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Entire functions that share values or small functions with their derivatives

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Abstract. We investigate the uniqueness of entire functions sharing values or small functions with their derivatives. One of our results gives a necessary condition on the Nevanlinna deficiency of the entire function f sharing one nonzero finite value CM with its derivative f'. Some applications of this result are provided. Finally, we prove some further results on small function sharing.

1. Introduction. In this paper, a meromorphic function always means meromorphic in the whole complex plane. We assume the reader is familiar with the basic notions of Nevanlinna theory (see [5, 10, 14, 17]). For a meromorphic function f and a constant a, we define the *Nevanlinna deficiency* by

$$\delta(a, f) = 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)},$$

and use the notation

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

We denote by $S(r, f)$ any quantity satisfying

$$\lim_{r\to\infty} \frac{S(r, f)}{T(r, f)} = 0, \qquad r \notin E,$$

where $E \subset (0, \infty)$ is of finite logarithmic measure. A meromorphic function $a(z)$ is said to be a *small function* of $f(z)$ if $T(r, a) = S(r, f)$. In addition, we say that two meromorphic functions $f(z)$ and $g(z)$ *share a finite value* a (resp. *a small function* $b(z)$) *IM* (ignoring multiplicities) when $f(z) - a$ and $g(z) - a$ (resp. $f(z) - b(z)$ and $g(z) - b(z)$) have the same zeros. And we say that $f(z)$ and $g(z)$ *share the finite value* a (resp. *the small function* $b(z)$)

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CM (counting multiplicities) when $f(z) - a$ and $g(z) - a$ (resp. $f(z) - b(z)$ and $g(z) - b(z)$) have the same zeros counting multiplicities.

Uniqueness of the entire function f sharing values with its derivative f' was first investigated by Rubel and Yang [13]. They proved that $f \equiv f'$ if f and f' share two distinct finite constants CM. Mues and Steinmetz [12] pointed out that the same conclusion holds if the two CM shared values are replaced by two IM shared values (for another proof when the two IM shared values are nonzero, see Gundersen [6]).

For one CM shared value, Brück [1] posed the following question: what can be said if one assumes that f and f' share only one value CM plus some growth condition? He proposed the following well-known conjecture.

Conjecture ([1]). Let f be a nonconstant entire function. Suppose that

$$\rho_2(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinite. If f and f' share one finite value a CM, then $f' - a = c(f - a)$ for some nonzero constant c.

Brück [1] proved that this conjecture holds when $a = 0$ and that $N(r, 1/f') = S(r, f)$. The case where f is of finite order was proved by Gundersen and Yang [8]. For further results concerning f sharing one finite value CM with $f^{(k)}$ $(k \geq 2)$, the readers can refer to [16]. Here we recall the following two results.

Theorem 1.1 ([1]). *Let f be a nonconstant entire function. If $\rho_2(f) < \infty$, $\rho_2(f) \notin \mathbb{N}$, and if f and f' share the value 0 CM, then $f' = cf$ for some nonzero constant c.*

Theorem 1.2 ([1]). *Let f be a nonconstant entire function. If f and f' share the value 1 CM, and if $N(r, 1/f') = S(r, f)$, then $f' - 1 = c(f - 1)$ for some nonzero constant c.*

The following two results were proved by Jank, Mues and Volkmann.

Theorem 1.3 ([9]). *Let f be a nonconstant meromorphic function, and let $a \neq 0$ be a finite constant. If f, f', and f'' share the value a CM, then $f' \equiv f$.*

Theorem 1.4 ([9]). *Let f be a nonconstant entire function, and let $a \neq 0$ be a finite constant. If f and f' share the value a IM, and if $f'' = a$ whenever $f = a$, then $f' \equiv f$.*

As mentioned in [8], Theorem 1.3 suggests the following question of Yang and Yi (see [14]).

Question. Let f be a nonconstant meromorphic function, let $a \neq 0$ be a finite constant, and let n and m be positive integers satisfying $n < m$. If f, $f^{(n)}$ and $f^{(m)}$ share the value a CM, where n and m are not both even or both odd, does it follow that $f \equiv f^{(n)}$?

Yang [15] gave the following example to show that the answer to Question above is, in general, negative. Let n and m be positive integers satisfying $m > n+1$, and let $b \neq 0$ be a constant satisfying $b^n = b^m \neq 1$. Set $a = b^n$ and $f(z) = e^{bz} + a - 1$. Then f, $f^{(n)}$ and $f^{(m)}$ share the value a CM, but $f \not\equiv f^{(n)}$.

For entire functions of finite order, Gundersen and Yang [8] generalized Theorem 1.3 and gave a positive answer to Yang and Yi's question when $m = n+1$ by the following result.

Theorem 1.5 ([8]). *Let f be a nonconstant entire function of finite order, let $a \neq 0$ be a finite constant, and let n be a positive integer. If f, $f^{(n)}$ and $f^{(n+1)}$ share the value a CM, then $f \equiv f'$.*

In 2009, Chang and Zhu [2] investigated the small function sharing problem. We recall one of their results:

Theorem 1.6 ([2]). *Let f be an entire function of finite order and a an entire function of order lower than f's. If f and f' share a CM, then $f' - a = c(f - a)$ for some nonzero constant c.*

In Section 2 we will give two results related to Theorems 1.1 and 1.2. An improvement of Theorem 1.2 is shown in Section 3 by applying Theorem 2.2. An alternative improvement of Theorem 1.5 is provided in Section 4. We give some complement to Theorem 1.6 in the final section.

2. Nevanlinna deficiency and value sharing. Looking into Theorems 1.1 and 1.2 of [9], we find that the Nevanlinna deficiency plays an important role. In fact, if f and f' share the value 0 CM, then f has no zeros and hence $\delta(0, f) = 1$ in Theorem 1.1, while in Theorem 1.2, the condition $N(r, 1/f') = S(r, f)$ implies that $\delta(0, f') = 1$. Moreover, we obtain the following result.

Theorem 2.1. *Let f be a nonconstant entire function and $k \geq 1$ be a positive integer. Suppose that f and $f^{(k)}$ share the value 0 CM. Then each zero of f is of order at most $k-1$. In addition, if $k = 1$, then f has no zeros and $\delta(0, f) = \Theta(0, f) = 1$.*

Remark. Theorem 2.1 is obvious and hence its proof is omitted. We should point out that if $k \geq 2$, then $\delta(0, f) = \Theta(0, f) = 1$ may not hold. Take $f(z) = \sin z$ and $k = 2$ for example; then $f(z)$ and $f''(z) = -\sin z$ share 0 CM, but $\delta(0, f) = \Theta(0, f) = 0$.

For the shared value $a \neq 0$ case, we give the following result.

THEOREM 2.2. *Let f be a nonconstant entire function and $k \geq 1$ be a positive integer. Suppose that f and $f^{(k)}$ share the value 1 CM. Then $f-1$ has infinitely many zeros such that each zero of $f-1$ is of order at most k, and*

$$N\left(r, \frac{1}{f-1}\right) \geq \frac{1}{2}T(r,f), \qquad \overline{N}\left(r, \frac{1}{f-1}\right) \geq \frac{1}{k+1}T(r,f),$$

and hence

$$\delta(1,f) \leq \frac{1}{2}, \qquad \Theta(1,f) \leq \frac{k}{k+1}.$$

To prove Theorem 2.2, we need the following Milloux inequality.

LEMMA 2.3 ([5, 11, 14, 17]). *Suppose that $f(z)$ is a nonpolynomial meromorphic function in $|z| < R$ $(\leq \infty)$ and k is a positive integer. If $f(0) \neq 0, \infty$, $f^{(k)}(0) \neq 1$, $f^{(k+1)}(0) \neq 0$, then for $0 < r < R$, we have*

$$T(r,f) < \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)}-1}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f),$$

where

$$S(r,f) = m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{f^{(k+1)}}{f}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-1}\right) + \log\left|\frac{f(0)(f^{(k)}(0)-1)}{f^{(k+1)}(0)}\right| + \log 2.$$

REMARK. In Lemma 2.3, the conditions on f at $z = 0$ can be omitted after amending the constant in $S(r,f)$.

Proof of Theorem 2.2. Suppose that f and $f^{(k)}$ share 1 CM. Then obviously each zero of $f-1$ is of order at most k. Set $g = f-1$. Then by assumption, we have

$$N\left(r, \frac{1}{g}\right) = N\left(r, \frac{1}{f-1}\right) = N\left(r, \frac{1}{f^{(k)}-1}\right) = N\left(r, \frac{1}{g^{(k)}-1}\right). \tag{2.1}$$

From Lemma 2.3 and (2.1) we obtain

$$\begin{aligned} T(r,f) &\leq T(r,g) + O(1) \\ &< N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g^{(k)}-1}\right) - N\left(r, \frac{1}{g^{(k+1)}}\right) + S(r,g) \\ &< 2N\left(r, \frac{1}{f-1}\right) + S(r,g) = 2N\left(r, \frac{1}{f-1}\right) + S(r,f), \end{aligned}$$

which yields

$$N\left(r, \frac{1}{f-1}\right) \geq \frac{1}{2}T(r,f), \quad \text{so} \quad \delta(1,f) \leq \frac{1}{2}.$$

Note that each m-order zero of $g^{(k)}-1$ is counted $m-1$ times in $N(r,1/g^{(k+1)})$ and each zero of g is of order at most k. Applying Lemma 2.3 again, we see that

$$T(r,f)\le T(r,g)+O(1)<k\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)+S(r,g)$$
$$=(k+1)\overline{N}\left(r,\frac{1}{f-1}\right)+S(r,g)=(k+1)\overline{N}\left(r,\frac{1}{f-1}\right)+S(r,f).$$

Thus, we obtain

$$\overline{N}\left(r,\frac{1}{f-1}\right)\ge\frac{1}{k+1}T(r,f),\quad \text{so}\quad \Theta(1,f)\le\frac{k}{k+1}.$$

3. An application of Theorem 2.2

THEOREM 3.1. *Let f be a nonconstant entire function. If f and f' share the value 1 CM, and if $N(r,1/f')<\alpha T(r,f)$, where $\alpha\in[0,1/4)$, then $f'-1=c(f-1)$ for some nonzero constant c.*

To prove Theorem 3.1, besides Theorem 2.2, we also use the following lemma.

LEMMA 3.2 ([14]). *Let f be a meromorphic function, and k be a positive integer. Then*

$$N\left(r,\frac{1}{f^{(k)}}\right)\le N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f).$$

Proof of Theorem 3.1. Since f is an entire function, by Lemma 3.2 we have

$$N\left(r,\frac{1}{f''}\right)\le N\left(r,\frac{1}{f'}\right)+S(r,f)<\alpha T(r,f)+S(r,f). \tag{3.1}$$

Set

$$F=\frac{f'''}{f''}-\frac{f''}{f'}-2\frac{f''}{f'-1}+2\frac{f'}{f-1}. \tag{3.2}$$

Then F is a meromorphic function and hence from the fundamental estimate of the logarithmic derivative, we have

$$m(r,F)=S(r,f). \tag{3.3}$$

Since the poles of F coincide with the zeros of f' and f'', by assumption and (3.1) we have

$$N(r,F)\le N\left(r,\frac{1}{f'}\right)+N\left(r,\frac{1}{f''}\right)+S(r,f)<2\alpha T(r,f)+S(r,f). \tag{3.4}$$

Combining (3.3) with (3.4), we see that $T(r,F)<2\alpha T(r,f)+S(r,f)$.

We assume that $F \not\equiv 0$. Notice that all zeros of $f-1$ and $f'-1$ are simple. Let z_0 be a common zero of $f-1$ and $f'-1$. Then $f''(z_0) \neq 0$, and it is easy to see that F is holomorphic at z_0, and $F(z_0)=0$. Thus we have

$$\begin{aligned} N\left(r, \frac{1}{f-1}\right) &\le N\left(r, \frac{1}{F}\right) \le T(r,F) + O(1) \\ &\le 2\alpha T(r,F) + S(r,f) < \frac{1}{2} T(r,f). \end{aligned}$$

However, by Theorem 2.2, we have

$$N\left(r, \frac{1}{f-1}\right) \ge \frac{1}{2} T(r,f), \tag{3.5}$$

a contradiction.

Thus we have $F \equiv 0$. Integration of (3.2) yields

$$A\frac{f''}{f'} = \left(\frac{f'-1}{f-1}\right)^2, \tag{3.6}$$

where A is a nonzero constant. Since f and f' share 1 CM, for a point z_1 satisfying $f(z_1) = f'(z_1) = 1$ we have $f''(z_1) \neq 0$, and hence $A = 1/f''(z_1)$. Thus if we assume that f''/f' is not a constant function, we see from (3.6) that

$$N\left(r, \frac{1}{f-1}\right) = O\left(N\left(r, \frac{f''}{f'}\right)\right) = S(r,f),$$

a contradiction to (3.5).

Therefore, f''/f' is a constant function and hence there exists a nonzero constant c such that

$$\frac{f'-1}{f-1} = c.$$

This completes the proof of Theorem 3.1.

4. Another improvement of Theorem 1.5. As another application of Theorem 2.2 (in fact, we use Theorem 3.1 directly), we give an alternative improvement of Theorem 1.5 by the following result, in which the order restriction on f is omitted.

THEOREM 4.1. *Let f be a nonconstant entire function, let $a \neq 0$ be a finite constant, and let n be a positive integer. If $f^{(n)}$ and $f^{(n+1)}$ share the value a CM, then $N(r, 1/f') < \alpha T(r,f)$ where $\alpha \in [0, 1/8)$, and if there is some finite value z_0 such that $f(z_0) \in \{f^{(n)}(z_0), f^{(n+1)}(z_0)\}$, then $f \equiv f'$.*

To prove Theorem 4.1, we recall the following lemma, which is a generalization of the second main theorem of Nevanlinna theory.

Lemma 4.2 ([3, 17]). *Let f and $\varphi_\nu(z)$ $(\nu = 1, \dots, q)$ be meromorphic functions. Suppose $\varphi_\nu(z)$ $(\nu = 1, \dots, q)$ are distinct and satisfy*

$$T(r, \varphi_\nu) = o\{T(r, f)\} = S(r, f) \qquad (\nu = 1, \dots, q).$$

Then

$$\{q - 1 - o(1)\}T(r, f) < \sum_{\nu=1}^{q} N\left(r, \frac{1}{f - \varphi_\nu}\right) + q\overline{N}(r, f) + S(r, f).$$

Proof of Theorem 4.1. We can assume that $a = 1$. From Lemma 3.2, we get

$$\begin{aligned} N\left(r, \frac{1}{f^{(n+1)}}\right) &\leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f) \\ &\leq \alpha T(r, f) + S(r, f) = \alpha T\left(r, \frac{f^{(n)}}{f} f\right) + S(r, f) \\ &\leq 2\alpha T(r, f^{(n)}) + S(r, f^{(n)}). \end{aligned}$$

Now Theorem 3.1 is valid for $f^{(n)}$ and $f^{(n+1)}$, thus if we set $g = f^{(n)}$, then for some nonzero constant c, we have

$$g' - 1 = c(g - 1), \tag{4.1}$$

which gives

$$g = b + de^{cz} \tag{4.2}$$

for some constants b and $d \neq 0$. From integration of (4.2), we obtain

$$f(z) = P(z) + \frac{d}{c^n} e^{cz}, \tag{4.3}$$

where $P(z)$ is a polynomial with $\deg P \leq n$. Therefore,

$$f'(z) = P'(z) + \frac{d}{c^{n-1}} e^{cz}. \tag{4.4}$$

We claim that $b = 0$. Indeed, if $b \neq 0$, then from (4.2) and (4.3), we find that $\deg P = n \geq 1$. Thus $P'(z) \not\equiv 0$. By Lemma 5.1 and our assumption, (4.4) yields

$$\begin{aligned} (1 - o(1))T(r, f') &< N\left(r, \frac{1}{f'}\right) + N\left(r, \frac{1}{f' - P'}\right) + S(r, f) \\ &< \frac{1}{8}T(r, f) + S(r, f) \leq \frac{1}{4}T(r, f') + S(r, f'), \end{aligned} \tag{4.5}$$

a contradiction.

Now we have proved that $b = 0$. From (4.1) and (4.2), we see that $c = 1$. Then from (4.4), we obtain

$$f(z) = P(z) + de^z \quad \text{and} \quad f'(z) = P'(z) + de^z. \tag{4.6}$$

If $P'(z) \not\equiv 0$, then the same contradiction as in (4.5) can be deduced. Therefore, $P'(z) \equiv 0$, which implies that $P(z) \equiv p \in \mathbb{C}$. By assumption, we can suppose that $f(z_0) = f^{(n)}(z_0)$, and we get

$$p = f(z_0) - f^{(n)}(z_0) = 0.$$

This together with (4.6) indicates that our conclusion is true.

5. Further results for small function sharing. Chang and Zhu [2] gave the following example to show that the condition that the order of the shared function a is smaller than f's is necessary.

EXAMPLE. Let

$$f = e^{2z} - (z-1)e^z, \qquad a = e^{2z} - ze^z.$$

Then $f' - a = e^z(f - a)$.

If we check this example carefully, we find that the order of the function $b = a' - a = e^{2z} - z$ equals f's. Thus, we ask: what can be said if the order of $b = a' - a$ is smaller than f's? Concerning this question, we first prove the following result.

THEOREM 5.1. *Let f and a be entire functions of finite order. If f and f' share the entire function a CM, and if $\rho(f - a) > \rho(a' - a)$, then $f' - a = c(f - a)$ for some nonzero constant c.*

EXAMPLE. Let

$$f = e^{2z} + e^z + 1/2, \qquad a = e^z + 1.$$

Then $\rho(f) = \rho(a) = 1$, $b = a' - a = -1$, and $f' - a = 2(f - a)$. This example satisfies the assumption of Theorem 5.1 but it does not satisfy the assumption of Theorem 1.6.

To prove Theorem 5.1, we need the following lemmas; Lemma 5.3 is proved in [2] with a reasoning similar to that in the proof of Lemma 4 in [7].

LEMMA 5.2 ([7]). *Let f be a nonconstant meromorphic function of finite order ρ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that if $\varphi_0 \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = r(\varphi_0)$ such that for all z satisfying $\arg z = \varphi_0$ and $|z| > R_0$, we have*

$$\left| \frac{f'(z)}{f(z)} \right| \le |z|^{\rho - 1 + \varepsilon}.$$

LEMMA 5.3 ([2]). *Let f be an analytic function on some ray $\arg z = \theta$ starting from $z_0 = r_0 e^{i\theta}$ and $K(x)$ a positive, decreasing, continuous function on the interval $[r_0, +\infty)$. Suppose that $|f'(z)|K(|z|)$ is unbounded on the ray*

$\arg z = \theta$ starting from $z_0 = r_0 e^{i\theta}$. Then there exists an infinite sequence of points $z_n = r_n e^{i\theta}$, where $r_n \to \infty$, such that $|f'(z_n)|K(|z_n|) \to \infty$ and

$$\left|\frac{f(z_n)}{f'(z_n)}\right| \le (1 + o(1))|z_n|.$$

Proof of Theorem 5.1. Let $g = f - a$ and $b = a' - a$. By assumption,

$$\frac{g' + b}{g} = \frac{f' - a}{f - a} = e^{h(z)}, \tag{5.1}$$

where $h(z)$ is an entire function such that $\rho(e^h) \le \max\{\rho(f), \rho(a)\} < \infty$ and b is a small function of g such that $\rho(b) = \sigma < \rho = \rho(g)$. Since $\rho(e^h) < \infty$, we see that $h(z)$ is a polynomial such that $\deg h(z) \le \rho$.

If $b(z) \equiv 0$, then from (5.1), we have

$$T(r, e^h) = m(r, e^h) = m(r, g'/g) = S(r, g) = O(r^{\rho+1}).$$

Thus $h(z)$ is a constant function, and hence our assertion holds.

Next we assume that $b(z) \not\equiv 0$ and $h(z)$ is a nonconstant polynomial such that $\rho \ge \deg h(z) = n \ge 1$.

From Lemma 5.2, for any given $\varepsilon > 0$ $(0 < \varepsilon < (\rho - \sigma)/3)$, there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that if $\varphi_0 \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = R(\varphi_0)$ such that for all z satisfying $\arg z = \varphi_0$ and $|z| > R_0$, we have

$$\left|\frac{g'(z)}{g(z)}\right| \le |z|^{\rho-1+\varepsilon}. \tag{5.2}$$

Set $h(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, where $a_n = \alpha_n e^{i\varphi_n}$, $\alpha_n \ge 0$, $\varphi_n \in [0, 2\pi)$. Denote

$$\begin{aligned} \varOmega_0 &= \{\theta \in [0, 2\pi) : \cos(\varphi_n + n\theta) = 0\} \cup E, \\ \varOmega_+ &= \{\theta \in [0, 2\pi) : \cos(\varphi_n + n\theta) > 0\} \setminus E, \\ \varOmega_- &= \{\theta \in [0, 2\pi) : \cos(\varphi_n + n\theta) < 0\} \setminus E. \end{aligned}$$

Let $\theta \in \varOmega_+$. Then from (5.1) and (5.2), we have

$$\left|\frac{b(re^{i\theta})}{g(re^{i\theta})}\right| \ge |e^{h(re^{i\theta})}| - \left|\frac{g'(re^{i\theta})}{g(re^{i\theta})}\right| \ge \exp\{\mathrm{Re}\{h(re^{i\theta})\}\} - |z|^{\rho-1+\varepsilon} \to \infty,$$

which yields

$$|g(re^{i\theta})| \le |b(re^{i\theta})| \le \exp\{r^{\sigma+\varepsilon}\}. \tag{5.3}$$

Let now $\theta \in \varOmega_-$. We discuss two cases:

Case 1: *$|g'(z)| \exp\{-r^{\sigma+\varepsilon}\}$ is bounded on the ray $\arg z = \theta$.* Then there is some $M = M(\theta) > 0$ such that

$$|g'(re^{i\theta})| \exp\{-r^{\sigma+\varepsilon}\} \le M,$$

which gives

$$|g(re^{i\theta})| = |g'(0)| + \left|\int_0^z g'(\zeta)\,d\zeta\right| \le 2Mr\exp\{r^{\sigma+\varepsilon}\} \le \exp\{r^{\sigma+2\varepsilon}\}. \tag{5.4}$$

CASE 2: $|g'(z)|\exp\{-r^{\sigma+\varepsilon}\}$ *is unbounded on* $\arg z = \theta$. Then by Lemma 5.3, there exist $z_n = r_n e^{i\theta}$, where $r_n \to \infty$, such that $|g'(z_n)|\exp\{-r_n^{\sigma+\varepsilon}\} \to \infty$ and

$$\left|\frac{g(z_n)}{g'(z_n)}\right| \le (1+o(1))|z_n|. \tag{5.5}$$

Since $b(z) \not\equiv 0$, from (5.1) and (5.5) we get

$$|g'(r_n e^{i\theta})| = \left|\frac{b(r_n e^{i\theta})}{1 - \frac{g(r_n e^{i\theta})}{g'(r_n e^{i\theta})}e^{h(r_n e^{i\theta})}}\right| \le \left|\frac{b(r_n e^{i\theta})}{1-o(1)}\right| \le 2\exp\{r_n^{\sigma+\varepsilon}\},$$

which contradicts that $|g'(z_n)|\exp\{-r_n^{\sigma+\varepsilon}\} \to \infty$.

Now we deduce from (5.3) and (5.4) that for each $\theta \in \Omega_+ \cup \Omega_-$ and sufficiently large r, we have

$$|g(re^{i\theta})| \le \exp\{r^{\sigma+2\varepsilon}\}. \tag{5.6}$$

Notice that $\Omega_0 = [0, 2\pi) \setminus (\Omega_+ \cup \Omega_-)$ has linear measure zero. Therefore, we can deduce from (5.6) and the Phragmén–Lindelöf theorem (see [4, pp. 138–139]) that (5.6) holds for each $\theta \in [0, 2\pi)$. Then we get a contradiction that $\rho = \rho(g) \le \sigma$. This completes our proof.

By Theorem 5.1, we have two corollaries.

COROLLARY 5.4. *Let f be an entire function of finite order. If f and f' share an entire function a CM, and if $b = a' - a$ is a polynomial, then $f' - a = c(f - a)$ for some nonzero constant c.*

COROLLARY 5.5. *Let f be an entire function of finite order. If f and f' have the same fixed points with same multiplicities, then $f' - z = c(f - z)$ for some nonzero constant c.*

EXAMPLE. Let $f = e^{2z} + z/2 + 1/4$. Then f and f' share $a = z$ CM, and f and f' have the same fixed points with same multiplicities. In this case, we have $b = a' - a = 1 - z$, and $f' - a = 2(f - a)$.

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