $\nu \in \mathbb{Q}$ such that $\nu \ge \mathcal{L}(F)$, we denote by $l_\infty(F,\nu)$ the unique number $l \in \mathbb{Q}$ satisfying (15) and (16) of the assertion of Theorem 3.5. If $\nu = \mathcal{L}(F)$, then for simplicity we write $l_\infty(F)$ instead of $l_\infty(F,\mathcal{L}(F))$.

REMARK 3.6. In the case when V is finite, the Łojasiewicz exponent of F at infinity $\mathcal{L}_\infty(F)$ has been investigated, where

$$\mathcal{L}_\infty(F) = \sup\{\nu \in \mathbb{R} : \exists_{C,R>0}\ \forall_{x\in X}\ (|x| \ge R \Rightarrow |F(x)| \ge C|x|^\nu)\}.$$

It is easy to see that, in this case, we have $\mathcal{L}_\infty(F) = \mathcal{L}(F) - l_\infty(F)$.

This exponent has been applied in many problems concerning polynomial mappings (see for instance [5]–[8], [14], [16], [18], [30], [31], [33], [34]).

In the case of polynomial mappings $F : \mathbb{R}^n \to \mathbb{R}^m$, estimations from above of $\mathcal{L}_{x_0}(F)$ are very interesting. In the complex case this has been done ([4], [9], [10], [17], [18]). The real case is more difficult. We have the following:

PROPOSITION 3.7. *Let $F = (f_1,\ldots,f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial mapping, $F_\mathbb{C} : \mathbb{C}^n \to \mathbb{C}^m$ be the complexification of F, $V = F^{-1}(0)$ and $V_\mathbb{C} = F_\mathbb{C}^{-1}(0)$. Then, for any $x_0 \in V$,*

$$\mathcal{L}_{x_0}(F) \le \mathcal{L}_{x_0}(F_\mathbb{C})\mathcal{L}_{x_0}(V_\mathbb{C},\mathbb{R}^n). \tag{20}$$

Moreover, if $d = \max_{1\le j\le m} \deg f_j > 0$, then

$$\mathcal{L}_{x_0}(F) \le d\mathcal{L}_{x_0}(W_\mathbb{C},\mathbb{R}^n), \tag{21}$$

where $W_\mathbb{C} \subset \mathbb{C}^n$ is the zero-set of the complexification of $g = f_1^2 + \cdots + f_m^2$.

Proof. The inequality (20) follows immediately from the definition.

It is easy to observe that $\mathcal{L}_{x_0}(F) = \frac{1}{2}\mathcal{L}_{x_0}(g)$. As the degree of $\Gamma(g_\mathbb{C})$ is equal to $2d$, by Theorem 4.2 in [10] we obtain $\mathcal{L}_{x_0}(g_\mathbb{C}) \le 2d$. Hence and from (20) we get (21). ∎

EXAMPLE 3.8. For a polynomial function $g : \mathbb{R}^n \to \mathbb{R}$, and its complex zero-set $W_\mathbb{C} = g_\mathbb{C}^{-1}(0)$, the exponent $\mathcal{L}_{x_0}(W_\mathbb{C},\mathbb{R}^n)$ for $x_0 \in \mathbb{R}^n \cap W_\mathbb{C}$ can be large.

Indeed, we take the Masser and Philippon example ([17, Example 15]). Let $f_1(x) = x_2 - x_1^d$, $f_2(x) = x_3 - x_2^d, \ldots, f_{n-1}(x) = x_n - x_{n-1}^d$, $f_n(x) = x_n^d$, and $g = f_1^2 + \cdots + f_n^2$, for $x = (x_1,\ldots,x_n)$. Let $F = (f_1,\ldots,f_n) : \mathbb{R}^n \to \mathbb{R}^n$. Then $\mathcal{L}_0(F) \ge d^n$, and by (21), $d^n \le d\mathcal{L}_0(W_\mathbb{C},\mathbb{R}^n)$, i.e. $\mathcal{L}_0(W_\mathbb{C},\mathbb{R}^n) \ge d^{n-1}$.

4. Remarks on separation of two mappings. Let X, Y, Z be closed subanalytic sets and let $F : X \to Y$ and $g : X \to Z$ be subanalytic mappings, $x_0 \in X$, $y_0 = F(x_0)$, $V = F^{-1}(y_0) \subset g^{-1}(z_0)$, where $z_0 \in Z$. From Theorem 1.5 and Corollary 2.6, we easily obtain

Corollary 4.1. *$\mathcal{L}_{x_0}(F/g) = \mathcal{L}_{(x_0,y_0,z_0)}(\Gamma(F,g); X \times \{y_0\} \times Z, X \times Y \times \{z_0\})$ is the smallest exponent ν satisfying* (LT) *for some $C > 0$ and a neighbourhood Ω of x_0. Moreover, the function $V \ni x \mapsto \mathcal{L}_x(F/g)$ is upper semi-continuous.*

Remark 4.2. If $g^{-1}(z_0) \cap \Omega = V \cap \Omega$ for some neighbourhood Ω of x_0, then $\mathcal{L}_{x_0}(F) \le \mathcal{L}_{x_0}(F/g)\mathcal{L}_{x_0}(g)$.

Indeed, $\varrho_{\mathcal{N}}(F(x), y_0) \ge D\varrho_{\mathcal{N}}(g(x), z_0)^{\mathcal{L}_{x_0}(F/g)} \ge D'\varrho(x, V)^{\mathcal{L}_{x_0}(F/g)\mathcal{L}_{x_0}(g)}$ in a neighbourhood of x_0 for some constants $D, D' > 0$.

Remark 4.3. Let $F : \mathcal{M} \to \mathbb{R}^k$ and $g : \mathcal{M} \to \mathbb{R}^m$ be analytic mappings, $V = F^{-1}(0) \subset g^{-1}(0)$, and let $x_0 \in V$.

(a) We have $\mathcal{L}_{x_0}(F) \ge \mathcal{L}_{x_0}(F/g)$. Indeed, g is a locally Lipschitz mapping, so $|g(x)|^{\mathcal{L}_{x_0}(F)} \le C\varrho(x, g^{-1}(0))^{\mathcal{L}_{x_0}(F)} \le C\varrho(x, V)^{\mathcal{L}_{x_0}(F)} \le C'|F(x)|$ in a neighbourhood of x_0 for some $C, C' > 0$.

(b) If x_0 is a smooth point of V, then $\mathcal{L}_{x_0}(F/g) = \mathcal{L}_{x_0}(F)$, provided the components of g generate the ideal of the germ of V at x_0. In particular $\mathcal{L}_{x_0}(F) = \sup_h \mathcal{L}_{x_0}(F/h)$, where h runs through all analytic mappings such that $V \cap \Omega \subset h^{-1}(0)$ for some neighbourhood Ω of x_0. Indeed, it is easy to see that $\mathcal{L}_{x_0}(g) = 1$, and then (a) and Remark 4.2 give the assertion.

(c) If x_0 is a singular point of V, then we can require neither $\mathcal{L}_{x_0}(F) = \sup_h \mathcal{L}_{x_0}(F/h)$ nor $\mathcal{L}_{x_0}(F) > \sup_h \mathcal{L}_{x_0}(F/h)$.

Indeed, for $F(x, y) = xy$, $(x, y) \in \mathbb{R}^2$, we have $\mathcal{L}_0(F) = 2$. Moreover, for any nonzero analytic mapping h such that $h(x, y) = 0$ for $xy = 0$ in a neighbourhood of 0, we have $|h(x, y)| = |F(x, y)|\,|h_1(x, y)|$, where h_1 is an analytic mapping. Thus, $\mathcal{L}_0(F/h) \le 1 < 2 = \mathcal{L}_0(F)$.

On the other hand, for $F(x, y, z) = (x^2, yz)$ and $g(x, y, z) = (x, yz)$, we have $V = F^{-1}(0, 0) = (\{0\} \times \mathbb{R} \times \{0\}) \cup (\{0, 0\} \times \mathbb{R})$, and in the polycylindric norm, $\varrho((x, y, z), V) = \min\{\max\{|x|, |y|\}, \max\{|x|, |z|\}\}$. Then we easily deduce that $\mathcal{L}_0(F) = 2 = \mathcal{L}_0(F/g)$.

Let $F : X \to \mathbb{R}^k$ and $g : X \to \mathbb{R}^m$ be semi-algebraic mappings, $V = F^{-1}(0)$, and let $V \subset g^{-1}(0)$. By Corollaries 2.7 and 4.1, the set $\{\mathcal{L}_x(F/g) : x \in V\}$ is finite. Then we may define

$$\mathcal{L}(F/g) = \max\{\mathcal{L}_x(F/g) : x \in V\} \quad \text{if } V \ne \emptyset.$$

Additionally we put $\mathcal{L}(F/g) = 0$ if $V = \emptyset$.

Remark 4.4. (a) Obviously, $\mathcal{L}(F/g) = 0$ if and only if for each connected component W of X either $W \subset g^{-1}(0)$ or $V \cap W = \emptyset$.

(b) From the definition of $\mathcal{L}(F/g)$ it follows that if X is a compact set, then there exists $C > 0$ such that $|F(x)| \ge C|g(x)|^{\mathcal{L}(F/g)}$ for $x \in X$.

Repeating the proof of Theorem 3.5 (by considering $|g(x)|$ instead of $\varrho(x, V)$) we obtain

THEOREM 4.5. *If $X \setminus V$ is an unbounded set, then for any $\nu \in \mathbb{Q}$ such that $\nu \geq \mathcal{L}(F/g)$, there exists a unique $l \in \mathbb{Q}$ such that for some constant $C > 0$,*

$$|F(x)|(1+|x|)^l \geq C|g(x)|^\nu \quad \textit{for any } z \in X, \tag{22}$$

and for some curve $\varphi : [r,\infty) \to X\setminus V$ meromorphic at infinity, with $\deg\varphi > 0$,

$$|F(\varphi(t))|(1+|\varphi(t)|)^l \leq C'|g(\varphi(t))|^\nu, \quad t \in [r,\infty), \tag{23}$$

where $C' > 0$ is a constant.

For any $\nu \in \mathbb{Q}$ such that $\nu \geq \mathcal{L}(F/g)$, the unique number $l \in \mathbb{Q}$ satisfying the assertion of Theorem 4.5 is denoted by $l_\infty(F/g,\nu)$. If $\nu = \mathcal{L}(F/g)$, then, for simplicity, we write $l_\infty(F/g)$.

In the case of polynomial mappings $F : \mathbb{R}^n \to \mathbb{R}^k$ and $g : \mathbb{R}^n \to \mathbb{R}^m$, we have the following connection between $l_\infty(F/g,\nu)$, $l_\infty(F,\nu)$ and $\mathcal{L}(F)$.

COROLLARY 4.6. *Let $g_1,\ldots,g_m$ be the components of g, and $d = \max_{1\leq j\leq m}\deg g_j$. If $V \neq \emptyset$ and $d > 0$, then*

$$l_\infty(F/g,\nu) \leq l_\infty(F,\nu) + (d-1)\nu \quad \textit{for any } \nu \in \mathbb{Q},\ \nu \geq \mathcal{L}(F). \tag{24}$$

The proof will be preceded by a lemma.

LEMMA 4.7. *Let $h \in \mathbb{R}[x_1,\ldots,x_n]$, $d = \deg h > 0$, and $S = h^{-1}(0)$. If $S \neq \emptyset$, then there exists $C > 0$ such that $C|h(x)| \leq \varrho(x,S)(1+|x|^{d-1})$ for any $x \in \mathbb{R}^n$.*

Proof. It is well known that there exist polynomials $h_1,\ldots,h_n \in \mathbb{R}[x,y]$, where $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$, such that $\deg h_1,\ldots,\deg h_n \leq d-1$ and

$$h(x) - h(y) = \sum_{j=1}^{n}(x_j - y_j)h_j(x,y). \tag{25}$$

Let $z \in S$. Then there exists $D > 0$ such that for any $j = 1,\ldots,n$,

$$|h_j(x,y)| \leq D(1+|x|^{d-1}) \quad \text{for } x,y \in \mathbb{R}^n \text{ such that } |y| \leq |z| + 2|x|. \tag{26}$$

Take any $x \in \mathbb{R}^n$, and let $y \in S$ be such that $\varrho(x,S) = |x-y|$. Since $\varrho(x,S) \leq |x-z|$, we have $|y| \leq |z| + 2|x|$. So, by (25) and (26),

$$|h(x)| \leq \sum_{j=1}^{n}|x_i - y_j|\,|h_j(x,y)| \leq n|x-y|D(1+|x|^{d-1}).$$

Then, for $C = 1/nD$, we obtain the assertion. ∎

Proof of Corollary 4.6. By Lemma 4.7, there exists $C_1 > 0$ such that

$$C_1|g(x)| \leq \varrho(x,V)(1+|x|^{d-1}) \quad \text{for } x \in \mathbb{R}^n. \tag{27}$$

Then for $\nu \in \mathbb{Q}$ with $\nu \geq \mathcal{L}(F)$, by Theorem 3.5, there exists $C_2 > 0$ such that

$$C_1^\nu C_2 |g(x)|^\nu \leq C_2 \varrho(x, V)^\nu (1 + |x|^{d-1})^\nu \leq |F(x)|(1 + |x|)^{l_\infty(F,\nu)}(1 + |x|^{d-1})^\nu$$

for any $x \in \mathbb{R}^n$. Hence, by Theorem 4.5, we easily obtain (24). ∎

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