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On the solvability of a fourth-order multi-point boundary value problem

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Abstract. We are concerned with the solvability of the fourth-order four-point boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0,1], \\ u(0) = u(1) = 0, \\ au''(\zeta_1) - bu'''(\zeta_1) = 0, \quad cu''(\zeta_2) + du'''(\zeta_2) = 0, \end{cases}$$

where $0 \le \zeta_1 < \zeta_2 \le 1$, $f \in C([0,1] \times [0,\infty) \times (-\infty,0],[0,\infty))$. By using Guo–Krasnosel'skiĭ's fixed point theorem on cones, some criteria are established to ensure the existence, nonexistence and multiplicity of positive solutions for this problem.

1. Introduction. Boundary value problems for fourth-order differential equations have important applications in mechanics, hence have attracted considerable attention over the last several decades. Some new methods were developed and many general and beautiful results concerning the existence of solutions were established (see [1–10] and the references therein).

In this paper, we consider the following fourth-order four-point boundary value problem:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0,1], \\ u(0) = u(1) = 0, \\ au''(\zeta_1) - bu'''(\zeta_1) = 0, \quad cu''(\zeta_2) + du'''(\zeta_2) = 0. \end{cases} \tag{1.1}$$

This problem occurs in beam theory, e.g. for a beam with small deformation; a beam of a material which satisfies a nonlinear power-like stress law; or a beam with two-sided links which satisfies a nonlinear power-like elasticity law (for details, see [11]).

In this paper, we aim to establish criteria to ensure the existence, nonexistence and multiplicity of positive solutions for problem (1.1).

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Throughout this paper, we assume the following conditions hold:

(C_1) a, b, c, d are nonnegative constants, $0 \le \zeta_1 < \zeta_2 \le 1$.
(C_2) $\Upsilon = ad + bc + ac(\zeta_2 - \zeta_1) \ne 0$, $-a\zeta_1 + b \ge 0$, $c(\zeta_2 - 1) + d \ge 0$.
(C_3) $f : [0,1] \times [0,\infty) \times (-\infty,0] \to [0,\infty)$ is continuous.

This paper is organized as follows: In Section 2, some notation and preliminaries are introduced. The existence, nonexistence and multiplicity results are given in Section 3. In Section 4, we extend the existence theorems to the limit case. Some examples and remarks are presented in the last section to illustrate the applications of our results.

2. Preliminaries. Let E be a Banach space. A closed convex set $K \subset E$ is called a *cone* if $x \in K$ and $x \ne 0$ implies $\alpha x \in K$ for $\alpha \ge 0$ and $\alpha x \notin K$ for $\alpha < 0$. A cone defines a *partial order* in the Banach space E: $x \le y$ if and only if $y - x \in K$.

The well-known Guo–Krasnosel'skiĭ fixed point theorem on cones is useful to establish existence and multiplicity results for differential equations.

LEMMA 2.1. *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$ and let $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Tu\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

DEFINITION 2.2. A function $x \in C^{(4)}[0,1]$ is called a *positive solution* of Problem (1.1) if x is a solution of Problem (1.1) and $x(t) > 0$ in $(0,1)$.

In Problem (1.1), let $v(t) = -u''(t)$. Then we get the following differential-integral equation:

$$
(2.1)\qquad \begin{cases} -v''(t) = f\Big(t, \int_0^1 K(t,s)v(s)\,ds, -v(t)\Big), & t \in [0,1], \\ av(\zeta_1) - bv'(\zeta_1) = 0, \quad cv(\zeta_2) + dv'(\zeta_2) = 0, \end{cases}
$$

where

$$
K(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}
$$

From (2.1), we get the integral equation

$$
v(t) = \int_{\zeta_1}^{\zeta_2} G(t,s) f\Big(s, \int_0^1 K(s,\tau)v(\tau)\,d\tau, -v(s)\Big)\,ds,
$$

where

$$G(t,s)=\begin{cases}\dfrac{1}{\Upsilon}((a(s-\zeta_1)+b)(d+c(\zeta_2-t)), & s\le t\le 1,\ \zeta_1\le s\le\zeta_2,\\ \dfrac{1}{\Upsilon}((a(t-\zeta_1)+b)(d+c(\zeta_2-s)), & 0\le t\le s,\ \zeta_1\le s\le\zeta_2,\end{cases}$$

$$\Upsilon=ad+bc+ac(\zeta_2-\zeta_1).$$

Let $E=C[0,1]$. Then E is a Banach space endowed with the norm $\|x\|=\max_{0\le t\le 1}|x(t)|$. Let $C^+[0,1]=\{x\in E \mid x(t)\ge 0\}$. Then $C^+[0,1]$ is a normal cone in E.

Let $K=\{x\in C^+[0,1] \mid \min_{\eta_1\le t\le\eta_2}x(t)\ge \frac14\|x\|\}$, where $\eta_1=\zeta_1+\frac14(\zeta_2-\zeta_1)$ and $\eta_2=\zeta_2-\frac14(\zeta_2-\zeta_1)$. Then K is a normal cone in E.

Define an operator $T: C^+[0,1]\to E$ as

$$Tx(t)=\int_{\zeta_1}^{\zeta_2}G(t,s)f\Big(s,\int_0^1 K(s,\tau)x(\tau)\,d\tau,-x(s)\Big)\,ds.$$

Lemma 2.3.

(1) $0\le G(t,s)\le G(s,s)$ *for* $t\in[0,1], s\in[\zeta_1,\zeta_2]$.
(2) $T:C^+[0,1]\to E$ *is completely continuous.*
(3) $T(C^+[0,1])\subset K$, *in particular* $T(K)\subset K$.

Proof. (1) can be obtained by a simple computation. The proof of (2) is standard, and we omit it. Using Lemmas 2.3 and 2.4 of [10], we can easily verify (3). ■

Lemma 2.4. *A function* $x\in C^{(4)}[0,1]$ *is a positive solution of Problem* (1.1) *if and only if* $u=-x''$ *is a nonzero fixed point of* T *in* K.

Proof. Assume u is a nonzero fixed point of T in K. Let x be a solution of the linear boundary value problem

$$\begin{cases}-x''(t)=u(t), & t\in[0,1],\\ x(0)=x(1)=0.\end{cases}$$

Then $x(t)=\int_0^1 K(t,s)u(s)\,ds$.

Since u is a solution of the integral equation

$$v(t)=\int_{\zeta_1}^{\zeta_2}G(t,s)f\Big(s,\int_0^1 K(s,\tau)v(\tau)\,d\tau,-v(s)\Big)\,ds,$$

it is a solution of the boundary value problem (2.1). Hence we get

$$\begin{cases}x^{(4)}(t)=f(t,x(t),x''(t)), & t\in[0,1],\\ x(0)=x(1)=0,\\ ax''(\zeta_1)-bx'''(\zeta_1)=0, \quad cx''(\zeta_2)+dx'''(\zeta_2)=0,\end{cases}$$

i.e. x is a solution of Problem (1.1). Since $K(t,s) > 0, (t,s) \in (0,1) \times (0,1)$, and $u(t) \geq 0$ for $t \in [0,1]$ and $u(t) > 0$ for $t \in [\eta_1, \eta_2]$, it follows that $x(t) > 0$ in $(0,1)$, i.e. $x(t)$ is a positive solution of Problem (1.1).

Conversely, if $x \in C^{(4)}[0,1]$ is a positive solution of Problem (1.1), then $x(t) \geq 0, x''(t) \leq 0$ in $[0,1]$ and $x(t) > 0$ in $(0,1)$.

Let $u = -x''$. Then $x(t) = \int_0^1 K(t,s)u(s)\,ds,\ u(t) \geq 0, u(t) \neq 0$ in $[0,1]$ and $u(t)$ satisfies

$$\begin{cases} -u''(t) = f\Big(t, \int_0^1 K(t,s)u(s)\,ds, -u(t)\Big), & t \in [0,1], \\ au(\zeta_1) - bu'(\zeta_1) = 0, \quad cu(\zeta_2) + du'(\zeta_2) = 0, \end{cases}$$

which implies u is a fixed point of T in $C^+[0,1]$. By Lemma 2.3(3), we have $u \in K$. ∎

LEMMA 2.5. *Let $x \in K$. Then for $s \in [0,1]$,*

$$\frac{h(s)}{4}\|x\| \leq \int_0^1 K(s,\tau)x(\tau)\,d\tau \leq g(s)\|x\|,$$

where $h(s) = \frac{1}{2}s(1-s) - \frac{1}{2}(1-s)\eta_1^2 - \frac{1}{2}s(1-\eta_2)^2$ and $g(s) = \frac{1}{2}s(1-s)$.

Proof. For $s, \tau \in [0,1]$, noting that $K(s,\tau) \geq 0$, for $x \in K$ we have

$$\int_{\eta_1}^{\eta_2} K(s,\tau)x(\tau)\,d\tau \leq \int_0^1 K(s,\tau)x(\tau)\,d\tau \leq \int_0^1 K(s,\tau)\|x\|\,d\tau.$$

Since $\min_{\eta_1 \leq t \leq \eta_2} x(t) \geq \frac{1}{4}\|x\|$, we obtain

$$\frac{1}{4}\int_{\eta_1}^{\eta_2} K(s,\tau)\,d\tau\,\|x\| \leq \int_0^1 K(s,\tau)x(\tau)\,d\tau \leq \int_0^1 K(s,\tau)\,d\tau\,\|x\|.$$

i.e., $\frac{h(s)}{4}\|x\| \leq \int_0^1 K(s,\tau)x(\tau)\,d\tau \leq g(s)\|x\|$. ∎

3. Existence, multiplicity and nonexistence of positive solutions. Let

$$\gamma_1 = \tfrac{1}{4}\min\{h(\eta_1), h(\eta_2)\}, \qquad \gamma_2 = \tfrac{1}{4}\min\{h(\zeta_1), h(\zeta_2)\},$$

where h is defined in Lemma 2.5. For $l > 0$, define

$$H(l) = \min\big\{f(t,u,v) \bigm| (t,u,v) \in [\eta_1,\eta_2] \times [\gamma_1 l, \tfrac{1}{8}l] \times \big[-l, -\tfrac{1}{4}l\big]\big\},$$
$$F(l) = \max\big\{f(t,u,v) \bigm| (t,u,v) \in [\zeta_1,\zeta_2] \times \big[\gamma_2 l, \tfrac{1}{8}l\big] \times [-l, 0]\big\}.$$

Then $H, F : (0,\infty) \to [0,\infty)$ are continuous. Let

$$\alpha = \frac{1}{\int_{\zeta_1}^{\zeta_2} G(s,s)\,ds}, \qquad \beta = \frac{4}{\int_{\eta_1}^{\eta_2} G(s,s)\,ds}.$$

The main result of this paper reads as follows:

THEOREM 3.1. *If there exist distinct positive numbers a, b such that $F(a) \le a\alpha$ and $H(b) \ge b\beta$, then Problem* (1.1) *has a positive solution x^* satisfying* $\min\{a,b\} \le \|x^{*\prime\prime}\| \le \max\{a,b\}$.

Proof. Firstly, it is easy to verify that when $s \in [\eta_1, \eta_2]$,

$$\gamma_1 \|x\| \le \int_0^1 K(s,\tau)x(\tau)\,d\tau \le \frac{1}{8}\|x\|,$$

while $s \in [\zeta_1, \zeta_2]$ implies

$$\gamma_2 \|x\| \le \int_0^1 K(s,\tau)x(\tau)\,d\tau \le \frac{1}{8}\|x\|.$$

Without loss of generality, we assume that $a < b$. Let $K_a = \{x \in K \mid \|x\| = a\}$, $K_b = \{x \in K \mid \|x\| = b\}$, and $K_{a,b} = \{x \in K \mid a \le \|x\| \le b\}$.

If $x \in K_a$, then

$$\begin{aligned}
\|Tx\| &= \max_{0\le t\le 1} \int_{\zeta_1}^{\zeta_2} G(t,s) f\Big(s, \int_0^1 K(s,\tau)x(\tau)\,d\tau, -x(s)\Big)\,ds \\
&\le \int_{\zeta_1}^{\zeta_2} G(s,s) f\Big(s, \int_0^1 K(s,\tau)x(\tau)\,d\tau, -x(s)\Big)\,ds \\
&\le \int_{\zeta_1}^{\zeta_2} G(s,s)F(a)\,ds \le a\alpha \int_{\zeta_1}^{\zeta_2} G(s,s)\,ds = \|x\|.
\end{aligned}$$

If $x \in K_b$, then

$$\begin{aligned}
\|Tx\| &\ge \min_{\eta_1\le t\le \eta_2} \int_{\zeta_1}^{\zeta_2} G(t,s) f\Big(s, \int_0^1 K(s,\tau)x(\tau)\,d\tau, -x(s)\Big)\,ds \\
&\ge \frac{1}{4}\int_{\zeta_1}^{\zeta_2} G(s,s) f\Big(s, \int_0^1 K(s,\tau)x(\tau)\,d\tau, -x(s)\Big)\,ds \\
&\ge \frac{1}{4}\int_{\eta_1}^{\eta_2} G(s,s) f\Big(s, \int_0^1 K(s,\tau)x(\tau)\,d\tau, -x(s)\Big)\,ds \\
&\ge \frac{1}{4}\int_{\eta_1}^{\eta_2} G(s,s)H(b)\,ds \ge \frac{1}{4} b\beta \int_{\eta_1}^{\eta_2} G(s,s)\,ds = \|x\|.
\end{aligned}$$

It follows from Lemma 2.1 that T has a fixed point $u^* \in K_{a,b}$, i.e., Problem (2.1) has a solution $u^* \ge 0$ with $a \le \|u^*\| \le b$. Let $x^*(t) = \int_0^1 K(t,s)u^*(s)\,ds$. Then x^* is a positive solution of Problem (1.1). ∎

By Theorem 3.1, we can give the following multiplicity result:

COROLLARY 3.2. *Assume there exist* $n+1$ *positive numbers* $0 < a_1 < \cdots < a_{n+1}$ *such that either*

(i) $F(\alpha_{2k-1}) < \alpha a_{2k-1}$ *and* $H(\alpha_{2k}) > \beta a_{2k}$, $k = 1, \ldots, [(n+1)/2]$, *or*
(ii) $F(\alpha_{2k}) < \alpha a_{2k}$ *and* $H(\alpha_{2k-1}) > \beta a_{2k-1}$, $k = 1, \ldots, [(n+1)/2]$.

Here $[\lambda]$ *denotes the integer part of* $\lambda \in \mathbb{R}$. *Then Problem* (1.1) *has at least* n *positive solutions* $x_1, \ldots, x_n$, *which satisfy*

$$a_k < \|x_k''\| < a_{k+1}, \qquad k = 1, \ldots, n.$$

Proof. Suppose case (i) holds. Since $F, H : (0,\infty) \to [0,\infty)$ are continuous, for every pair (α_k, α_{k+1}) there exists a par (b_k, c_k) such that $\alpha_k < b_k < c_k < \alpha_{k+1}$ and

$$\begin{aligned} F(b_{2k-1}) &\le b_{2k-1}\alpha, & H(c_{2k-1}) &\ge c_{2k-1}\beta, & k &= 1, \ldots, [(n+1)/2], \\ H(b_{2k}) &\ge b_{2k}\beta, & F(c_{2k}) &\le c_{2k}\alpha, & k &= 1, \ldots, [(n+1)/2]. \end{aligned}$$

According to Theorem 3.1, every pair (b_k, c_k) gives a positive solution x of Problem (1.1) such that

$$b_k < \|x''\| < c_k, \qquad k = 1, \ldots, n.$$

When case (ii) holds, the proof is similar. ∎

COROLLARY 3.3. *Assume there exists* $a > 0$ *such that either*

(i) $F(a) < a\alpha$, *or*
(ii) $H(a) > a\beta$.

Then T *has no fixed point* $u \in K$ *satisfying* $\|u\| = a$.

Proof. Suppose case (i) holds.

On the contrary, assume that $x \in K$ is such that $\|x\| = a$ and $Tx = x$. Then

$$\begin{aligned} a = \|Tx\| &= \max_{0\le t\le 1} \int_{\zeta_1}^{\zeta_2} G(t,s) f\Big(s, \int_0^1 K(s,\tau)x(\tau)\,d\tau, -x(s)\Big)\,ds \\ &\le \int_{\zeta_1}^{\zeta_2} G(s,s) f\Big(s, \int_0^1 K(s,\tau)x(\tau)\,d\tau, -x(s)\Big)\,ds \\ &\le \int_{\zeta_1}^{\zeta_2} G(s,s)F(a)\,ds < a\alpha \int_{\zeta_1}^{\zeta_2} G(s,s)\,ds = a, \end{aligned}$$

a contradiction.

Case (ii) is handled similarly. ∎

By Corollary 3.3 and Lemma 2.4, we have the following nonexistence result.

THEOREM 3.4. *If*

$$\sup_{l>0} \frac{F(l)}{l} < \alpha \quad \textit{or} \quad \inf_{l>0} \frac{H(l)}{l} > \beta,$$

then Problem (1.1) *has no positive solution.*

4. Limit case. Let

$$H_0 = \lim_{l\to 0^+} \frac{H(l)}{l}, \quad H_\infty = \lim_{l\to\infty} \frac{H(l)}{l}, \quad F_0 = \lim_{l\to 0^+} \frac{F(l)}{l}, \quad F_\infty = \lim_{l\to\infty} \frac{F(l)}{l}.$$

The following existence and multiplicity results in the limit case follow directly from Theorem 3.1.

COROLLARY 4.1. *Assume that either*

(1) $F_0 < \alpha$, $H_\infty > \beta$, *or*
(2) $F_\infty < \alpha$, $H_0 > \beta$.

Then Problem (1.1) *has at least one positive solution.*

COROLLARY 4.2. *Let* $a > 0$ *be a constant. Assume that either*

(1) $F_0 < \alpha$, $F_\infty < \alpha$, $H(a) > a\beta$, *or*
(2) $F(a) < a\alpha$, $H_0 > \beta$, $H_\infty > \beta$.

Then Problem (1.1) *has at least two positive solutions* x_1, x_2 *satisfying*

$$0 < \|x_1''\| < a < \|x_2''\|.$$

5. Examples and remarks. In this section, we present some examples to illustrate the applications of our results.

Let $a = b = c = d = 1$, $\zeta_1 = \frac{1}{4}$, $\zeta_2 = \frac{3}{4}$. Then $-a\zeta_1 + b = \frac{3}{4} > 0$, $c(\zeta_2 - 1) + d = \frac{3}{4} > 0$, $\varUpsilon = \frac{5}{2}$, $\eta_1 = \frac{3}{8}$, $\eta_2 = \frac{5}{8}$,

$$G(t,s) = \begin{cases} \frac{2}{5}\left(\frac{3}{4} + s\right)\left(\frac{7}{4} - t\right), & s \le t \le 1,\ \frac{1}{4} \le s \le \frac{3}{4}, \\ \frac{2}{5}\left(\frac{3}{4} + t\right)\left(\frac{7}{4} - s\right), & 0 \le t \le s,\ \frac{1}{4} \le s \le \frac{3}{4}. \end{cases}$$

Hence

$$\max_{0\le t\le 1} \int_{\zeta_1}^{\zeta_2} G(t,s)\,ds = \frac{9}{32}, \qquad \min_{0\le t\le 1} \int_{\zeta_1}^{\zeta_2} G(t,s)\,ds = \frac{5}{32},$$

and $\alpha = \frac{240}{74}, \beta = \frac{7680}{299}, \gamma_1 = \frac{3}{256}, \gamma_2 = \frac{3}{512}$.

EXAMPLE 5.1. Let $f(t,u,v) = te^{-u} - \frac{160v}{v^2+1}$. Then $f : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$ is continuous and there exist positive numbers $l_1 = 1$ and $l_2 = 160$ such that

$$H(l_1) > \beta l_1, \qquad F(l_2) < \alpha l_2.$$

Theorem 3.1 implies that the boundary value problem

$$\begin{cases} u^{(4)}(t) = te^{-u(t)} - \dfrac{160u''(t)}{(u''(t))^2+1}, \quad t \in [0,1], \\ u(0) = u(1) = 0, \\ u''\left(\frac{1}{4}\right) - u'''\left(\frac{1}{4}\right) = 0, \quad u''\left(\frac{3}{4}\right) + u'''\left(\frac{3}{4}\right) = 0, \end{cases}$$

has at least one positive solution.

Remark 5.2. (1) It is easy to check that when f is either sublinear, i.e.

$$\min f_0 = \lim_{-v\to 0^+} \min_{t\in[0,1]} \inf_{u\in[0,\infty)} \frac{f(t,u,v)}{-v} = \infty,$$

$$\max f_\infty = \lim_{-v\to\infty} \max_{t\in[0,1]} \sup_{u\in[0,\infty)} \frac{f(t,u,v)}{-v} = 0,$$

or superlinear, i.e.

$$\max f_0 = \lim_{-v\to 0^+} \max_{t\in[0,1]} \sup_{u\in[0,\infty)} \frac{f(t,u,v)}{-v} = 0,$$

$$\min f_\infty = \lim_{-v\to\infty} \min_{t\in[0,1]} \inf_{u\in[0,\infty)} \frac{f(t,u,v)}{-v} = \infty,$$

as defined in [10], there always exist distinct positive numbers a, b such that $F(a) \le a\alpha$ and $H(b) \ge b\beta$. Hence by Theorem 3.1, Problem (1.1) has at least one positive solution.

(2) In Example 5.1, we can see that

$$\min f_0 = 160, \qquad \max f_\infty = 0, \qquad \max f_0 = \infty, \qquad \min f_\infty = 0,$$

i.e. f is neither sublinear nor superlinear. Thus the conditions of Theores 3.1 or 3.2 in [10] do not hold. However, by Theorem 3.1 of this paper, we can obtain the existence of a positive solution.

Example 5.3. Let $f(t,u,v) = te^{-u} + \min\{v^2 - v, 170156\}$. Then $f : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$ is continuous and there exist numbers $l_1 = 2, l_2 = 412, l_3 = 60000$ such that

$$F(l_1) < \alpha l_1, \qquad H(l_2) > \beta l_2, \qquad F(l_3) < \alpha l_3.$$

Corollary 3.2 implies that the problem

$$\begin{cases} u^{(4)}(t) = te^{-u(t)} + \min\{(u''(t))^2 - u''(t), 170156\}, \quad t \in [0,1], \\ u(0) = u(1) = 0, \\ u''\left(\frac{1}{4}\right) - u'''\left(\frac{1}{4}\right) = 0, \quad u''\left(\frac{3}{4}\right) + u'''\left(\frac{3}{4}\right) = 0, \end{cases}$$

has at least two positive solutions.

Example 5.4. Let $f(t,u,v) = tu + \frac{v^2}{1+v^2}$. Then $f : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$ is continuous. For $l > 0$,

$$0 \le F(l) = \max\left\{f(t,u,v) \mid (t,u,v) \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right] \times \left[\tfrac{3}{512}l, \tfrac{1}{8}l\right] \times [-l, 0]\right\} \le \tfrac{19}{32}l.$$

Then

$$\sup_{l>0} \frac{F(l)}{l} \le \frac{19}{32} < \alpha.$$

Hence an application of Theorem 3.4 shows that the problem

$$\begin{cases} u^{(4)}(t) = tu(t) + \dfrac{(u''(t))^2}{1+(u''(t))^2}, \quad t \in [0,1], \\ u(0) = u(1) = 0, \\ u''\left(\frac{1}{4}\right) - u'''\left(\frac{1}{4}\right) = 0, \quad u''\left(\frac{3}{4}\right) + u'''\left(\frac{3}{4}\right) = 0, \end{cases}$$

has no positive solution.

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